Radiation Reaction for a Scalar Charged Particle in Schwarzschild Eccentric Orbits

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With

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1. Introduction

An era of gravitational wave astronomy has almost arrived.

- GWs are directly related to a particle motion. So, in order to detect GWs, we have to know more accurate particle motion including radiation reaction.

By using assumption of adiabatic orbital evolution and conservation law (balance argument)

$$\left\langle \left(\frac{d\mathcal{E}}{dt} \right) \right\rangle_{\rm particle} = - \left\langle \left(\frac{d\mathcal{E}}{dt} \right) \right\rangle_{\rm GW}, \quad \left\langle \left(\frac{d\mathcal{L}}{dt} \right) \right\rangle_{\rm particle} = - \left\langle \left(\frac{d\mathcal{L}}{dt} \right) \right\rangle_{\rm GW},$$

we can obtain a particle motion from the flux of GW.

In this time, for simplicity, we focus on a scalar charged particle case. But our calculation can be applicable to the gravitational case.

Flux on the Flat Background

We consider a point particle which has a scalar charge q, moving on the flat background. The scalar field ψ induced by this point particle obeys;

$$\Box_{\rm flat}\psi = -\rho, \quad \rho = q\,\delta(\boldsymbol{x} - \boldsymbol{z}(\tau))$$

Here $\boldsymbol{x}$ and $\boldsymbol{z}$ are a field and orbital position, respectively. This equation can be solved easily. A solution is

$$\psi = \int \frac{\rho|_{t'=t-R/c}}{4\pi R} dV, \quad (\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{z},\ R = |\boldsymbol{R}|).$$

Moreover, if we assume $|\boldsymbol{x}| \gg |\boldsymbol{z}|$, then

$$R = |\boldsymbol{x} - \boldsymbol{z}| \approx |\boldsymbol{x}| - \boldsymbol{n}\cdot\boldsymbol{z}, \quad (\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|).$$

Therefore,

$$\begin{aligned}\psi &= \frac{1}{4\pi|\boldsymbol{x}|}\int \left[\rho|_{t'=t-R_0/c} + \frac{\boldsymbol{n}\cdot\boldsymbol{z}}{c}\partial_t\rho|_{t'=t-R_0/c} + \cdots\right] dV \\ &= \frac{1}{4\pi|\boldsymbol{x}|}\left[Q + \frac{1}{c}(\dot{d}\cdot\boldsymbol{n}) + \cdots\right], \quad (|\boldsymbol{x}| \gg |\boldsymbol{z}|)\end{aligned}$$

Here the dot represents the derivatives with respect to t

Dipole Approximation and Circular Case

The energy-momentum tensor of ψ is written by

$$T_{\mu\nu} = \left[\partial_\mu \psi \, \partial_\nu \psi - \frac{1}{2} \eta_{\mu\nu} (\partial \psi)^2 \right].$$

So, using the relation $\boldsymbol{n} \cdot \nabla \approx -\partial_0$, the energy loss can be obtained by

$$\left(\frac{dE}{dt} \right) = c \int_{|\boldsymbol{x}| \to \infty} (-T_{0\,i}\, n^i) |\boldsymbol{x}|^2 d\Omega \approx \int_{|\boldsymbol{x}| \to \infty} \frac{(\ddot{\boldsymbol{d}} \cdot \boldsymbol{n})^2}{16\pi^2 c^3} d\Omega$$
$$= \int_{|\boldsymbol{x}| \to \infty} \frac{|\ddot{\boldsymbol{d}}|^2 \cos^2\theta}{16\pi^2 c^3} 2\pi \sin\theta d\theta = \frac{q^2}{4\pi c^3} \left[\frac{|\ddot{\boldsymbol{z}}|^2}{3} \right]$$

Circular case,

$$|\ddot{\boldsymbol{z}}|^2 = r_0^2 \Omega^4$$

Therefore

$$\left(\frac{dE}{dt} \right) = \frac{q^2}{4\pi r_0^2 c^3} \left[\frac{V^4}{3} \right]. \quad \left(V^2 \equiv \frac{GM}{r_0} \right)$$

Eccentric Case

Eccentric case

$$\boldsymbol{z} = \frac{a}{1 - e\cos\chi}, \qquad |\ddot{\boldsymbol{z}}|^2 = \frac{a^2\Omega^4}{(1-e^2)^4}(1+e\cos\chi)^4 . \qquad \left(\Omega^2 \equiv \frac{GM}{a^3}\right)$$

Therefore, the averaged value of energy loss over one period is

$$\left\langle \left(\frac{dE}{dt}\right) \right\rangle = \frac{q^2}{4\pi c^3}\frac{1}{3}\frac{a^2\Omega^4}{(1-e^2)^4}\left\langle (1+e\cos\chi)^4 \right\rangle$$
$$= \frac{q^2}{4\pi a^2 c^3}\frac{(a\Omega)^4}{3}\frac{(2+e^2)}{2(1-e^2)^{5/2}} .$$

In order to compare with the self-force we show later, we define r_0 as

$$a = r_0(1 + e^2 + \cdots).$$

Then we can obtain

$$\left\langle \left(\frac{dE}{dt}\right) \right\rangle = \frac{q^2}{4\pi r_0^2 c^3}\frac{V^4}{3}\frac{(2+e^2)}{2(1+e^2-2e^3)^4(1-e^2)^{5/2}}$$
$$\approx \frac{q^2}{4\pi a^2 c^3}\frac{V^4}{3}\left(1 - e^2 + \cdots\right) . \qquad \left(V^2 \equiv \frac{GM}{r_0}\right)$$

Beyond Dipole Approx. and the Self-Force

Because a GW signal is very weak, we need more accurate waveform than that obtained by the dipole approximation (quadrapole approximation in the gravitational case).

