Towards Metric Perturbations of Kerr

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The Plan

- Introduction
- A Little GHP
- Einstein's New Clothes
- Easy Applications
- Progress in Schwarzschild
- On the Horizon: Future Work
- Conclusion

Introduction

Important sources for LISA are thought to be compact objects orbiting *rotating* supermassive black holes.

http://lisa.jpl.nasa.gov/gallery/stellar-mass-black-hole.html

Introduction:
Current Methods for Perturbing Kerr

Teukolsky (1973)

$$[(\text{Þ} - 4\rho - \bar{\rho})(\text{Þ}' - \rho') - (\eth - 4\tau - \bar{\tau}')(\eth' - \tau') - 3\psi_2]\dot{\psi}_0 = T_0$$

$$[(\text{Þ}' - \bar{\rho}')(\text{Þ} + 3\rho) - (\eth' - \bar{\tau})(\eth + 3\tau) - 3\psi_2]\psi_2^{-\frac{4}{3}}\dot{\psi}_4 = \psi_2^{-\frac{4}{3}}T_4$$

Introduction:
Current Methods for Perturbing Kerr

Cohen & Kegeles (1975,1979), Chrzanowski (1975), Stewart (1979)

$$h_{ab} l^b = h^a{}_a = 0$$

$$\begin{aligned} h_{ab} = \{ & l_a l_b (\eth - \tau)(\eth + 3\tau) + m_a m_b (\text{Þ} - \rho)(\text{Þ} + 3\rho) \\ & - l_{(a} m_{b)} [(\text{Þ} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\text{Þ} + 3\rho)] \} \Psi + \text{c.c.} \end{aligned}$$

Introduction:
Current Methods for Perturbing Kerr

Lousto & Whiting (2002), Ori (2003), Lousto (2005)

$$\dot{\psi}_0 = \frac{1}{2}\Big\{(\eth - \bar{\tau}')(\eth - \bar{\tau}')h_{ll} + (\text{Þ} - \bar{\rho})(\text{Þ} - \bar{\rho})h_{mm} - [(\text{Þ} - \bar{\rho})(\eth - 2\bar{\tau}') + (\eth - \bar{\tau}')(\text{Þ} - 2\bar{\rho})]h_{(lm)}\Big\}$$

$$\dot{\psi}_4 = \frac{1}{2}\Big\{(\eth' - \bar{\tau})(\eth' - \bar{\tau})h_{nn} + (\text{Þ}' - \bar{\rho}')(\text{Þ}' - \bar{\rho}')h_{\bar{m}\bar{m}} - [(\text{Þ}' - \bar{\rho}')(\eth' - 2\bar{\tau}) + (\eth' - \bar{\tau})(\text{Þ}' - 2\bar{\rho}')]h_{(n\bar{m})}\Big\}$$

A Little GHP

Newman & Penrose (1962), Geroch, Held & Penrose (1973)

Introduce a normalized null tetrad:

$$l_a n^a = -m_a \bar{m}^a = 1$$

A Little GHP

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Metric takes the form:

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$$

A Little GHP

In Type D, the only tetrad freedom is a spin-boost:

$$l^a \longrightarrow r l^a$$

$$n^a \longrightarrow r^{-1} n^a$$

$$m^a \longrightarrow e^{i\theta} m^a$$

$$\bar{m}^a \longrightarrow e^{-i\theta} \bar{m}^a$$

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A quantity has spin-weight s and boost-weight b if under a spin-boost:

$$\chi \longrightarrow r^b e^{is\theta} \chi$$

A Little GHP

One derivative in each direction:

$$\begin{aligned}
\langle 0,1\rangle &: \text{Þ} \longleftrightarrow l^a \\
\langle 0,-1\rangle &: \text{Þ}' \longleftrightarrow n^a \\
\langle 1,0\rangle &: \eth \longleftrightarrow m^a \\
\langle -1,0\rangle &: \eth' \longleftrightarrow \bar{m}^a
\end{aligned}$$

A Little GHP

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The full covariant derivative is written:

$$\Theta_a = l_a \text{Þ}' + n_a \text{Þ} - m_a \eth' - \bar{m}_a \eth \equiv \nabla_a$$

