Time-Domain (Finite-Element) Simulations of Extreme-Mass-Ratio Binaries

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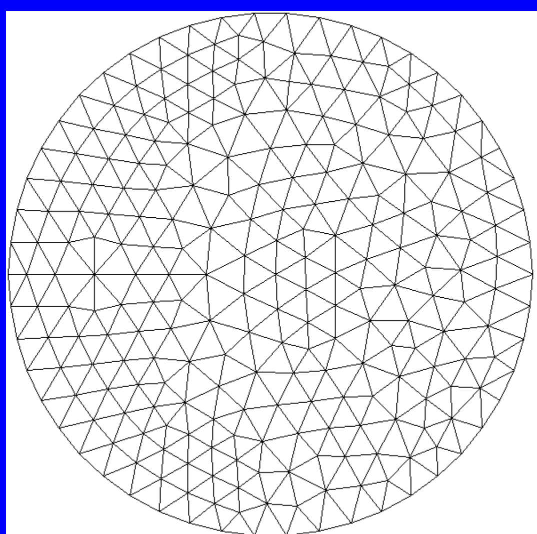

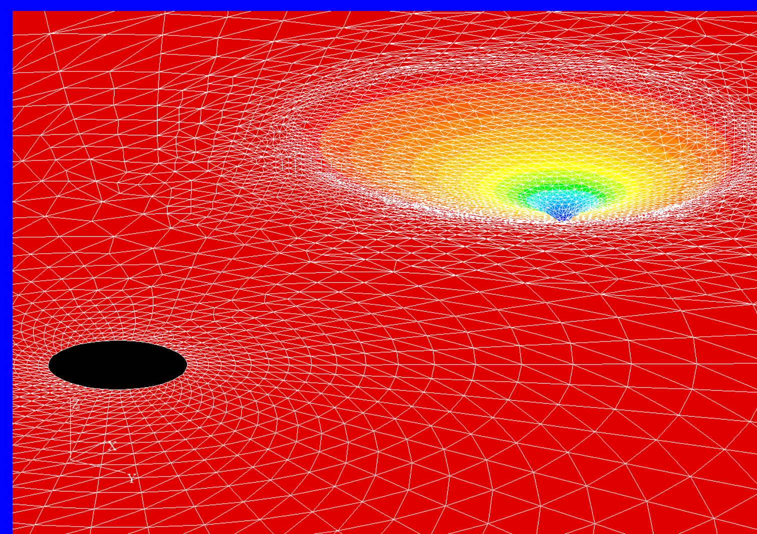

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Motivation for the Time-Domain Numerical Approach to EMRBs

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$$\mu = \frac{m}{M_\bullet} \sim 10^{-3} - 10^{-8}$$

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  - GW wavelength: $r_w \sim \pi(r_o/M_\bullet)^{3/2}M_\bullet$ (circular orbit estimation)
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$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

Strong need for (dynamical) Adaptivity!

Why can Finite Element Methods help to solve this problem?

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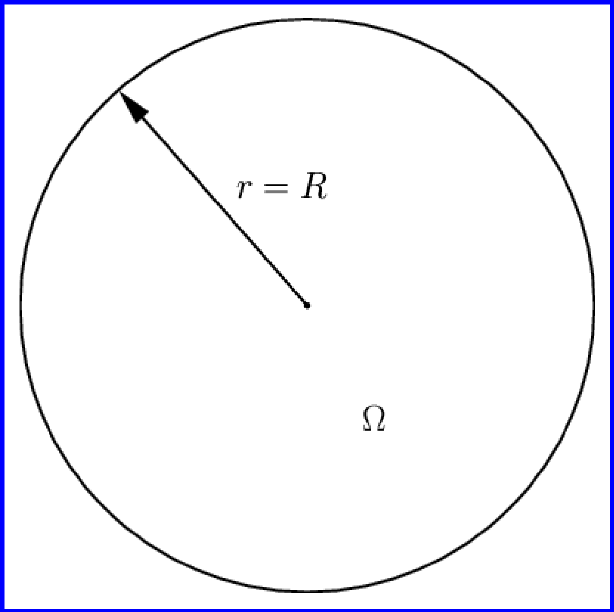
- Let us illustrate the arguments with the example of the wave equation:

$$\left[-\partial_t^2 + \nabla^2 - V(r)\right]\Psi(t,\boldsymbol{x}) = \mathcal{S}(t,\boldsymbol{x})\,, \quad \boldsymbol{x}\in\Omega\,, \quad r^2 = \boldsymbol{x}\cdot\boldsymbol{x}\,,$$

$$\left(\partial_t + \partial_r + \frac{1}{2r}\right)\Psi\Big|_{\partial\Omega} = 0\,,$$



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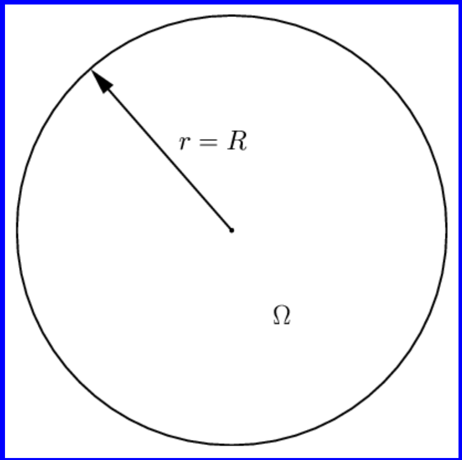

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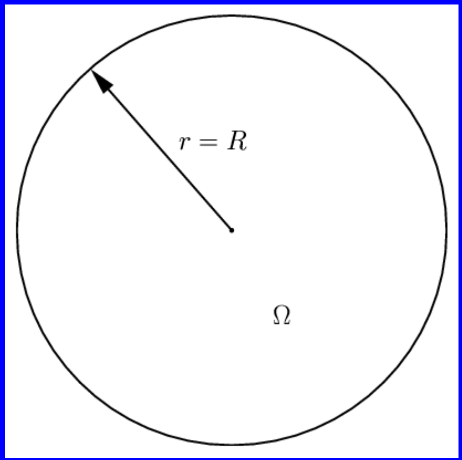

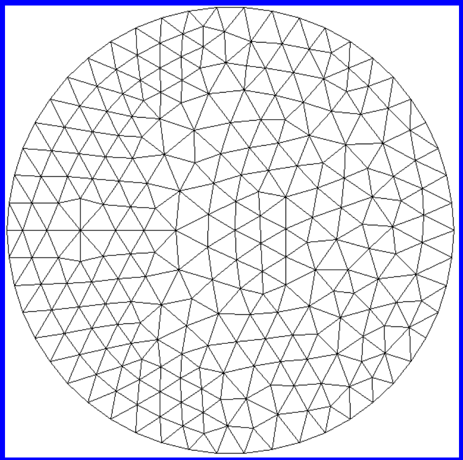

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- Choosing *linear elements* (i.e. $a + bx + cy$) leads, in general, to second-order convergence in the L^2 norm.