- Current State (Gravitational case)

1). Schwarzschild, Circular (5.5PN) [Ref. Tanaka *et al.*('96)]
2). Schwarzschild, Eccentric (4PN) [Ref. Mino *et al.*('97)]
3). By using the method Mano *et al.* developed, we can calculate by arbitrary PN order [Ref. Mano *et al.*('96), Fujita and Tagoshi('04)]

It is also important to consider beyond the balance argument, i.e. self-force. This is because that

1). A particle, orbiting a Kerr black hole, has the Carter constant.
$\Rightarrow$ Recently, we formulate the time dependence of Carter constant $< \dot{\mathcal{C}} >$. [Ref. Mino ('04), Sago *et al.*('05), Sago-san's talk]
2). This argument is not valid when the orbit has large value of e.
3). In fact, the reaction force on a particle contains not only dissipative part [$F^\alpha \to -F^\alpha$ as $t \to -t$] but also conservative part [$F^\alpha \to F^\alpha$ as $t \to -t$].

2. S.F. in the Schwarzschild Spacetime

Next we consider a point particle which has a scalar charge q, moving on the Schwarzschild background. The scalar field ψ obeys;

$$\nabla^\alpha \nabla_\alpha \psi(x) = -\rho(x), \qquad \rho(x') = q \int_{-\infty}^{\infty} d\tau (-g)^{-\frac{1}{2}} \delta^{(4)}(x' - z(\tau)).$$

Considering a backreaction of this scalar field, the equation of motion that includes the self-force can be written by

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = \frac{1}{\mu} F^\alpha[\psi], \quad F^\alpha[\psi] \equiv q P^\alpha{}_\beta \partial^\beta \psi(x),$$

The our purpose is to calculate the self-force F^α. But the self-force F^α diverges at the location of the particle.

↓

It is necessary to regularize the self-force.

Regularized Self-Force and S-Part

The full field can be formally divided into two parts;

$$\psi^{\text{full}} = \psi^{S\,(\text{or dir})} + \psi^{R\,(\text{or tail})}, \quad \text{s.t. } F^{\text{S.F.}} = F^{\alpha}[\psi^{R\,(\text{or tail})}].$$

- S-Part (or direct part) = Singular term
  - This comes from the field propagating along the light cone directly from the source particle.
  - It is possible to calculate directly, using the local expansion.
- R-Part (or tail part) = Regularized self-force
  - This arises from the curvature scattering of the field.
  - It is not possible to calculate directly, since this part depends on the whole history of the particle.

In the last few years, it is the S-part that a large number of studies have been made on.

Barack *et al.*('02) calculated the S-part, using $Y_{\ell m}$ decomposition, in the following form:

$$F^{\alpha}[\psi^{S}] = \sum_{\ell=0}^{\infty} \left(A^{\alpha} L + B^{\alpha} + C^{\alpha}/L + D^{\alpha}\right), \; L = \ell + \frac{1}{2} \Rightarrow \; C^{\alpha} = 0, \; \sum_{\ell=0}^{\infty} D^{\alpha} = 0$$

New Analytical Regularization Method

1). divided the full field into two parts; [next page in detail]

$$\psi^{\rm full} = \psi^{\tilde{S}} + \psi^{\tilde{R}},$$

$\tilde{S}$-Part : Inhomogeneous solution, Singular
$\tilde{R}$-Part : Homogeneous solution, Regular

2). From the study of S-Part, the $\tilde{S}$-Part can be calculated, using $Y_{\ell m}$ decomposition, in the following form

$$F^\alpha[\psi^{\tilde{S}}] = \sum_{\ell=0}^{\infty} \left(A^\alpha L + B^\alpha + C^\alpha / L + \tilde{D}^\alpha \right),\ C^\alpha = 0.$$

A, B are same as those of S-Part, while $\tilde{D}$ is not same as D.

3). our way to regularize the self-force is

$$F^{\rm S.F.} = F^\alpha[\psi^R] = F^\alpha[\psi^{\rm full}] - F^\alpha[\psi^S] = \left(F^\alpha[\psi^{\tilde{S}}] - F^\alpha[\psi^S] \right) + F^\alpha[\psi^{\tilde{R}}]$$
$$= \sum_{\ell=0}^{\infty} F_\ell^{(\tilde{S}-S)\alpha} + \sum_{\ell=0}^{\ell_{\rm max}} F_\ell^{(\tilde{R})\alpha} = \sum_{\ell=0}^{\infty} \tilde{D}_\ell^\alpha + \sum_{\ell=0}^{\ell_{\rm max}} F_\ell^{(\tilde{R})\alpha}$$

Here, if we would like the n PN accuracy, we set $\ell_{\rm max} = n+1$.

Method of Our Decomposition

Green fcn. method + $Y_{\ell m}$ decomposition + Fourier transformation

$$\psi^{\rm full} = \int dx'\, G^{\rm ret}(x,x')j(x'), \quad j(x') = q\int d\tau \delta^{(4)}(x' - z(\tau)),$$
$$G^{\rm ret}(x,x') \propto \left[R^{\rm in}(r)R^{\rm up}(r')\theta(r'-r) + (r \leftrightarrow r')\right] YY^* \exp(-i\omega(t-t')).$$

Using the MST's solution, homogeneous solutions are written by

$$R^{\rm in} = \phi_c^{\nu} + \tilde{\beta}\phi_c^{-\nu-1}, \quad R^{\rm up} = \tilde{\gamma}\phi_c^{\nu} + \phi_c^{-\nu-1},$$
$$\phi_c^{\nu} = (2z)^{\nu}\Phi_c^{\nu}(\ell),\ \phi_c^{-\nu-1} = (2z)^{-\nu-1}\Phi_c^{\nu}(-\ell-1),$$
$$\Phi_c^{\nu} = 1 - \frac{z^2}{2(2\ell+3)} - \frac{\ell z}{2z} + \cdots,\ \nu = \ell - \frac{15\ell^2+15\ell-11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2 + \cdots$$

According to the property of $\tilde{S}$-Part and $\tilde{R}$-Part, we define

$$G^{\tilde{S}}(x,x') \propto \left[\phi_c^{\nu}(r)\phi_c^{-\nu-1}(r')\theta(r-r') + (r \leftrightarrow r')\right] YY^* \exp(-i\omega(t-t')),$$
$$G^{\tilde{R}}(x,x') = G^{\rm full}(x,x') - G^{\tilde{S}}(x,x')$$

and

$$\psi^{\tilde{S}/\tilde{R}} = \int dx'\, G^{\tilde{S}/\tilde{R}}(x,x')j(x').$$

More Detail ...