A Little GHP

The spin coefficients:

$$\kappa = l^a m^b \nabla_a l_b \qquad \kappa' = n^a \bar{m}^b \nabla_a n_b$$
$$\sigma = m^a m^b \nabla_a l_b \qquad \sigma' = \bar{m}^a \bar{m}^b \nabla_a n_b$$
$$\rho = \bar{m}^a m^b \nabla_a l_b \qquad \rho' = m^a \bar{m}^b \nabla_a n_b$$
$$\tau = n^a m^b \nabla_a l_b \qquad \tau' = l^a \bar{m}^b \nabla_a n_b$$

$$\beta = \tfrac{1}{2}(m^a n^b \nabla_a l_b - m^a \bar{m}^b \nabla_a m_b)$$
$$\beta' = -\tfrac{1}{2}(\bar{m}^a m^b \nabla_a \bar{m}_b - \bar{m}^a l^b \nabla_a n_b)$$
$$\epsilon = \tfrac{1}{2}(l^a n^b \nabla_a l_b - l^a \bar{m}^b \nabla_a m_b)$$
$$\epsilon' = -\tfrac{1}{2}(n^a m^b \nabla_a \bar{m}_b - n^a l^b \nabla_a n_b)$$

A Little GHP

The Weyl scalars:

$$\psi_0 = -C_{lmlm}$$
$$\psi_1 = -C_{lnlm}$$
$$\psi_2 = -C_{lm\bar{m}n}$$
$$\psi_3 = -C_{ln\bar{m}n}$$
$$\psi_4 = -C_{n\bar{m}n\bar{m}}$$

A Little GHP

Prime operation:

$$(l^a)' = n^a \qquad (n^a)' = l^a$$
$$(m^a)' = \bar{m}^a \qquad (\bar{m}^a)' = m^a$$

Very useful discrete symmetry!

A Little GHP

Goldberg-Sachs, et al

In Type D with the tetrad aligned with PNDs:

$$\kappa = \kappa' = \sigma = \sigma' = \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$$

Formalism doesn't distinguish between different black hole solutions!

Einstein's New Clothes

The metric perturbation:

$$
\begin{aligned}
h_{ab} = \quad & h_{ll} n_a n_b + h_{nn} l_a l_b + 2 h_{ln} l_{(a} n_{b)} + 2 h_{m\bar{m}} m_{(a} \bar{m}_{b)} \\
& - 2 h_{lm} n_{(a} \bar{m}_{b)} - 2 h_{l\bar{m}} n_{(a} m_{b)} - 2 h_{n\bar{m}} l_{(a} m_{b)} - 2 h_{nm} l_{(a} \bar{m}_{b)} \\
& + h_{mm} \bar{m}_a \bar{m}_b + h_{\bar{m}\bar{m}} m_a m_b
\end{aligned}
$$

Einstein's New Clothes

The metric perturbation:

$$h_{ab} = \quad h_{ll}n_a n_b + h_{nn}l_a l_b + 2h_{ln}l_{(a}n_{b)} + 2h_{m\bar{m}}m_{(a}\bar{m}_{b)}$$
$$- 2h_{lm}n_{(a}\bar{m}_{b)} - 2h_{l\bar{m}}n_{(a}m_{b)} - 2h_{n\bar{m}}l_{(a}m_{b)} - 2h_{nm}l_{(a}\bar{m}_{b)}$$
$$+ h_{mm}\bar{m}_a\bar{m}_b + h_{\bar{m}\bar{m}}m_a m_b$$

The gauge vector:

$$\xi_a = \xi_l n_a + \xi_n l_a - \xi_{\bar{m}} m_a - \xi_m \bar{m}_a$$

Einstein's New Clothes

Compute perturbed Einstein equations via:

$$\begin{aligned}\mathcal{E}_{ab} = & -\tfrac{1}{2}\Theta^c\Theta_c h_{ab} - \tfrac{1}{2}\Theta_a\Theta_b h^c{}_c + \Theta^c\Theta_{(a}h_{b)c} \\ & + \tfrac{1}{2}g_{ab}(\Theta^c\Theta_c h^d{}_d - \Theta^c\Theta^d h_{cd})\end{aligned}$$

Einstein's New Clothes

GHPTools

- Handles contractions
- Does commutators
- Keeps track of spin- and boost- weights
- GHP <=> NP
- NP <=> Coordinates

