Metric of a tidally distorted, nonrotating black hole

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Goals and motivation

- To calculate the metric of a nonrotating black hole moving in an external universe and subjected to a tidal gravitational field.

This is motivated by recent attempts to use post-Newtonian theory to construct astrophysically realistic initial data sets for the numerical evolution of a binary black-hole spacetime.

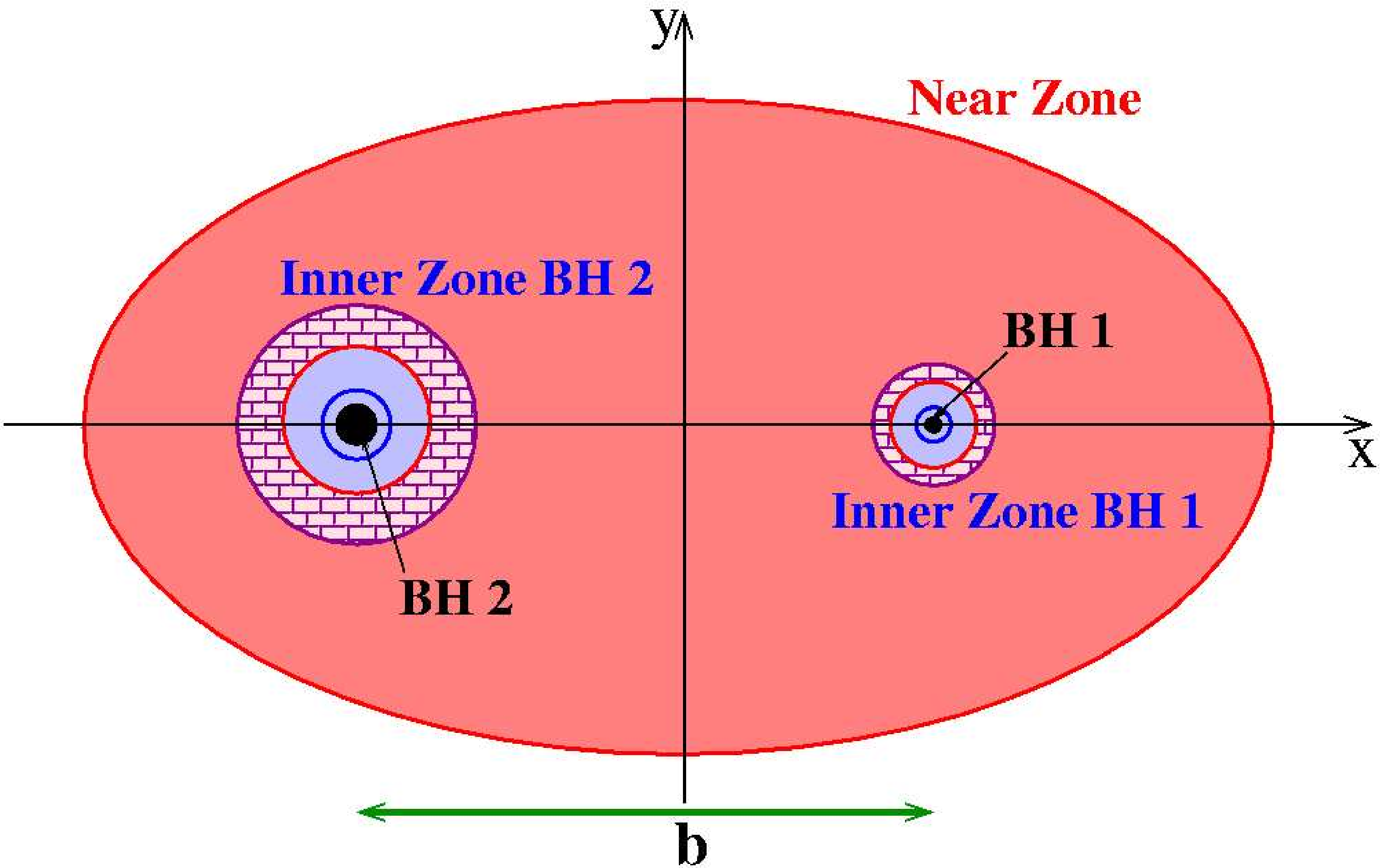
[Tichy et al, gr-qc/0207011]

The PN two-body metric fails near each black hole, but it can be matched to a perturbed Schwarzschild solution.

[Alvi, gr-qc/9912113; Yunes et al, gr-qc/0503011].

Goals and motivation

Black-hole zone: $r \ll b$
Post-Newtonian zone: $r \gg M$
Matching zone: $M \ll r \ll b$



Goals and motivation

- To calculate (and better understand) the tidal heating of a black hole.