The important fact is that the $\tilde{S}$-part in the frequency domain is given in the form of a simple Taylor series with respect to ω multiplied by $\exp[-i\omega(t-t')]$.

Therefore, the integration over ω just produces $\delta(t-t')$ and its derivatives. Using this technique we can obtain the $\tilde{S}$-part in the time domain relatively easily.

Summary of Our Method

- We propose the $(\tilde{S}-\tilde{R})$-decomposition
- From the asymptotic form of $\tilde{S}$-Part, we can obtain the S-Part without direct calculation.
- Our regularization method is applicable to general orbits and gravitational case, in Schwarzschild spacetime.

Result (Circular Case)

$(\tilde{S}-S)$-Part (Regularization part)

$$F_t^{\tilde{S}-S} = 0\,,$$
$$F_r^{\tilde{S}-S} = \frac{q^2}{4\pi r_0^2}\left[-\frac{73}{133} + \frac{16151}{21014}V^2 + \frac{395567}{106808}V^4 + \left(\frac{1107284037660637}{400151300487120} + \frac{7}{64}\pi^2\right)V^6 + \cdots\right].$$

This part does not contain the dissipative part $[F^\alpha \to -F^\alpha$ as $t \to -t]$.

- $\tilde{R}$-Part (Remained part)

$$F_t^{\tilde{R}} = \frac{q^2 V}{4\pi r_0^2}\left[\frac{1}{3}V^3 - \frac{1}{6}V^5 + \frac{2\pi}{3}V^6 + \cdots\right],$$
$$F_r^{\tilde{R}} = \frac{q^2}{4\pi r_0^2}\left[\frac{73}{133} - \frac{16151}{21014}V^2 - \frac{395567}{106808}V^4 + \left(-\frac{4}{3}\gamma - \frac{4}{3}\ln(2V) - \frac{1196206548879997}{400151300487120}\right)V^6 + \cdots\right].$$

R-Part (Regularized self-force)

$$F_t^{R} = \frac{q^2 V}{4\pi r_0^2}\left[\frac{1}{3}V^3 - \frac{1}{6}V^5 + \frac{2\pi}{3}V^6 + \cdots\right],$$
$$F_r^{R} = \frac{q^2}{4\pi r_0^2}\left[\left(-\frac{4}{3}\gamma + \frac{7}{64}\pi^2 - \frac{4}{3}\ln(2V) - \frac{2}{9}\right)V^6 + \cdots\right].$$

The leading term of energy loss is consistent with the previous analysis.

Consistency Check

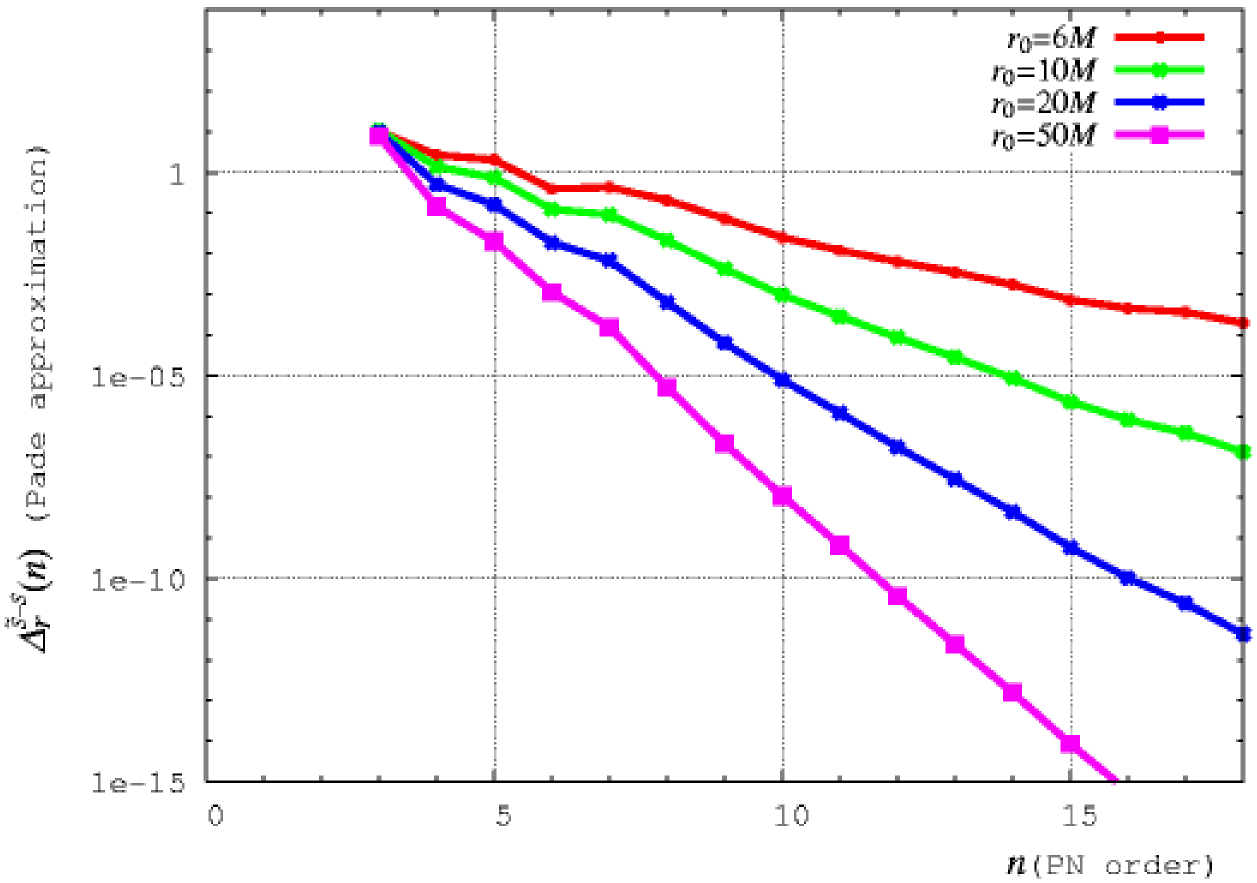
The value of F_r^{R} (Our result: 18PN+$\ell_{\mathrm{max}} = 19$)

r_0	$6M$	$10M$	$20M$
Our result	$1.67620878 \times 10^{-4}$	$1.378448171 \times 10^{-5}$	$4.937905866 \times 10^{-7}$
Detweiler et al.	1.6772834×10^{-4}	$1.37844828 \times 10^{-5}$	4.937906×10^{-7}

In our computation the accuracy is limited by $(\tilde{S} - S)$-part. Hence, the accuracy of the full regularized force can be read from the Figure.