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$$\mathcal{L}[\phi, \Psi] \equiv \int_{\Omega} \phi \left\{ \left[-\partial_t^2 + \nabla^2 - V \right] \Psi - \mathcal{S} \right\} d\Omega \quad (= 0 \text{ if } \Psi \text{ is a solution})$$

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$$\mathcal{L}[\phi, \Psi] = \int_\Omega \nabla(\phi\nabla\Psi)d\Omega - \int_\Omega \left\{\phi\partial_t^2\Psi + \nabla\phi\nabla\Psi + V\phi\Psi + \mathcal{S}\phi\right\} d\Omega$$

(Integration by parts)

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$$\mathcal{L}[\phi, \Psi] = \int_{\partial\Omega} \phi\, \boldsymbol{n}\cdot\nabla\Psi\, ds - \int_{\Omega} \left\{ \phi\partial_t^2\Psi + \nabla\phi\nabla\Psi + V\phi\Psi + \mathcal{S}\phi \right\} d\Omega$$

(Gauss Theorem. $\boldsymbol{n}$ is the normal to $\partial\Omega$)

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$$\mathcal{L}[\phi, \Psi] = -\int_{\partial\Omega} \phi \left(\partial_t + \frac{1}{2r}\right) \Psi \, ds - \int_{\Omega} \left\{ \phi \partial_t^2 \Psi + \nabla\phi \nabla\Psi + V\phi\Psi + \mathcal{S}\phi \right\} d\Omega$$

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  - Now we approximate the solution by an expansion in terms of nodal functions $n_I(\boldsymbol{x})$ [For a given node $\boldsymbol{x}_J$, $n_I(\boldsymbol{x}_J) = \delta_{IJ}$]

$$\Psi_h(t, \boldsymbol{x}) = \sum_I \Psi_I(t) \, n_I(\boldsymbol{x})$$

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$$\begin{aligned}
\mathbb{M}_{IJ} &= (n_I, n_J) = \int_\Omega n_I\, n_J\, d\Omega && \text{(\textit{Mass} matrix)} \\
\mathbb{G}_{IJ} &= [n_I, n_J] = \int_{\partial\Omega} n_I\, n_J\, ds && \text{(\textit{Damping} matrix)} \\
\mathbb{K}_{IJ} &= (\nabla n_I, \nabla n_J) + (V n_I, n_J) + \left[\frac{1}{2r} n_I, n_J\right] && \text{(\textit{Stiffness} matrix)} \\
\boldsymbol{F}_I &= -(n_I, \mathcal{S}) && \text{(\textit{Force} vector)}
\end{aligned}$$

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  - The structure of the discretization process makes it suitable for modular programming.

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  - *Versatility*. The FEM can be applied to a wide range of problems: static, quasi-static, transient, highly dynamical, linear and nonlinear, etc. Moreover, the modular character of the FEM implementation makes possible to have multi-purpose Finite Element frameworks.
  - Many of the procedures that one uses in the framework of the FEM have solid theoretical foundations based on rigorous mathematical analysis.

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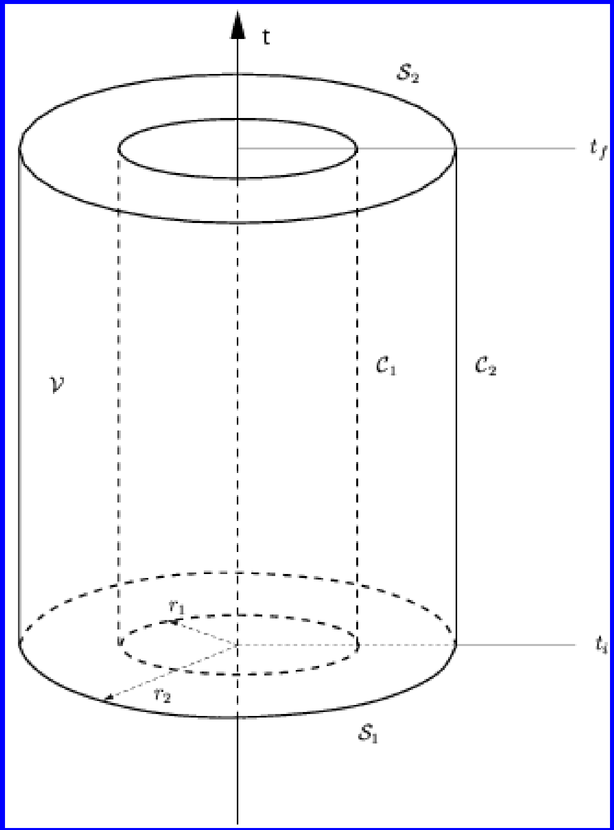
- The source describing the SO is regularized as:

$$\delta[x - z(\tau)] \quad \rightarrow \quad \frac{1}{2\pi\sigma}e^{-(x-z(\tau))^2/(2\sigma^2)}$$

Scalar gravity Toy model and the Adaptive-FEM

- There is a conservation law for this system:



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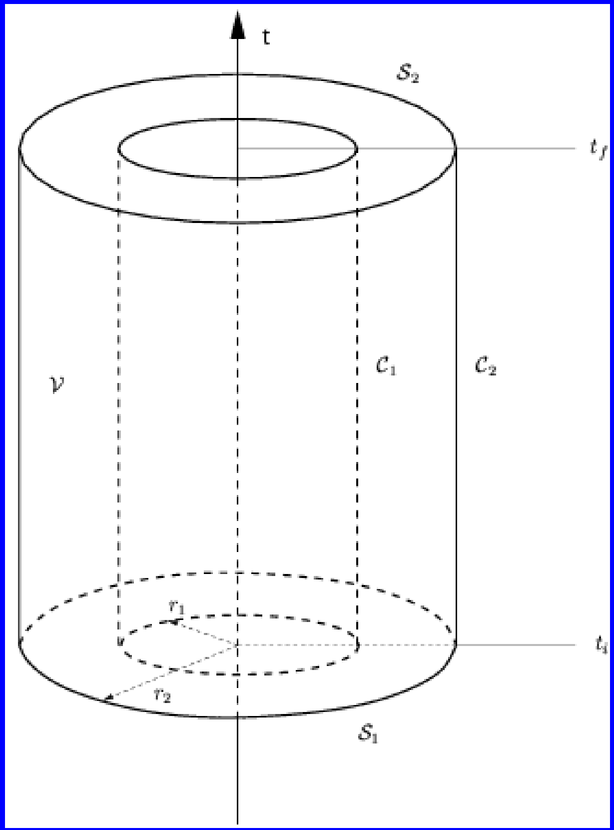

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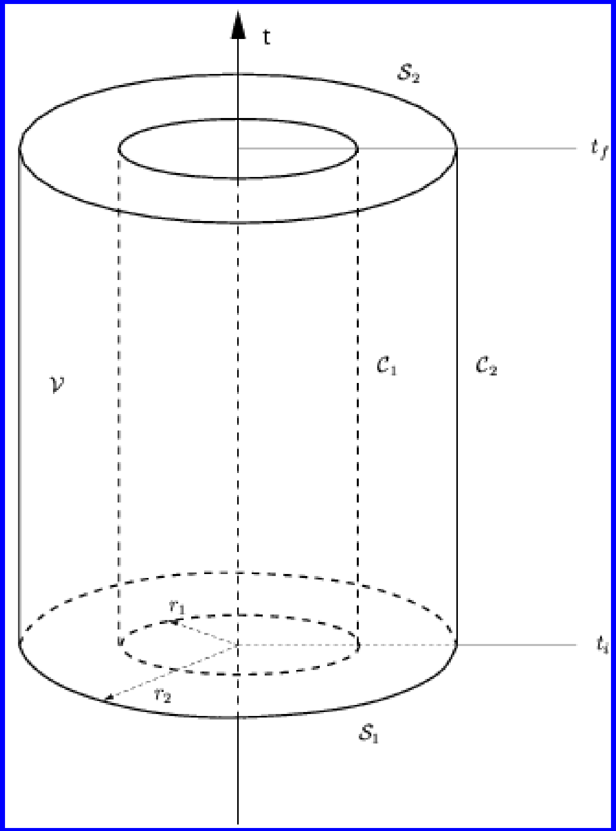
- Using the Timelike Killing $\boldsymbol{\xi} = \boldsymbol{\partial}_t$ of the background we can derive global conservation laws that can be used to check the numerical calculations:

$$\int_{\partial\mathcal{V}} T^{\mu\nu}\xi_\mu d\Sigma_\nu = 0\,.$$



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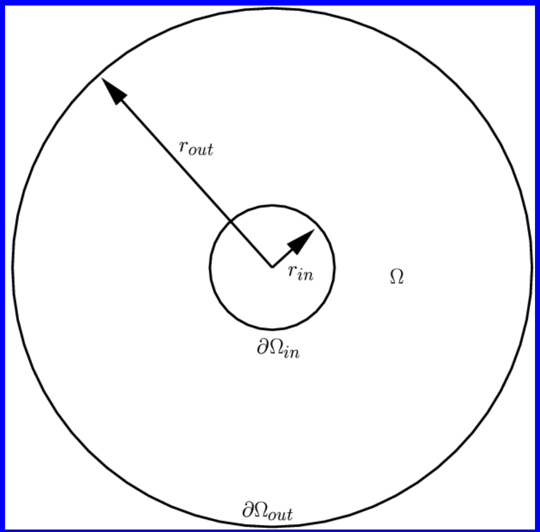