Einstein's New Clothes

Flatspace (Minkoswki Coordinates):

$$\begin{aligned}
\mathcal{E}_{ll} &= 2\text{Þ}\text{Þ}h_{m\bar{m}} + 2\eth'\eth h_{ll} - 2\text{Þ}\eth h_{l\bar{m}} - 2\text{Þ}\eth' h_{lm} \\
\mathcal{E}_{ln} &= -3\text{Þ}\eth' h_{nm} - 3\text{Þ}'\eth' h_{lm} + \eth'\eth' h_{mm} + 6\eth'\eth h_{m\bar{m}} \\
&\quad - 3\text{Þ}'\eth h_{l\bar{m}} + 4\text{Þ}'\text{Þ}h_{ln} + 2\text{Þ}\text{Þ}h_{nn} - 2\text{Þ}'\text{Þ}h_{m\bar{m}} \\
&\quad - 2\eth'\eth h_{ln} - 3\text{Þ}\eth h_{n\bar{m}} + 2\text{Þ}'\text{Þ}' h_{ll} + \eth\eth h_{\bar{m}\bar{m}} \\
\mathcal{E}_{lm} &= \text{Þ}\eth h_{ln} + \eth\eth h_{l\bar{m}} - \text{Þ}'\eth h_{ll} - \text{Þ}\eth h_{m\bar{m}} \\
&\quad + \text{Þ}\eth' h_{mm} - \text{Þ}\text{Þ}h_{nm} + \text{Þ}'\text{Þ}h_{lm} - \eth'\eth h_{lm} \\
\mathcal{E}_{mm} &= 2\text{Þ}\eth h_{nm} - 2\eth\eth h_{ln} - 2\text{Þ}'\text{Þ}h_{mm} + 2\text{Þ}'\eth h_{lm} \\
\mathcal{E}_{m\bar{m}} &= 3\text{Þ}\eth' h_{nm} + 3\text{Þ}'\eth' h_{lm} - 2\eth'\eth' h_{mm} - 4\eth'\eth h_{m\bar{m}} \\
&\quad + 3\text{Þ}'\eth h_{l\bar{m}} - 6\text{Þ}'\text{Þ}h_{ln} - \text{Þ}\text{Þ}h_{nn} + 2\text{Þ}'\text{Þ}h_{m\bar{m}} \\
&\quad + 2\eth'\eth h_{ln} + 3\text{Þ}\eth h_{n\bar{m}} - \text{Þ}'\text{Þ}' h_{ll} - 2\eth\eth h_{\bar{m}\bar{m}}
\end{aligned}$$

Easy Applications

Understanding the radiation gauges

Gauge conditions: $h_{ab}l^b = h^a{}_a = 0$

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First condition: $h_{ll} = h_{ln} = h_{lm} = h_{l\bar{m}} = 0$

Trace condition: $h_{ln} = h_{m\bar{m}} = 0$

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Understanding the radiation gauges

Gauge conditions: $h_{ab}l^b = h^a{}_a = 0$

First condition: $h_{ll} = h_{ln} = h_{lm} = h_{l\bar{m}} = 0$

Trace condition: $h_{ln} = h_{m\bar{m}} = 0$

Consider:

$$2\text{Þ}\text{Þ}h_{m\bar{m}} + 2\eth'\eth h_{ll} - 2\text{Þ}\eth h_{l\bar{m}} - 2\text{Þ}\eth' h_{lm} = 8\pi T_{ll}$$

Easy Applications

Generalizing the Regge-Wheeler gauge

$$
\begin{aligned}
h_{\theta\phi} &= 0 \\
(\sin\theta)^2 h_{\theta\theta} - h_{\phi\phi} &= 0 \\
\sin\theta\partial_\theta(\sin\theta h_{t\theta}) + \partial_\phi h_{t\phi} &= 0 \\
\sin\theta\partial_\theta(\sin\theta h_{r\theta})\partial_\phi h_{r\phi} &= 0
\end{aligned}
$$

Easy Applications

Generalizing the Regge-Wheeler gauge

$$h_{\theta\phi} = 0$$

$$(\sin\theta)^2 h_{\theta\theta} - h_{\phi\phi} = 0$$

=>

$$h_{mm} = 0$$

$$h_{\bar{m}\bar{m}} = 0$$

$$\sin\theta\,\partial_\theta(\sin\theta\, h_{t\theta}) + \partial_\phi h_{t\phi} = 0$$

$$\sin\theta\,\partial_\theta(\sin\theta\, h_{r\theta})\partial_\phi h_{r\phi} = 0$$