In the context of a binary system, the increase of the black-hole mass by tidal heating impacts the energy balance between radiated energy and orbital energy.

This affects the phasing of the gravitational waves, and has measurable consequences.
[Poisson and Sasaki (1995); Tagoshi et al (1997); Alvi, gr-qc/0107080;

Poisson, gr-qc/0407050]

Goals and motivation

- To define and calculate the induced quadrupole moment of a tidally distorted black hole.

The hole's acquired internal structure will prevent it from moving on a geodesic of the background spacetime (excluding self-force effects).

How large is this effect?

Problem and strategy

We want to calculate the metric of a tidally distorted, nonrotating black hole by integrating the vacuum equations of black-hole perturbation theory.

We want to express the metric in geometrically meaningful coordinates that penetrate the event horizon.

We want to parameterize the perturbation with tidal fields defined in terms of the spacetime's Weyl tensor and its derivatives in the asymptotic region $r \gg M$.

Problem and strategy

We assume that the black hole is well isolated, so that

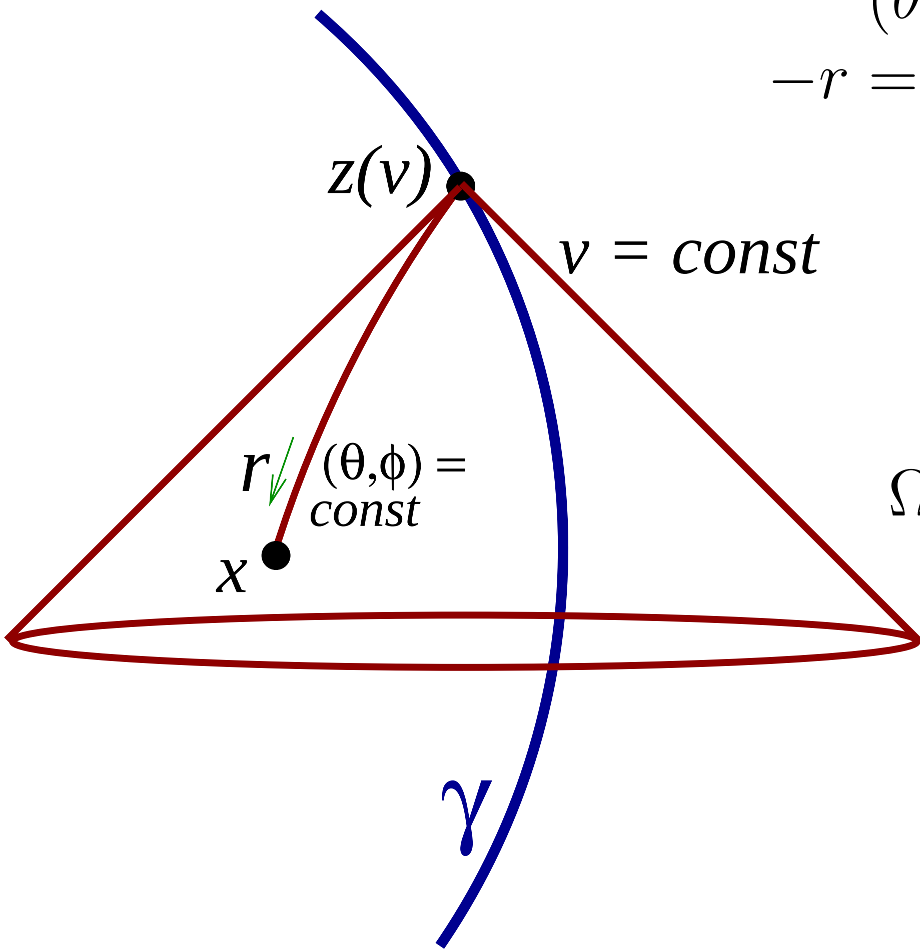
$$M \ll \mathcal{R} \equiv \text{local radius of curvature}$$

We shall first take $M = 0$ and construct the coordinates in a neighbourhood of a geodesic γ in a vacuum region of an arbitrary spacetime.

We will later place a nonrotating black hole on this world line.

Light-cone coordinates

$z^\mu(\tau) =$ description of geodesic
$v =$ const on past light cone of $z(v)$
$(\theta, \phi) =$ const on each generator
$-r =$ affine parameter on generator



$$\Omega^a := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$r := -u^\alpha(v)\sigma_\alpha(x, z(v))$$
$$\sigma(x, z(v)) = 0$$

Tidal fields

Let

$$(u^\alpha, e^\alpha_a) \qquad (a = 1, 2, 3)$$

be a tetrad of parallel-transported vectors on γ.