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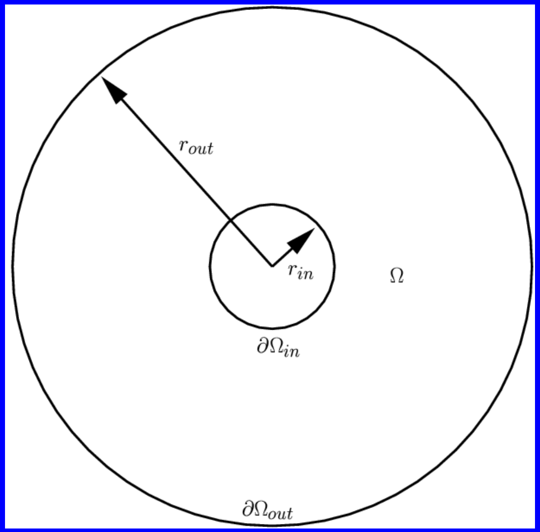

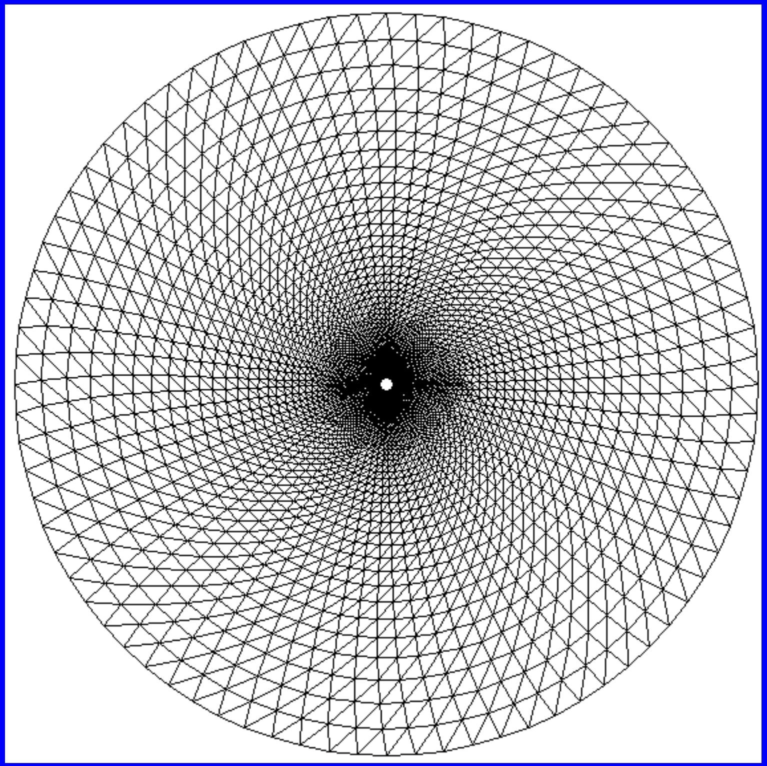

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- Simulations **without** adaptivity around the particle

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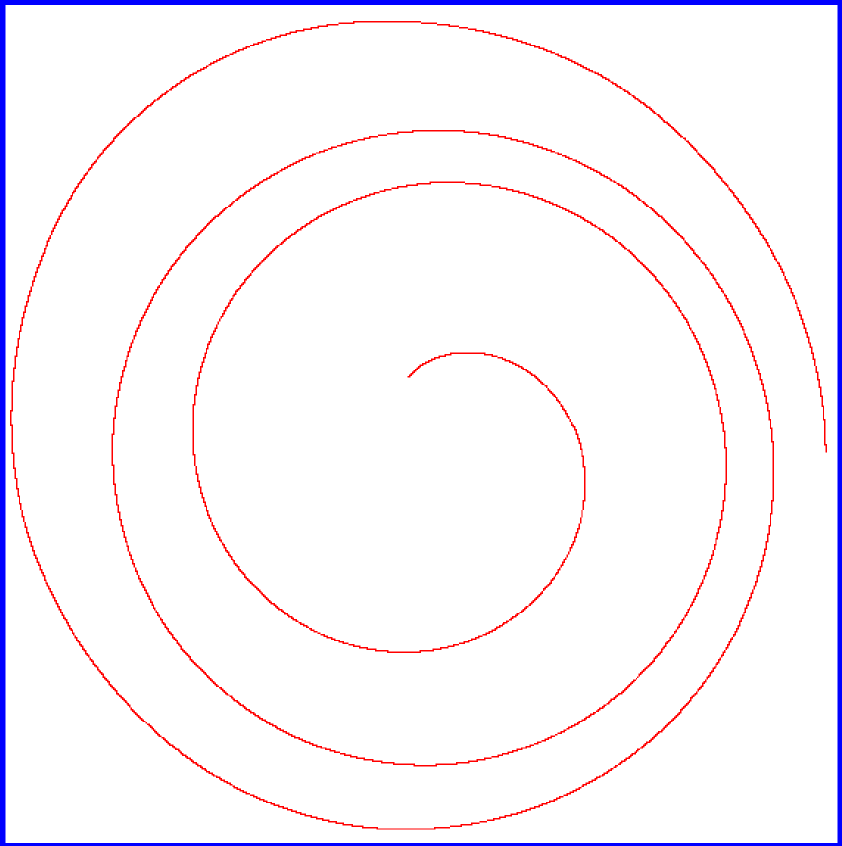
- Simulations **without** adaptivity around the particle
  - Typical trajectory of the SO [Initial conditions for Φ: $(\Phi_o = 0, \dot{\Phi}_o = 0)$]