=>

$$\eth h_{l\bar{m}} + \eth' h_{lm} = 0$$

$$\eth h_{nm} + \eth' h_{n\bar{m}} = 0$$

Easy Applications

Generalizing the Regge-Wheeler gauge

$$\begin{aligned} h_{\theta\phi} &= 0 \\ (\sin\theta)^2 h_{\theta\theta} - h_{\phi\phi} &= 0 \\ \sin\theta\partial_\theta(\sin\theta h_{t\theta}) + \partial_\phi h_{t\phi} &= 0 \\ \sin\theta\partial_\theta(\sin\theta h_{r\theta})\partial_\phi h_{r\phi} &= 0 \end{aligned}$$

=>

$$\begin{aligned} h_{mm} &= 0 \\ h_{\bar{m}\bar{m}} &= 0 \end{aligned}$$

=>

$$\begin{aligned} \eth h_{l\bar{m}} + \eth' h_{lm} &= 0 \\ \eth h_{nm} + \eth' h_{n\bar{m}} &= 0 \end{aligned}$$

Boostrap to Kerr via spin- and boost- weight:

$$\begin{aligned} h_{mm} = h_{\bar{m}\bar{m}} &= 0 \\ (\eth + \bar{a}\tau + b\bar{\tau}')h_{l\bar{m}} + (\eth' + a\bar{\tau} + \bar{b}\tau')h_{lm} &= 0 \\ (\eth + a\bar{\tau}' + \bar{b}\tau)h_{n\bar{m}} + (\eth' + \bar{a}\tau' + b\bar{\tau})h_{nm} &= 0 \end{aligned}$$

Progress in Schwarzschild

Parity without separation of variables

For the spacelike 2-surface orthogonal to l and n:

$$\sigma_{ab} = -m_a \bar{m}_b - \bar{m}_a m_b$$

$$\epsilon_{ab} \equiv \epsilon_{lnab} = i(m_a \bar{m}_b - \bar{m}_a m_b)$$

Then decompose vectors and stf tensors:

$$\begin{aligned}\xi_a &= \sigma_a{}^b \nabla_b \xi_{\text{even}} + \epsilon_a{}^b \nabla_b \xi_{\text{odd}} \\ &= -m_a(\eth' \xi_{\text{even}} - i\eth' \xi_{\text{odd}}) - \bar{m}_a(\eth \xi_{\text{even}} + i\eth \xi_{\text{odd}})\end{aligned}$$

$$\begin{aligned}\chi_{ab} &= \sigma_a{}^c \nabla_c \chi_b + \sigma_b{}^c \nabla_c \chi_a - \sigma_{ab} \sigma^{cd} \nabla_c \chi_d \\ &= 2m_{(a} m_{b)}(\eth'\eth' \chi_{\text{even}} - i\eth'\eth' \chi_{\text{odd}}) + 2\bar{m}_{(a} \bar{m}_{b)}(\eth\eth \chi_{\text{even}} + i\eth\eth \chi_{\text{odd}})\end{aligned}$$

Progress in Schwarzschild

The "Regge-Wheeler" equation:

$$[2(\text{Þ}' - \rho')(\text{Þ} - \rho) - 2\eth'\eth + 6\psi_2)]\psi_2^{-\frac{2}{3}}\,\dot{\psi}_2^{\mathrm{odd}} = 0$$

Progress in Schwarzschild

The “Regge-Wheeler” equation:

$$[2(\text{Þ}' - \rho')(\text{Þ} - \rho) - 2\eth'\eth + 6\psi_2)]\psi_2^{-\frac{2}{3}}\dot{\psi}_2^{\mathrm{odd}} = 0$$

In more familiar terms:

$$(\Box + 8\psi_2)\psi_2^{-\frac{2}{3}}\dot{\psi}_2^{\mathrm{odd}} = 0$$

On the Horizon: Future Work

- l=0,1 in Schwarzschild
- Even parity in Schwarzschild
- Reconstruction in Schwarzschild
- “Parity” in Kerr
- Gauge invariants

Conclusion

- Recent results are promising
- There remains much to be done...

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...and a lot to think about!

Photo by Curt Busse