Let

$$\mathcal{E}_{ab}(v) := C_{\alpha\mu\beta\nu} e^\alpha_a u^\mu e^\beta_b u^\nu = O(\mathcal{R}^{-2})$$

be frame components of the Weyl tensor evaluated on γ.

Tidal fields

Similarly, let

$$\dot{\mathcal{E}}_{ab}(v) := C_{\alpha\mu\beta\nu;\lambda} e^\alpha_a u^\mu e^\beta_b u^\nu u^\lambda = O(\mathcal{R}^{-3})$$

and

$$\begin{aligned} \mathcal{E}_{abc}(v) &:= \frac{1}{3}(C_{\alpha\mu\beta\nu;\gamma} + C_{\gamma\mu\alpha\nu;\beta} + C_{\beta\mu\gamma\nu;\alpha}) e^\alpha_a u^\mu e^\beta_b u^\nu e^\gamma_c \\ &= O(\mathcal{R}^{-3}) \end{aligned}$$

Tidal fields

The frame tensors $\mathcal{E}_{ab}$ and $\mathcal{E}_{abc}$ are symmetric and tracefree.

They give rise to quadrupolar and octupolar tidal potentials,

$$\mathcal{E}^{\mathsf{q}}(v, \theta, \phi) := \mathcal{E}_{ab}\Omega^a\Omega^b = (\mathcal{E}_{11} + \mathcal{E}_{22})Y^{2,0} + \cdots$$

and

$$\mathcal{E}^{\circ}(v, \theta, \phi) := \mathcal{E}_{abc}\Omega^a\Omega^b\Omega^c = (\mathcal{E}_{113} + \mathcal{E}_{223})Y^{3,0} + \cdots$$

These appear in the expression for the metric.

The magnetic components of the Weyl tensor also appear (but this is not shown in this talk.)

Metric near a world line

The metric of an arbitrary spacetime can be expressed, in a neighbourhood of a geodesic γ, as an expansion in powers of $r/\mathcal{R}$.

In the light-cone coordinates,

$$g_{vv} = -1 - r^2 \mathcal{E}^{\mathsf{q}} + \frac{1}{3} r^3 \dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3} r^3 \mathcal{E}^{\circ} + O(r^4/\mathcal{R}^4)$$

Metric near a black hole

Now place a nonrotating black hole of mass M on the geodesic γ and calculate its metric.

The geometrical meaning of the light-cone coordinates shall be preserved.

In the limit $\mathcal{R} \to \infty$ the black hole is fully isolated and its metric is

$$g_{vv} = -f, \qquad f = 1 - \frac{2M}{r}$$

In the limit $M \to 0$ the metric is given by our previous result.

Metric near a black hole

The “interpolating” metric must be of the form

$$\begin{aligned} g_{vv} &= -f - r^2 e_1(M/r)\mathcal{E}^{\mathsf{q}} + \frac{1}{3} r^3 e_2(M/r)\dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3} r^3 e_3(M/r)\mathcal{E}^{\circ} \\ &\quad + O(r^4/\mathcal{R}^4) \end{aligned}$$

which reproduces the correct limiting expressions when $M \to 0$ or $\mathcal{R} \to \infty$.

The radial functions are obtained by integrating the equations of black-hole perturbation theory.

They approach 1 as $M/r \to 0$.

Metric near a black hole

We find

$$
\begin{aligned}
e_1 &= f^2 \\
e_2 &= f\Big[1 + \frac{1}{2}\frac{M}{r}\Big(5 + 12\ln\frac{r}{2M}\Big) - 3\frac{M^2}{r^2}\Big(9 + 4\ln\frac{r}{2M}\Big) \\
&\qquad + 14\frac{M^3}{r^3} + 12\frac{M^4}{r^4}\Big] \\
e_3 &= f^2\Big(1 - \frac{M}{r}\Big)
\end{aligned}
$$

The metric is parameterized by the tidal fields $\mathcal{E}_{ab}(v)$ and $\mathcal{E}_{abc}(v)$, which represent components of the Weyl tensor in the hole's local asymptotic rest frame.

These can be determined, for example, by matching the metric to the PN two-body metric.

Perturbed event horizon

The coordinate system is such that the coordinate description of the perturbed event horizon is preserved,

$$r = 2M\Big[1 + O(M^4/\mathcal{R}^4)\Big]$$

But the intrinsic geometry of the horizon is nonspherical.

For example, the surface gravity is given by

$$\kappa = \frac{1}{4M}\Big[1 + \frac{16}{3}M^3\dot{\mathcal{E}}_{ab}(v)\Omega^a\Omega^b + \cdots\Big]$$

At this level of approximation the event horizon is an apparent horizon:

$$\Theta = O(M^4/\mathcal{R}^5)$$

Tidal heating

The slow growth of the event horizon is not revealed by a direct examination of the horizon's perturbed geometry.