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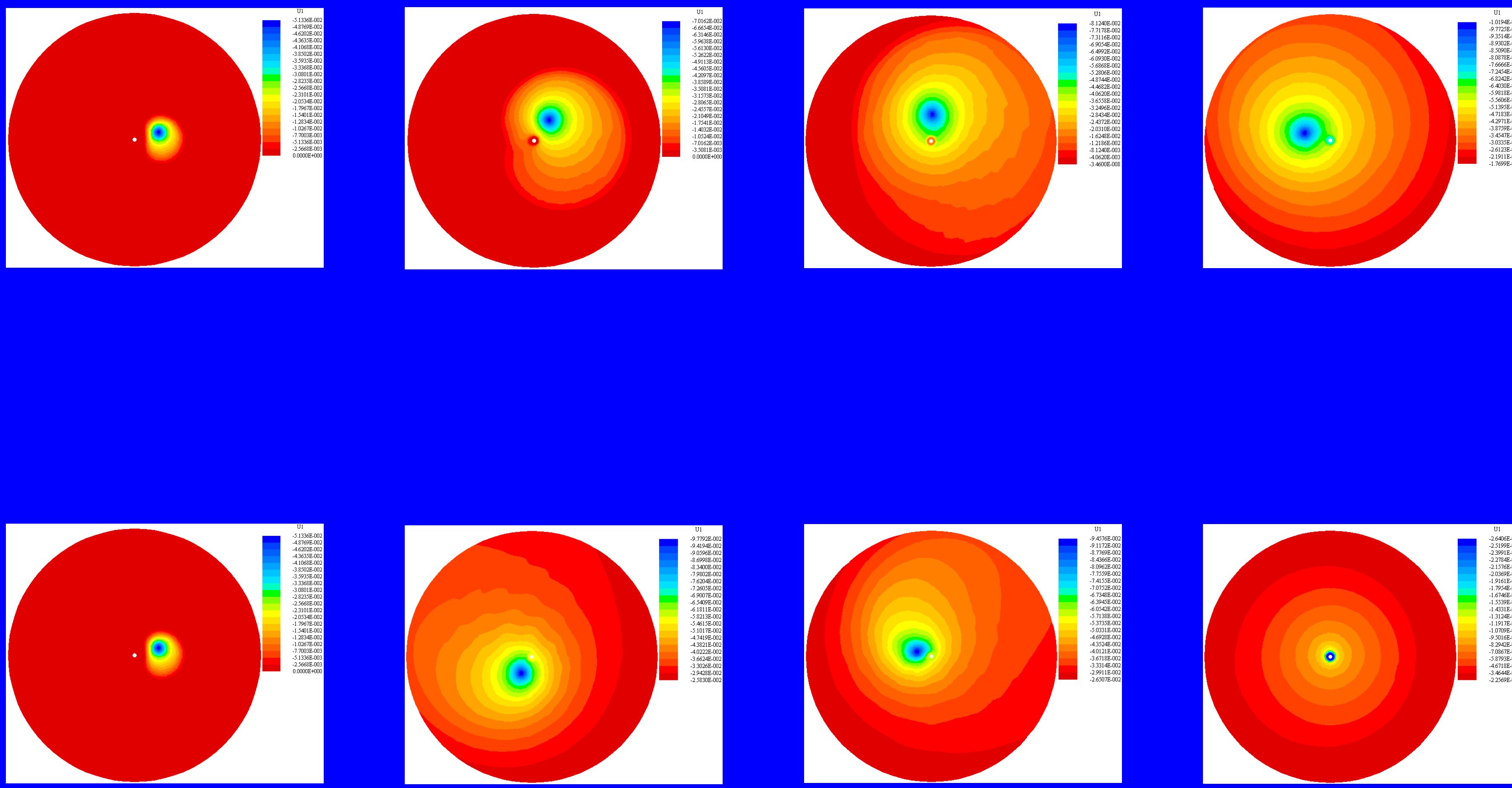

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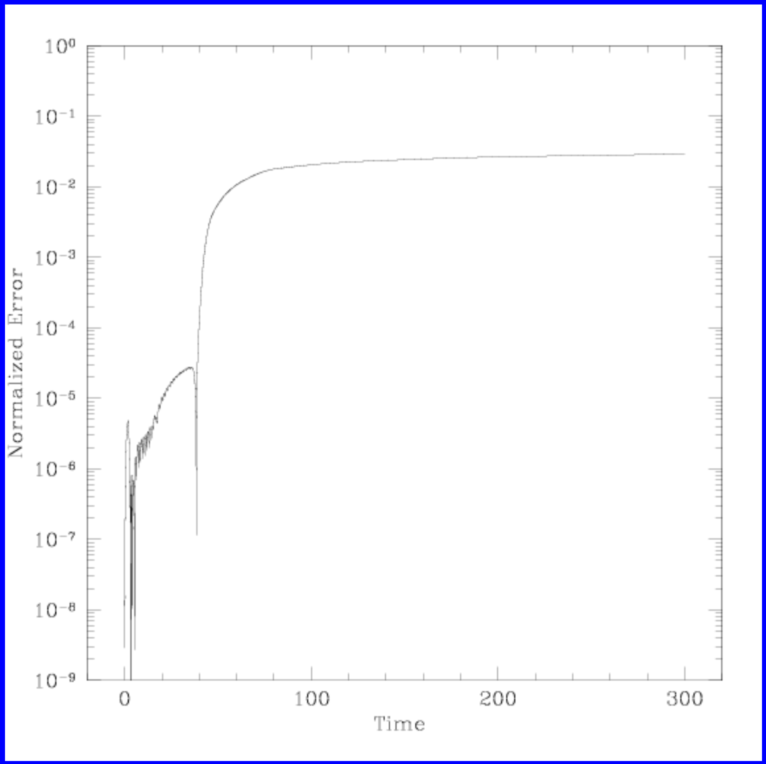

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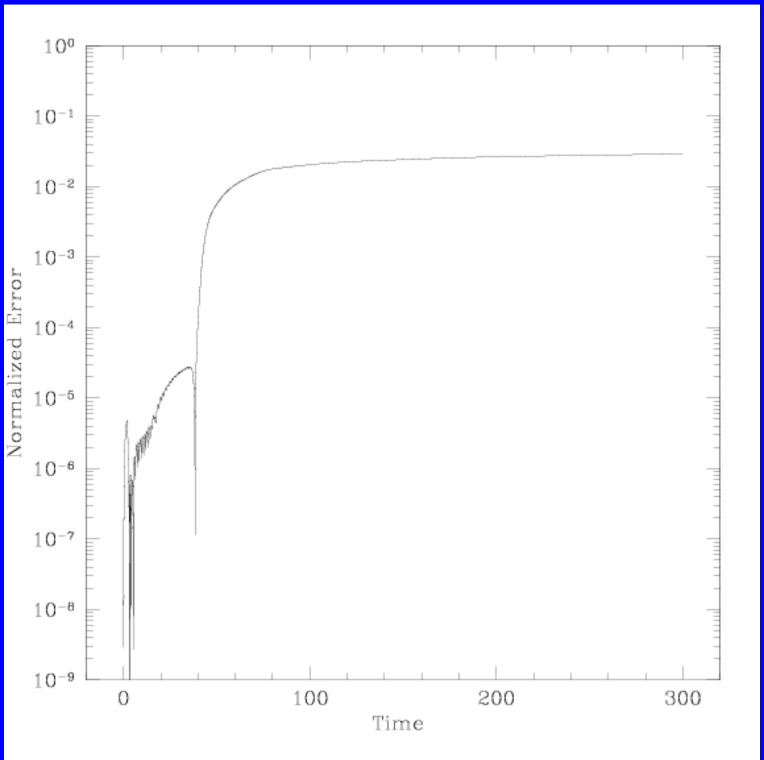

  - Using around 10^4 elements, these simulations work for $\sigma \gtrsim 1\,M$.

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- Simulations **with** adaptivity around the particle (AFEM)
  - The essence of the adaptive mesh technique is to produce real-time local mesh coarsening or refinement to achieve the desired level of smoothness in the solution.
  - To that end, we use an *a posteriori* error estimator to predict the regions in the computational domain where rapidly changes take place is extremely important.

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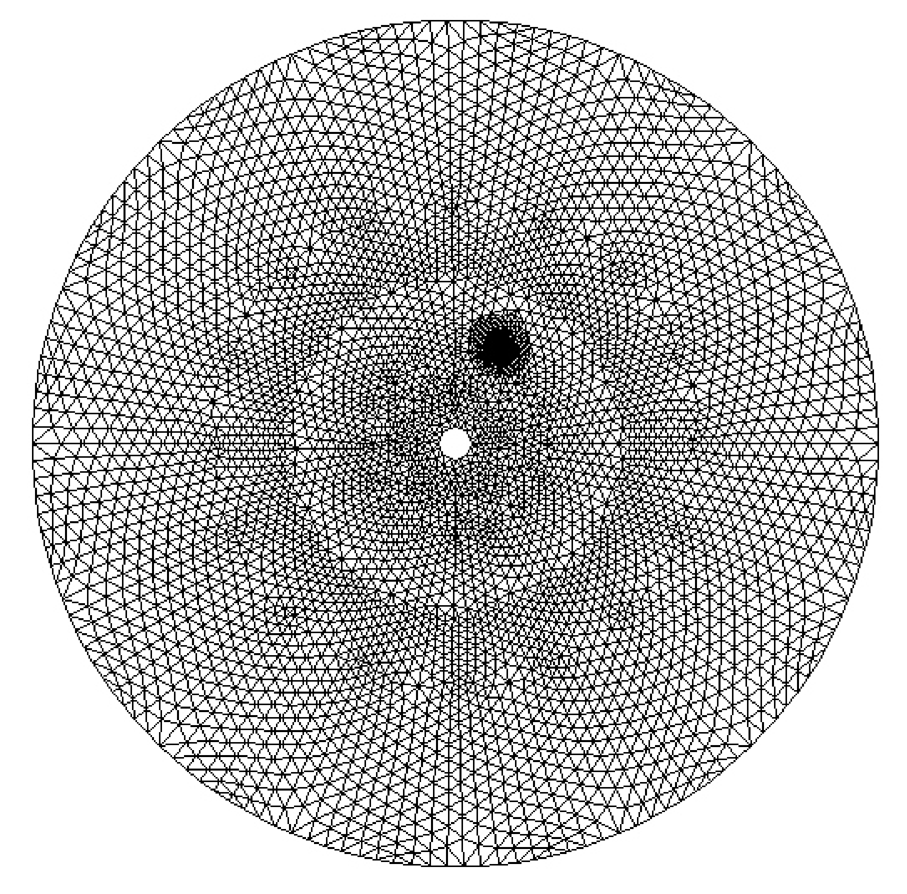