This error may seem large, but if we use our result as a template for LISA, it turns out that the error is small enough [Ref. W.H et al.(2004)]

Error of PN expansion



3. Bound Orbits

1). Glampedakis and Kennefick ('02)

$$r_p = \frac{p}{1-e}, \quad r_a = \frac{p}{1+e}$$

2). Apostolatos *et al.*('93)

$$\frac{\partial V(r)}{\partial r}(r = r_0) = 0, \quad r_p = r_0(1+e)$$

In this parametrization, r_a is written by

$$r_a(v = 1/\sqrt{10}) = 1 - e + 1.5e^2 - 2.25e^3 + 3.25e^4 - 4.50e^5 + 5.91e^6$$
$$-7.20e^7 + 7.80e^8 - 6.57e^9 + \mathcal{O}(e^{10}).$$

The convergence is very slow but, in this time, we used this convention.

3). (Future work) The convergence improve a little.

$$\frac{\partial V(r)}{\partial r}(r = r_0) = 0, \quad r_p = \frac{r_0}{1-e}$$

In this parametrization, r_a is written by

$$r_a(v = 1/\sqrt{10}) = 1 - e + 0.50e^2 - 0.25e^3 + 10^{-10}e^4 - 10^{-11}e^5$$
$$-0.093e^6 - 0.0156e^7 - 0.0859e^8 - 0.0742e^9 + \mathcal{O}(e^{10})$$

Parameterization

The solution of the geodesic equation for slightly eccentric orbits has been given by Apostolatos *et al.*('93). We define r_0 and e as

$$\frac{\partial V(r)}{\partial r}(r = r_0) = 0, \quad V(r = r_0(1+e)) = 0.$$

Here V is an potential for the radial motion. From these equations, $\mathcal{E}$ and $\mathcal{L}_z$ are expressed in terms of r_0 and e as

$$\mathcal{E}^2 = \frac{(1-2v^2)^2}{1-3v^2} + \frac{v^2(1-6v^2)}{1-3v^2}e^2 + \cdots, \ \mathcal{L}_z = \frac{M}{v^2}\frac{v}{\sqrt{1-3v^2}}, \text{ where } v \equiv \sqrt{M/r_0}.$$

Then expanding the geodesic equations in power of e, the solution is

$$r(t) = r_0[1 + e\cos\Omega_r t + \cdots] \quad \varphi(t) = \Omega_\varphi t - ep_1(v)\sin\Omega_r t + \cdots$$

$$\text{where} \quad \Omega_r = \frac{v^3}{M}[(1-6v^2)^{1/2} + \cdots], \quad \Omega_\varphi = \frac{v^3}{M}[1 - f(v)e^2 + \cdots].$$

and p_1 and f are some functions of v.

Result (Eccentric Case)

R-Part (Regularized self-force):

$$F_t^R = \frac{q^2 V}{4\pi r_0^2}\left[\frac{1}{3}V^3 - \frac{1}{6}V^5 + \frac{2}{3}\pi V^6 + e\left\{\left(-\frac{2}{3}V^3 - V^5 - \frac{20}{3}\pi V^6\right)\cos(\Omega_r t)\right.\right.$$
$$\left. + \left(-4V^4 + \left(\frac{70}{3} - 12\gamma - \frac{100}{3}\ln(2) - \frac{45}{64}\pi^2 - 6\ln(V)\right)V^6\right)\sin(\Omega_r t)\right\}$$
$$+e^2\left\{-\frac{1}{3}V^3 + \frac{20}{3}V^5 + \pi V^6 + \left(\frac{2}{3}V^3 + \frac{1}{3}V^5 + \frac{20}{3}\pi V^6\right)\cos(\Omega_r t)\right.$$
$$\left.\left. + \left(-\frac{1}{3}V^3 + \frac{9}{2}V^5 + \frac{37}{2}\pi V^6\right)\cos(2\Omega_r t) + \cdots\right\} + \cdots\right]$$
$$F_r^R = \frac{q^2}{4\pi r_0^2}\left[\left(-\frac{4}{3}\gamma + \frac{7}{64}\pi^2 - \frac{4}{3}\ln(2V) - \frac{2}{9}\right)V^6\right.$$
$$+e\left\{\left(-\frac{92}{9} - \frac{7}{32}\pi^2 + \frac{64}{3}\ln(2) + \frac{32}{3}\gamma + \frac{8}{3}\ln(V)\right)V^6\cos(\Omega_r t)\right.$$
$$\left.\left. + \left(-\frac{2}{3}V^3 + 8V^5 - 4\pi V^6\right)\sin(\Omega_r t)\right\} + \cdots\right]$$

This energy loss is also consistent with the previous analysis.

4． Bound Orbits Including R.R.

The equation of motion that includes the self-force can be written by

$$\frac{d\tilde{\mathcal{E}}}{d\tau} = -\frac{1}{\mu}F_t[\psi], \quad \frac{d\tilde{\mathcal{L}}}{d\tau} = \frac{1}{\mu}F_\varphi[\psi],$$

$$\frac{dt}{d\tau} = \frac{\tilde{\mathcal{E}}}{(1-2M/r)}, \quad \frac{d\varphi}{d\tau} = \frac{\tilde{\mathcal{L}}}{r^2},$$

$$\frac{dr}{d\tau} = \pm\left[\tilde{\mathcal{E}}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{\mathcal{L}}^2}{r^2}\right)\right]^{1/2} \equiv \pm V(\tilde{\mathcal{E}}, \tilde{\mathcal{L}}, r).$$

Here we translate our results in the scalar case to the gravitational case by identifying $q/\sqrt{G}$ with the mass μ of the particle.