It can, however, be calculated by applying the Hawking-Hartle formula, which gives

$$\frac{dM}{dv} = \frac{16}{45} M^6 \dot{\mathcal{E}}_{ab}(v) \dot{\mathcal{E}}^{ab}(v) + \text{higher-order terms}$$

For a circular orbit of radius b around an external body of mass $M_{\rm ext}$, this becomes

$$\frac{dM}{dv} \simeq \frac{32}{5} \frac{M^6 M_{\rm ext}^2}{(M + M_{\rm ext})^8} V^{18}$$

where $V = \sqrt{(M + M_{\rm ext})/b}$ is the orbital velocity.

Induced quadrupole moment

What is the tidally-induced quadrupole moment of a black hole?

On dimensional grounds we would expect

$$Q_{ab} \propto M^5 \mathcal{E}_{ab}$$

and we would expect to see a term $Q_{ab}\Omega^a\Omega^b/r^3$ in the metric.

Surprisingly, no such term appears:

$$g_{vv} = -1 + \frac{2M}{r} - (r^2 - 4Mr + 4M^2)\mathcal{E}_{ab}\Omega^a\Omega^b + \cdots$$

Induced quadrupole moment

To define Q_{ab} operationally we might use the general expression for tidal work in Newtonian or relativistic gravity,

[Purdue, gr-qc/9901086; Favata, gr-qc/0008061; Booth and Creighton, gr-qc/0003038]

$$\frac{dW}{dv} = -\frac{1}{2}\dot{Q}_{ab}\mathcal{E}^{ab} = \frac{1}{2}Q_{ab}\dot{\mathcal{E}}^{ab} + \frac{d}{dv}\Big(-\frac{1}{2}Q_{ab}\mathcal{E}^{ab}\Big)$$

Suppose

$$Q_{ab} = \lambda M^5 \mathcal{E}_{ab} + \frac{32}{45}M^6\dot{\mathcal{E}}_{ab} + \cdots \qquad (\lambda \text{ unknown})$$

Then

$$\frac{dW}{dv} = \frac{16}{45}\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab} + \frac{d}{dv}\Big(\frac{\lambda}{4}M^5\mathcal{E}_{ab}\mathcal{E}^{ab} - \frac{1}{2}Q_{ab}\mathcal{E}^{ab}\Big) + \cdots$$

Induced quadrupole moment

This calculation does not reveal the leading contribution to the induced quadrupole moment.

The leading term, $\lambda M^5 \mathcal{E}_{ab}$, is gauge dependent and ambiguous.

The same conclusion was reached by Fang and Lovelace (gr-qc/0505156), based on a different operational definition provided by

$$\dot{P}^a = -\frac{1}{2}\mathcal{E}^a{}_{bc} Q^{bc}$$

The correction term, $\frac{32}{45} M^6 \dot{\mathcal{E}}_{ab}$, is both gauge invariant and unambiguous.

Newtonian fluid sphere

There is an analogous situation in Newtonian theory, in the case of a viscous, incompressible fluid sphere, for which

$$Q_{ab}(t) = -\frac{1}{2} R^5 \mathcal{E}_{ab}(t-\tau), \qquad \tau = \frac{19}{2}\frac{R}{M}\,\nu$$

$R =$ sphere's radius $\qquad \nu =$ kinematical viscosity

Then

$$\begin{aligned} \frac{dW}{dt} &= \frac{1}{2} Q_{ab}\dot{\mathcal{E}}^{ab} + \frac{d}{dt}\Big(\cdots\Big) = -\frac{1}{4} R^5 \mathcal{E}_{ab}(t-\tau)\dot{\mathcal{E}}^{ab}(t) + \frac{d}{dt}\Big(\cdots\Big) \\ &\simeq \frac{1}{4} R^5 \tau \dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab} + \frac{d}{dt}\Big(\cdots\Big) \end{aligned}$$

The black hole therefore behaves as a viscous body with viscosity $\nu \sim M$. [Hartle, 1974]

Conclusion

- The metric of a tidally distorted, nonrotating black hole is expressed in light-cone coordinates (v, r, θ, ϕ), as an expansion in powers of $r/\mathcal{R}$.
- The coordinates are geometrically meaningful and convenient to work with — the perturbed event horizon is described by $r = 2M$.
- The metric is usefully parameterized by tidal fields $\mathcal{E}_{ab}(v)$ and $\mathcal{E}_{abc}(v)$, which represent components of the Weyl tensor and its covariant derivative in the hole's local asymptotic rest frame.
- The tidally-induced quadrupole moment of a black hole is intrinsically ambiguous at leading order.
- But its correction term can be operationally defined in terms of the tidal heating of the event horizon.