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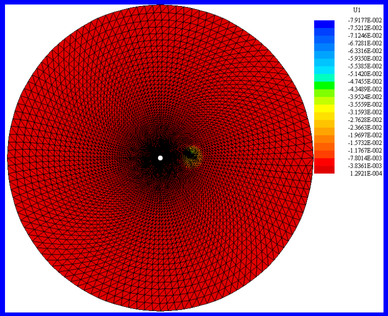

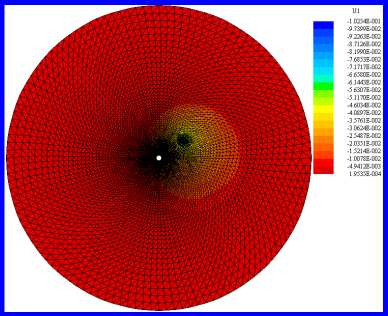

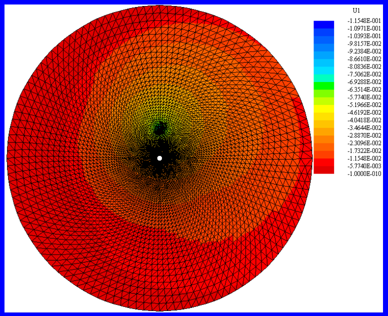

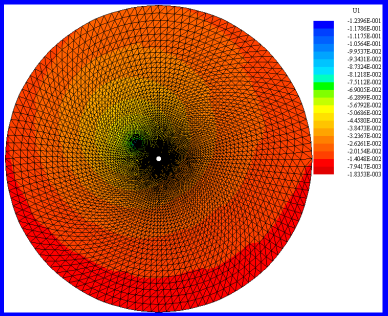

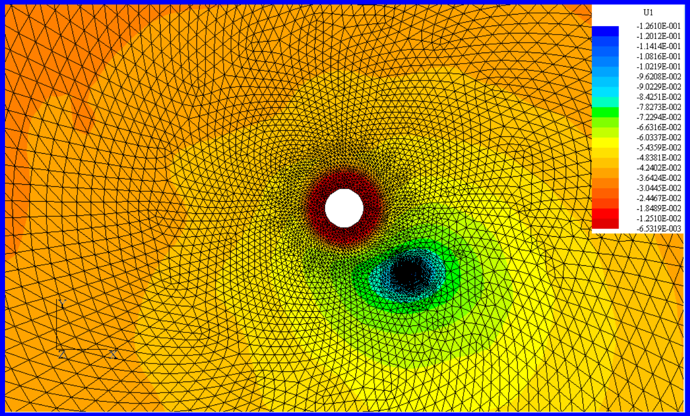

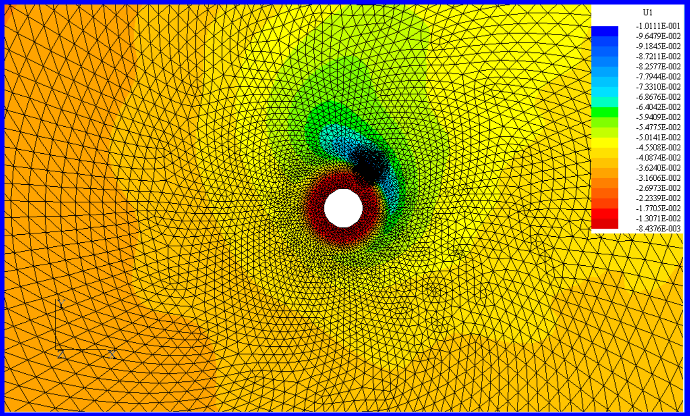

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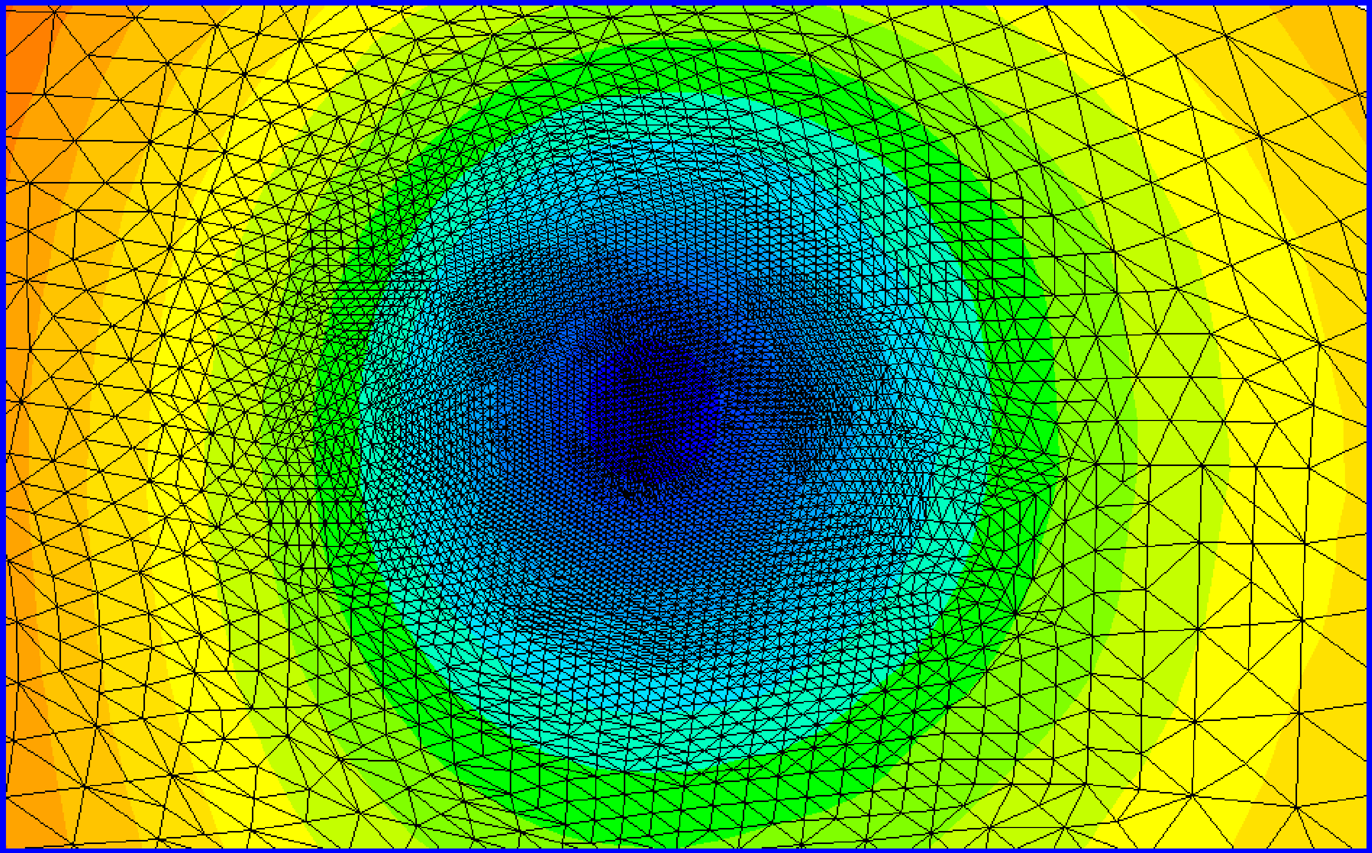