The signature of $\dot{r}$ changes at the turning points. So if we solve numerically, the error increases.

Modify the Equation of Motion

We modify the EOM in the following.

$$\frac{d}{d\tau}(\dot{r})^2 = \dot{r}\frac{\partial}{\partial r}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r) + \dot{\mathcal{E}}\frac{\partial}{\partial\tilde{\mathcal{E}}}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r) + \dot{\mathcal{L}}\frac{\partial}{\partial\tilde{\mathcal{L}}}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r)$$

$$2\frac{d}{d\tau}\dot{r} = \frac{\partial}{\partial r}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r) + (\dot{r})^{-1}\dot{\mathcal{E}}\frac{\partial}{\partial\tilde{\mathcal{E}}}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r) + (\dot{r})^{-1}\dot{\mathcal{L}}\frac{\partial}{\partial\tilde{\mathcal{L}}}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r).$$

It seems to diverge at the turning points, however, using the relation

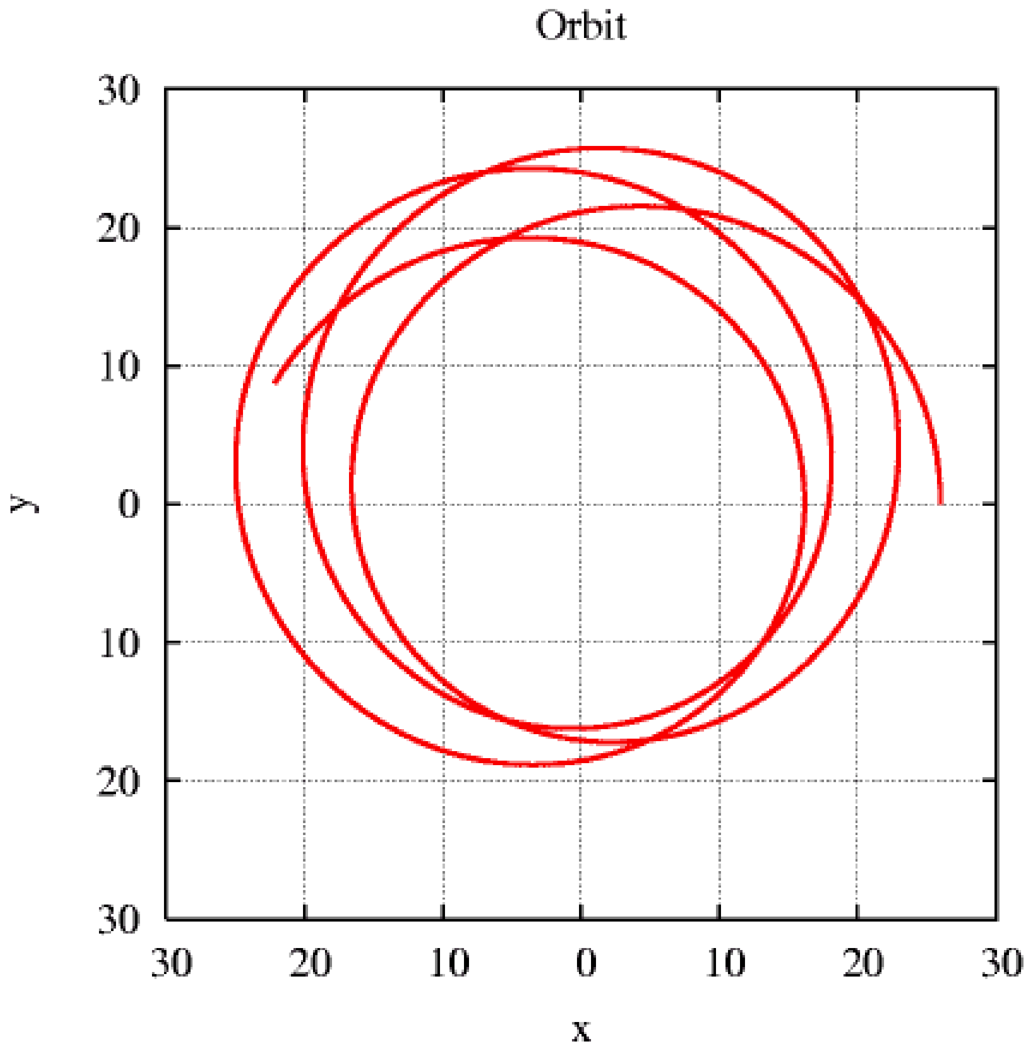
$$\frac{d}{d\tau}(\tilde{u}^{\mu}\tilde{u}_{\mu}) = 0, \;\Leftrightarrow\; -\mu\frac{\tilde{\mathcal{E}}}{(1-2M/r)}\dot{\tilde{\mathcal{E}}} + \dot{r}F_r + \mu\frac{\tilde{\mathcal{L}}}{r^2}\dot{\tilde{\mathcal{L}}} = 0$$