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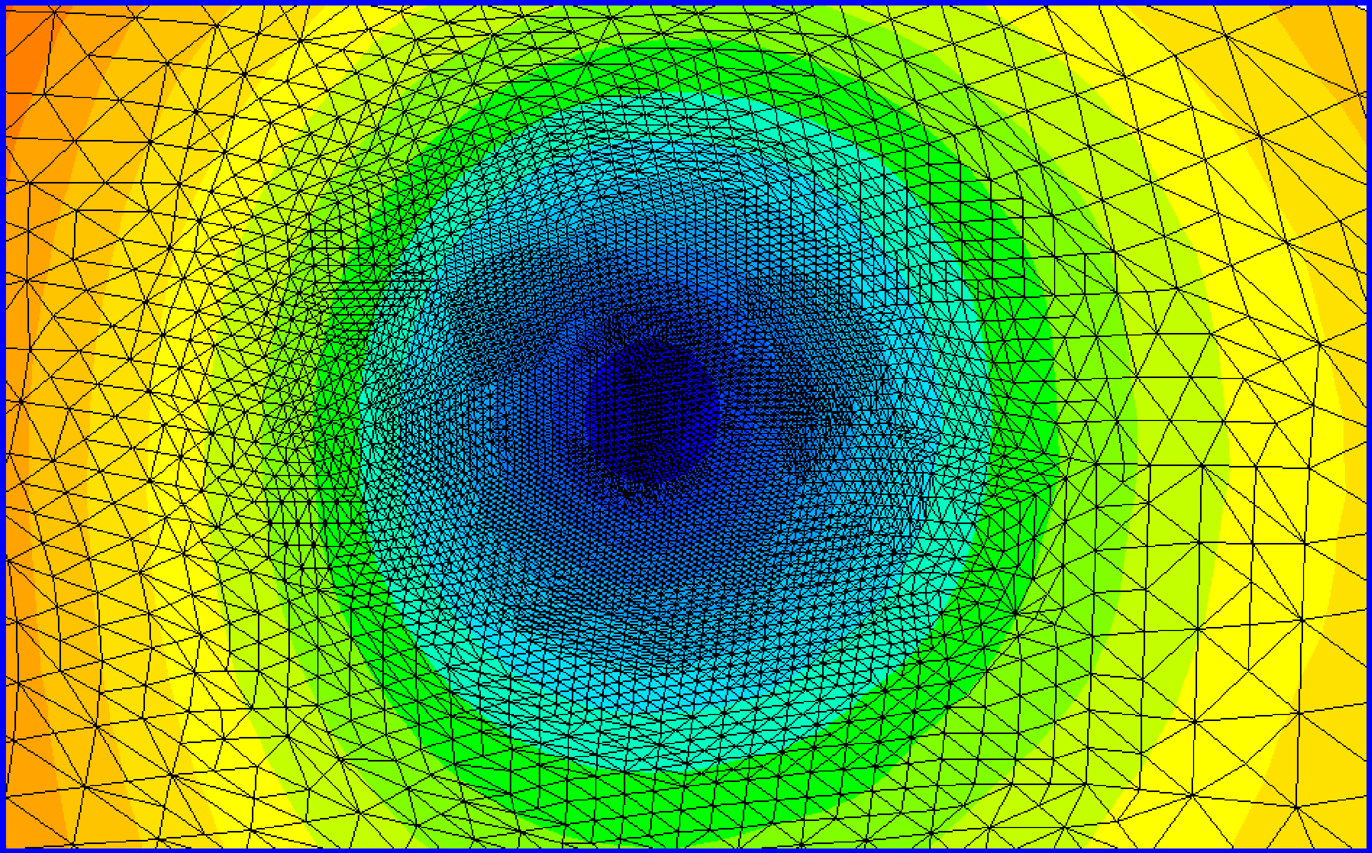

➤ In this case, with $\sim 10^4$ elements, the simulations work for $\sigma \gtrsim 0.1\, M$.

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$$\left(\partial_t \pm \partial_{r_*}\right)\psi_{lm}\Big|_{r_*\to\pm\infty} = 0$$

Perturbative Theory + FEM

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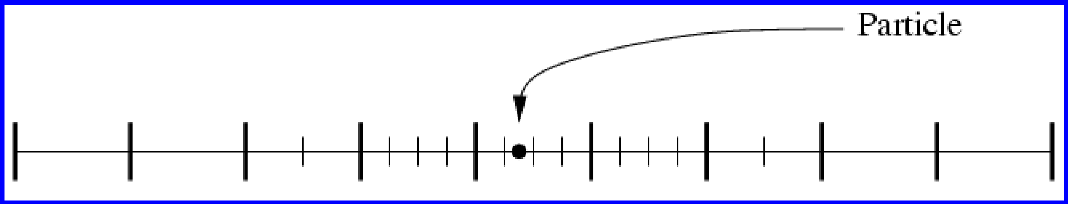

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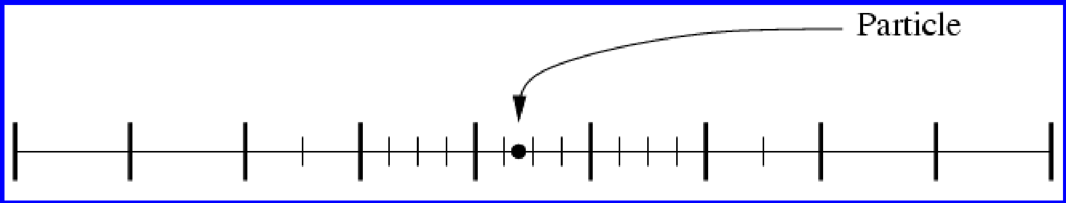

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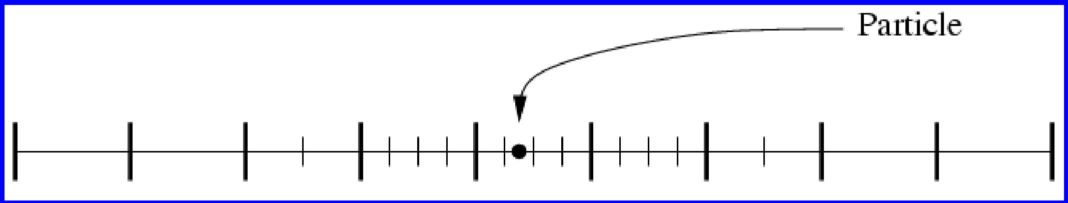

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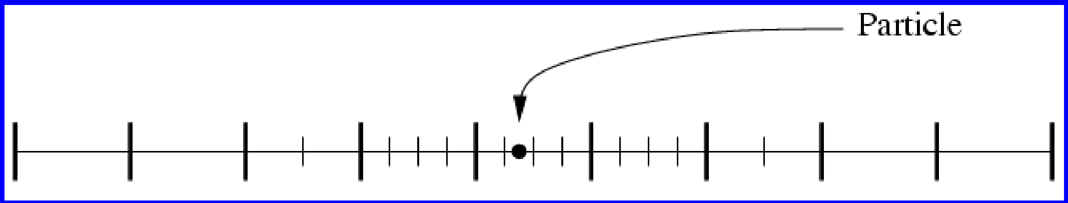

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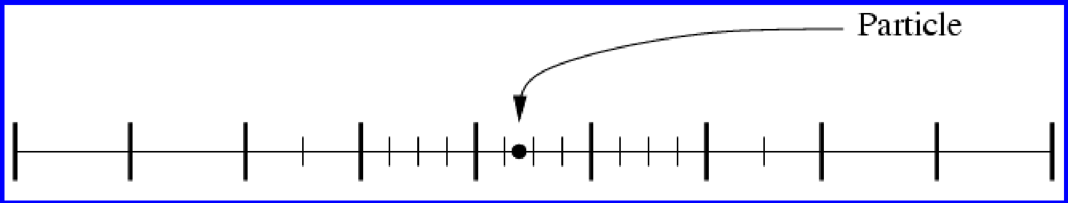

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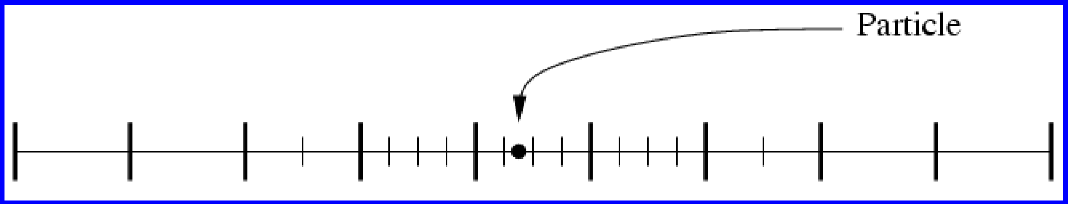

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- Discretization $[\psi_{lm} = \sum_I \psi_{lm,I}(t) n_I(r_*)]$ produces a system of ODEs:

$$\mathbb{M}\cdot\ddot{\mathbf{\Psi}} + \mathbb{G}\cdot\dot{\mathbf{\Psi}} + \mathbb{K}\cdot\mathbf{\Psi} = \boldsymbol{F}$$

where

Perturbative Theory + FEM

$$\begin{aligned}\mathbb{G}_{IJ} &= n_I(r_*^L)n_J(r_*^L) + n_I(r_*^R)n_J(r_*^R) \\ \boldsymbol{F}_I &= P(t, r_p)n_I(r_*^p) + Q(t, r_p)n_I'(r_*^p)\end{aligned}$$

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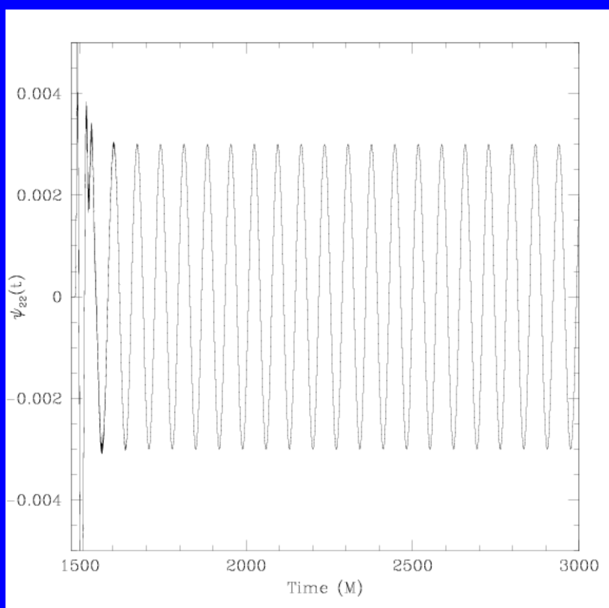
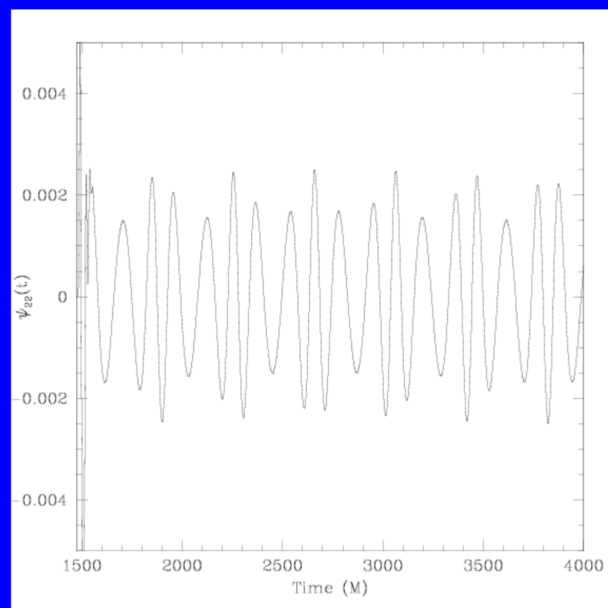
Perturbative Theory + FEM

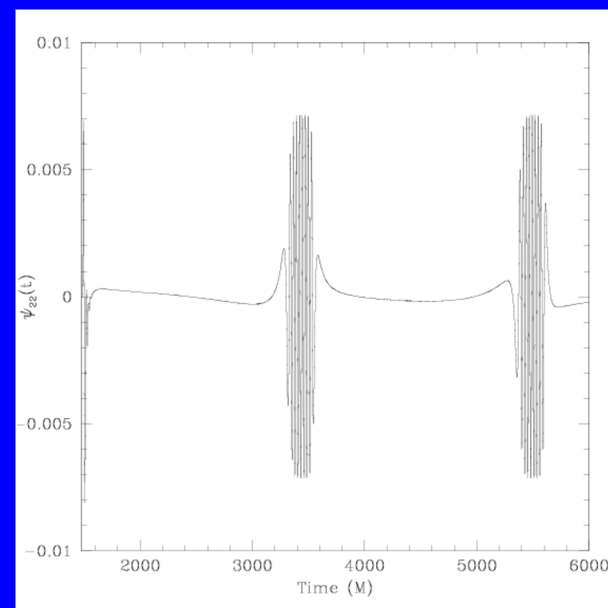
Some Waveforms



Circular orbits



Orbits with $e = 0.2$



Zoom-Whirl orbits

Perturbative Theory + FEM

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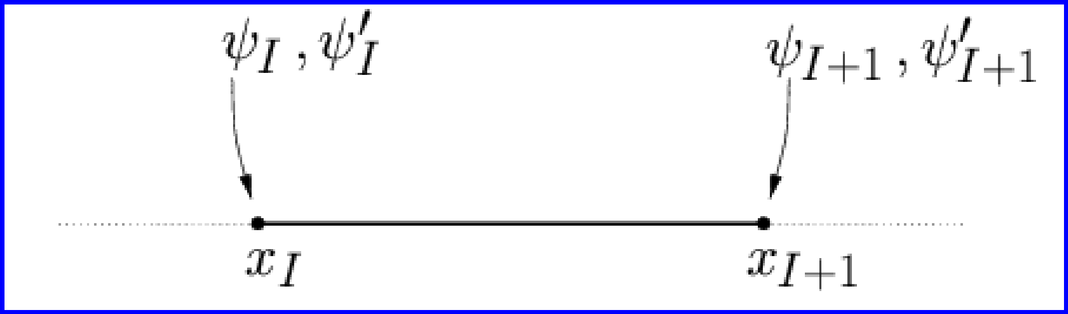

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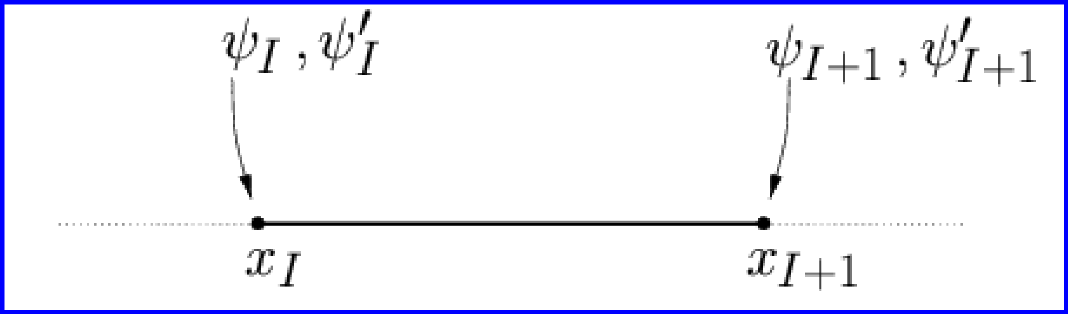