Then EOM is

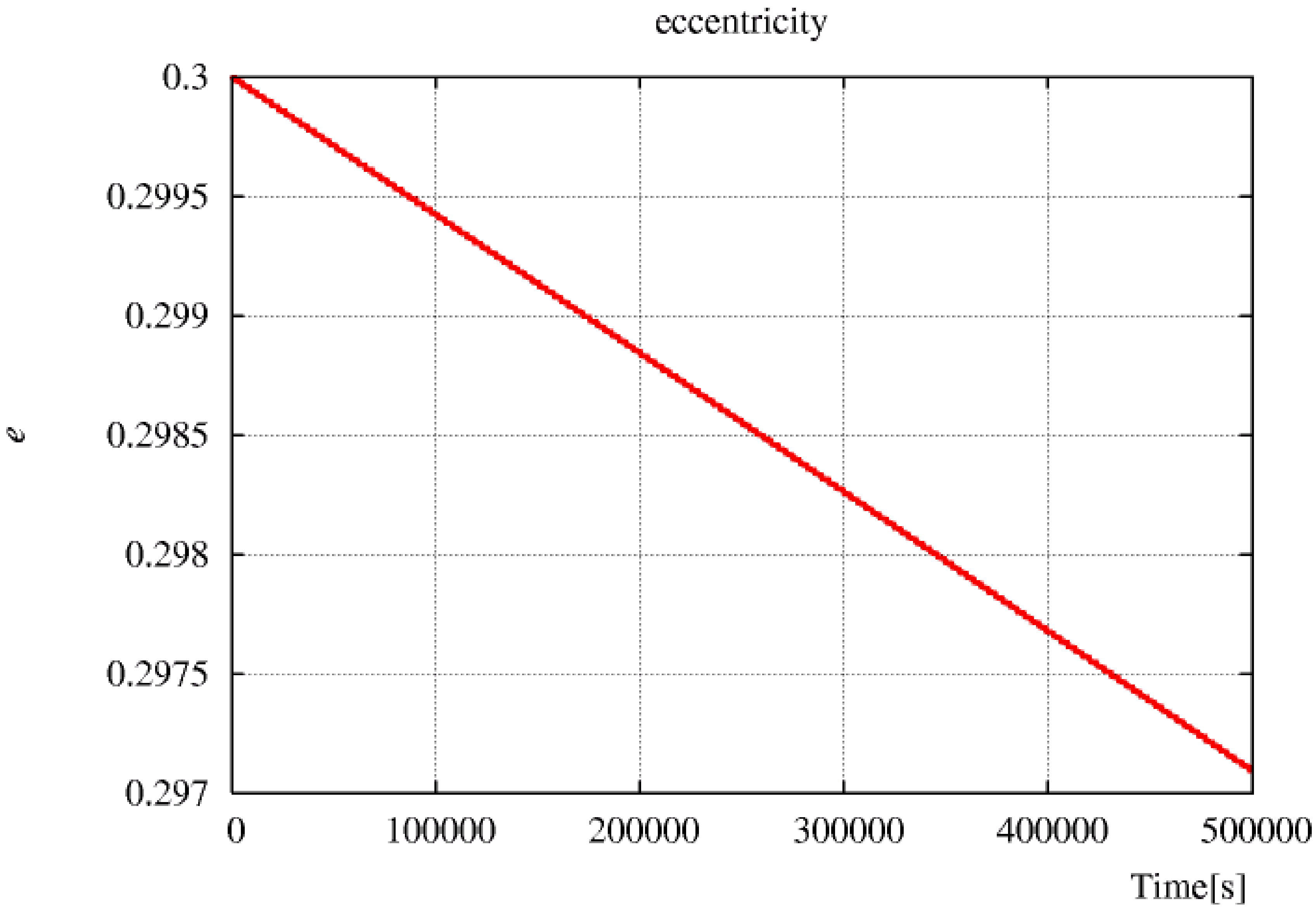
$$2\frac{d}{d\tau}\dot{r} = \frac{\partial}{\partial r}V(\tilde{\mathcal{E}},\tilde{\mathcal{L}},r) + \frac{2}{\mu}\left(1-\frac{2M}{r}\right)F_r.$$

The next figures are preliminary results of solution of this EOM, which are solved by using Runge-kutta method.

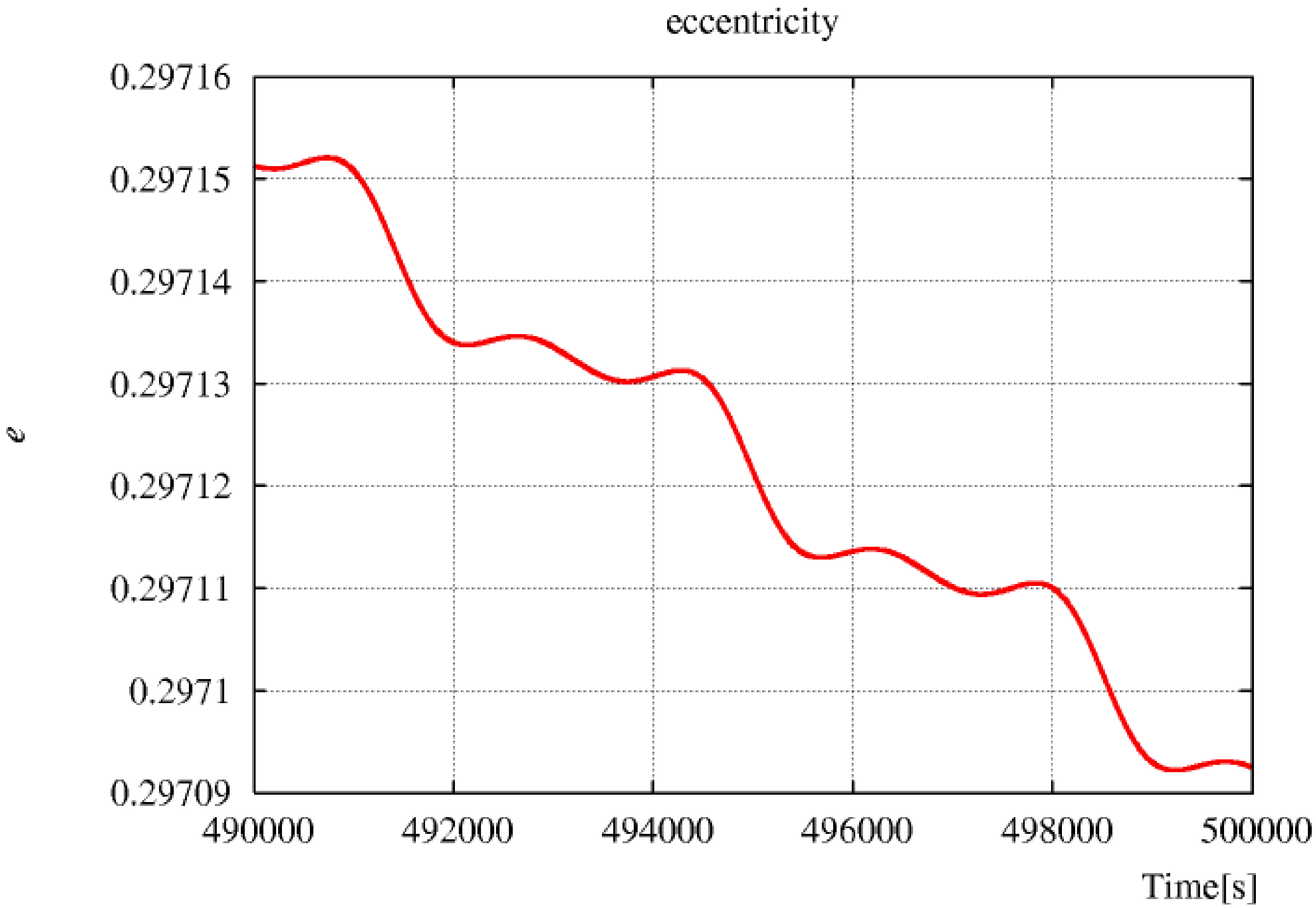
In this time, we set $(M, \mu) = (10^6 M_\odot, 10^3 M_\odot)$, $(r_0, e) = (20M, 0.3)$, initially and F^α is accurate up though 3PN order and 2nd order of e.