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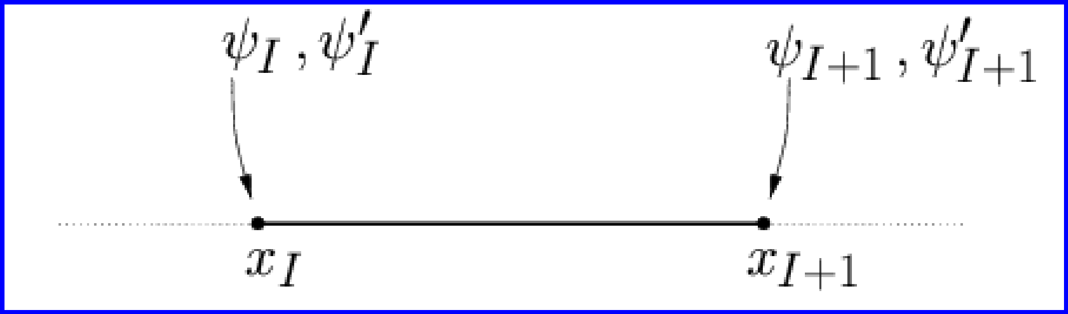

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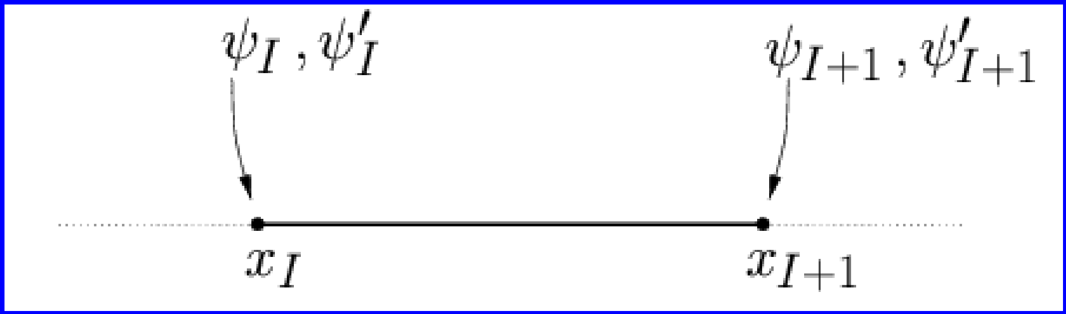

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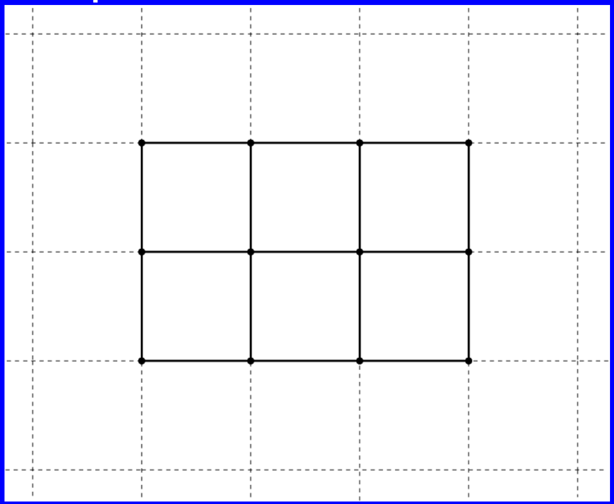

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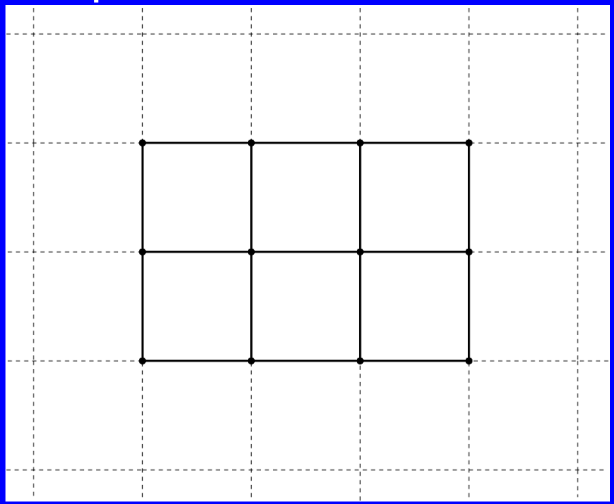

  - Then, the nodal functions can be constructed from the previous one-dimensional ones:

$$n(x,y) \quad \longrightarrow \quad n(x) \otimes n(y)$$

Full NR + Hydro without hydro (Finite Differences)

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- The idea is to explore non-linear effects (relevant for not too extreme mass-ratios) by describing the spacetime geometry with full GR, with the SO affecting it through the matter energy-momentum tensor, which we assume that depends only on its trajectory $z^\alpha(\tau)$ and a finite number of parameters $\boldsymbol{\lambda^I}$:

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- The simplest case is when the matter distribution moves *rigidly* ($T_{\mu\nu} = T_{\mu\nu}[z^\alpha]$) along spacetime geodesics [Bishop, Gomez, Husa, Lehner, & Winicour, PRD**68** 084015 (2003)].

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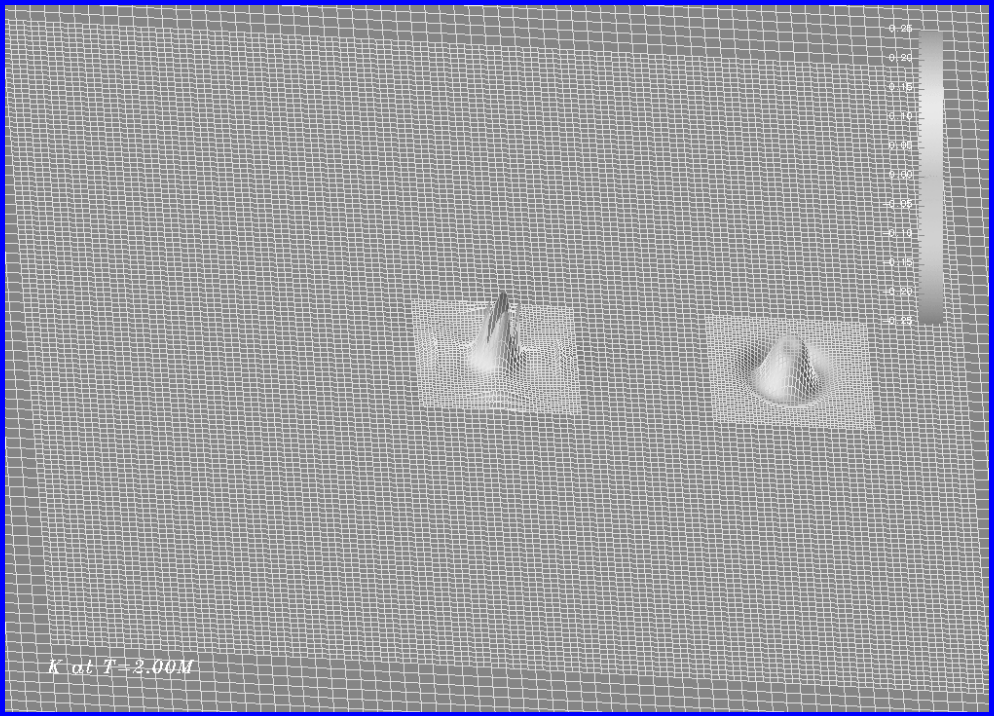

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