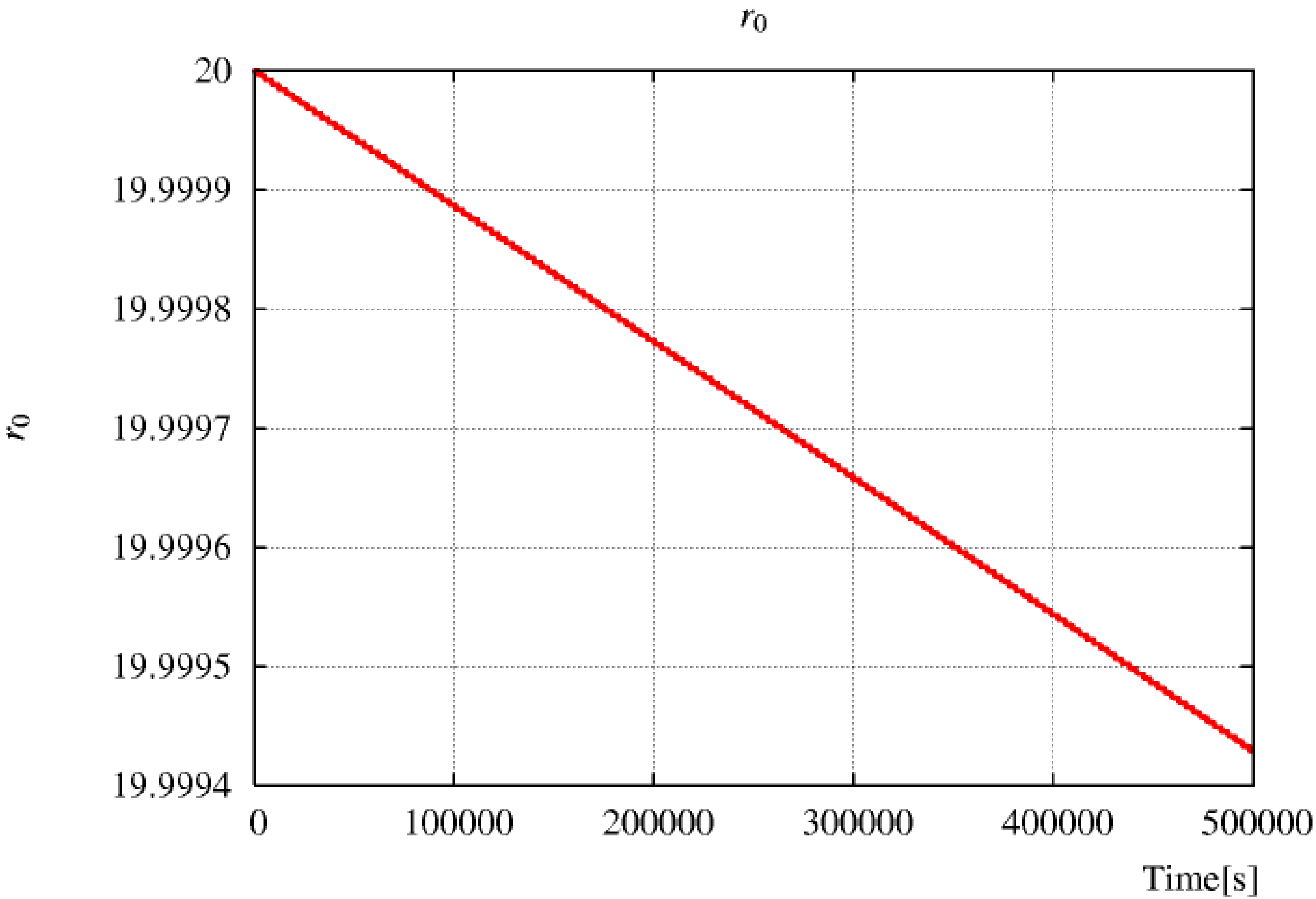
Eccentricity



Eccentricity



r_0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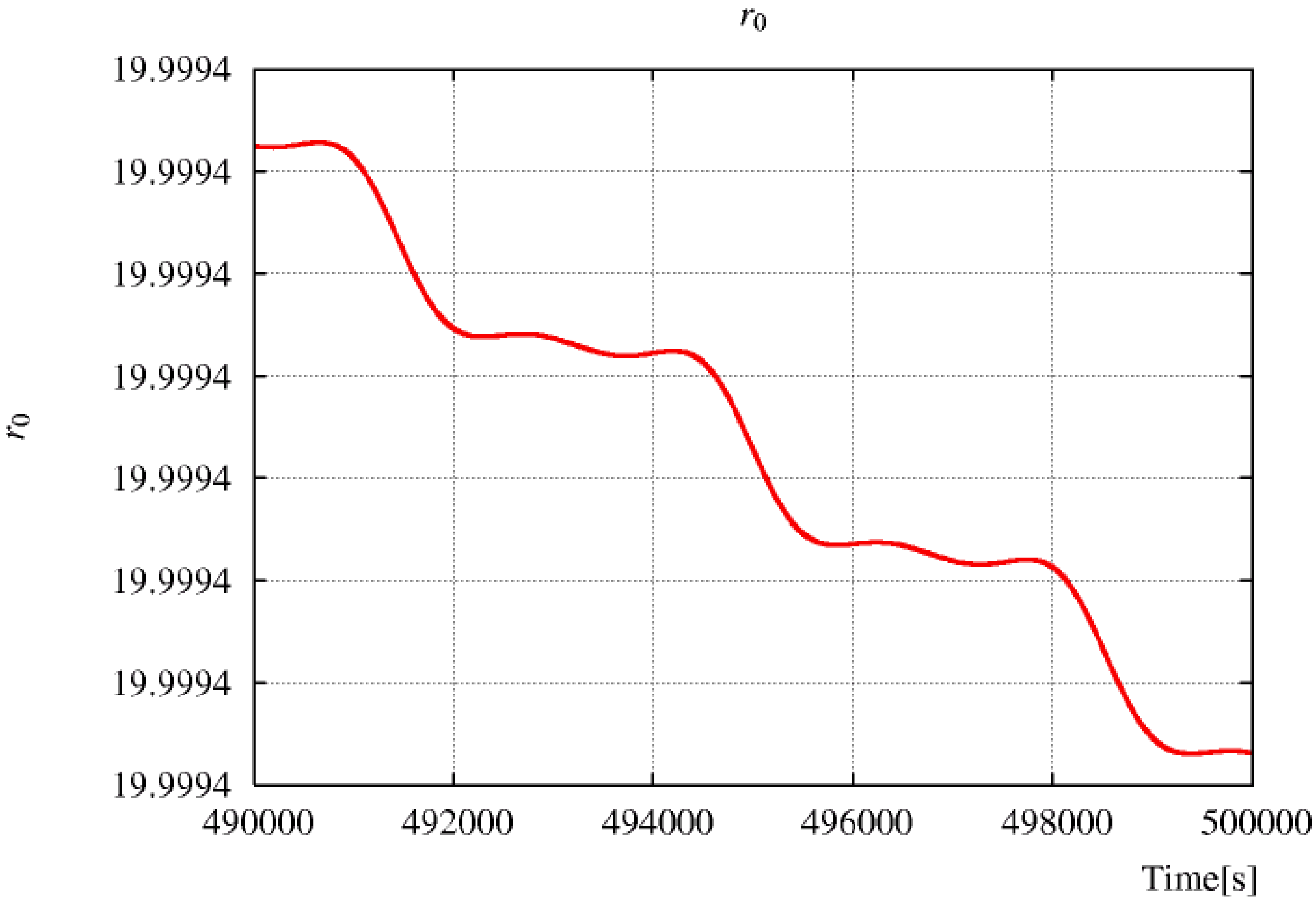

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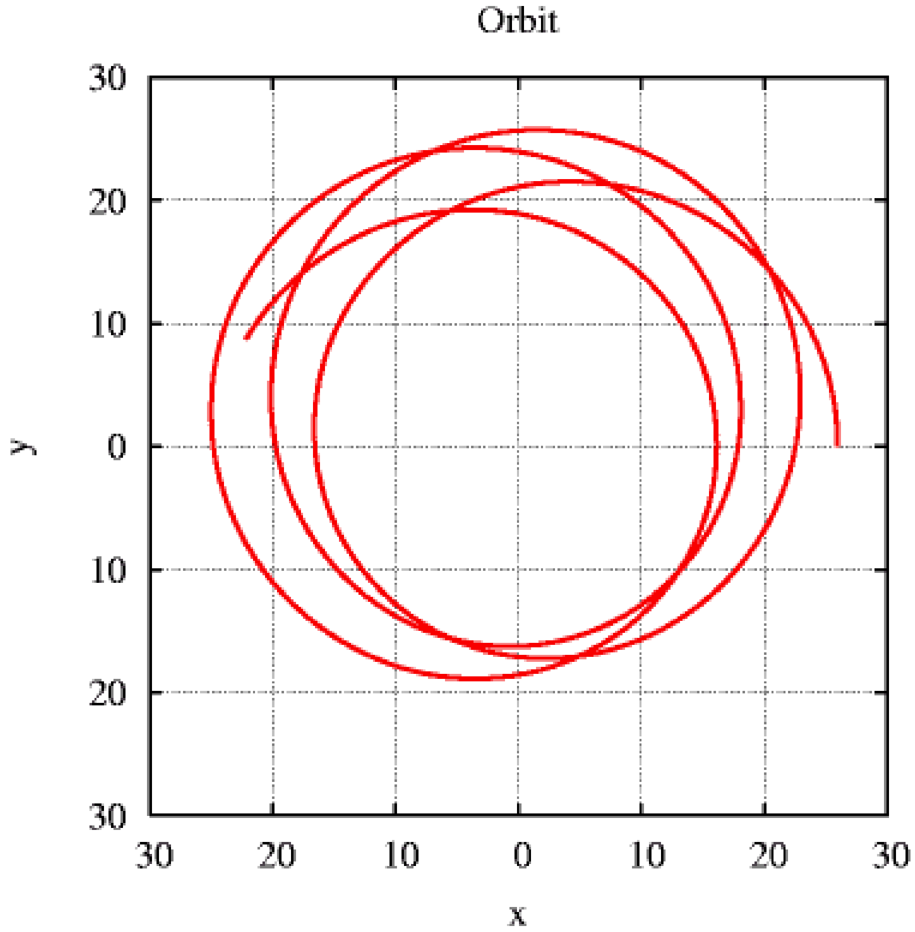
Motivation

What can we see the difference between full self-force and adiabatic self-force?

1). Energy loss
2). Perihelion Shift



5. Conclusion

Summary

- First we derive the dipole formula.
- We formulate the analytical method for regularizing the self-force in Schwarzschild spacetime.
- We actually calculate the self-force for eccentric orbit. This is the first time even if for scalar case.
- We solve the equation of motion including the self-force numerically.
- The application to gravitational case in Schwarzschild black hole is straightforward, as Nakano-san talked.

Discussion

- We are investigating the convergence of our expansion.
- We would like to find out the difference between adiabatic case and self-force case.