Mode-sum methods:
calculation of the gravitational self-force for orbits around black holes

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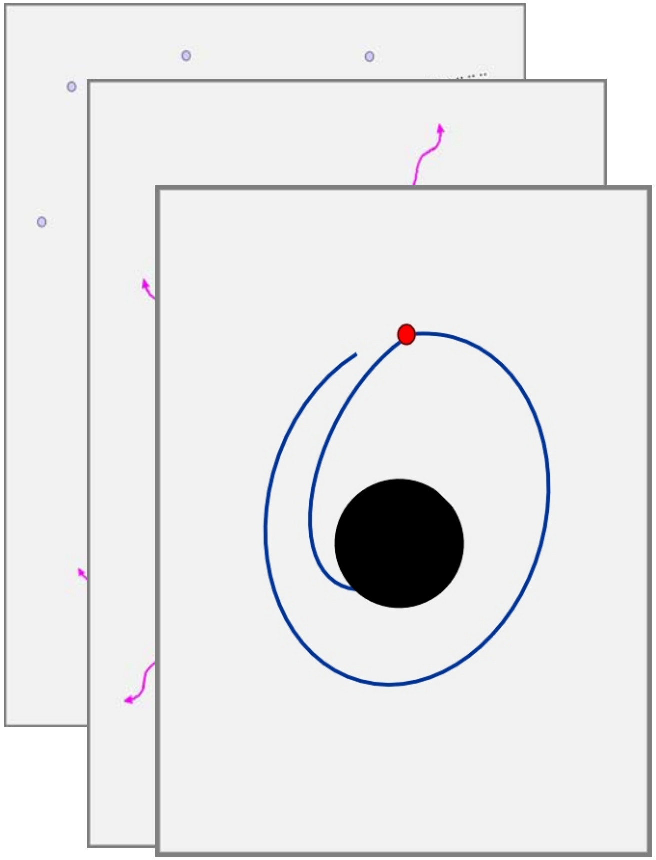
plan

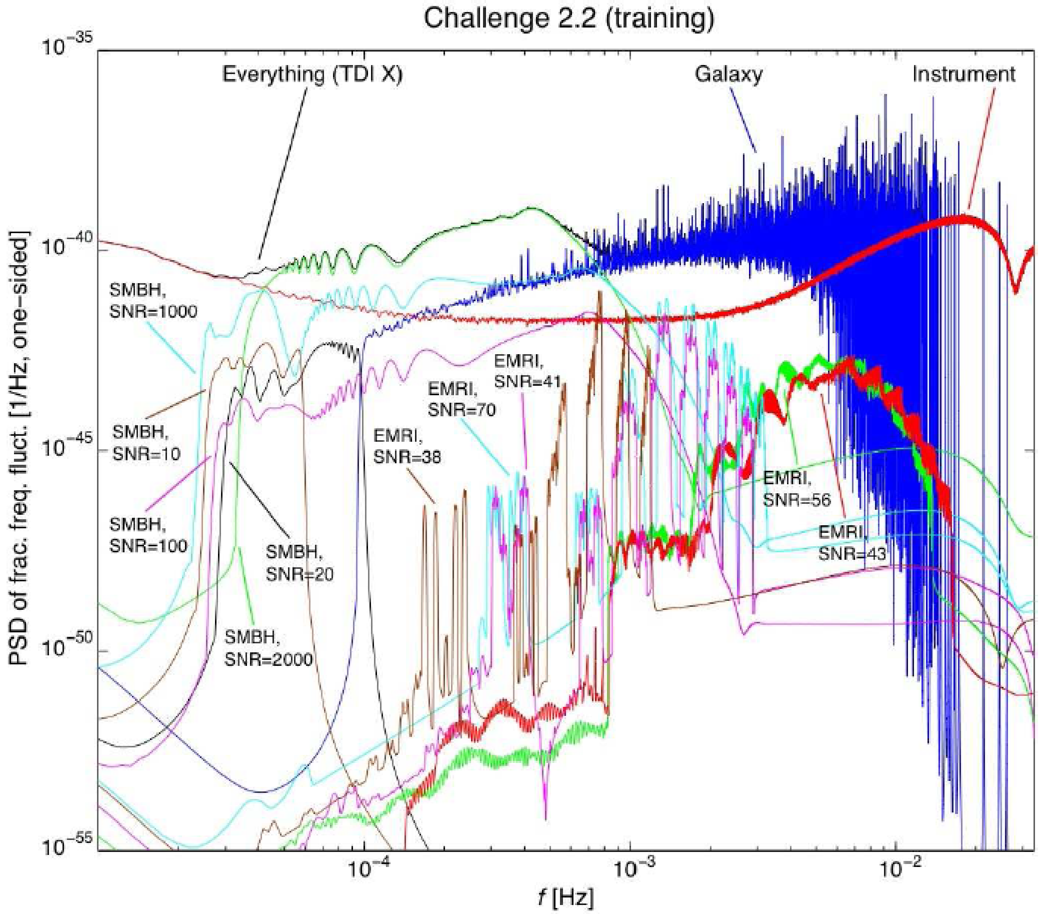
- From Self Force to LISA: EMRI theory challenge
- Practical methods for calculating the MiSaTaQuWa force for orbits around black holes:
  - l-mode method (Schwarzschild)
  - m-mode method (Kerr)
- Implementation for orbits in Schwarzschild using Lorenz gauge and time-domain evolution

LISA's EMRI sources



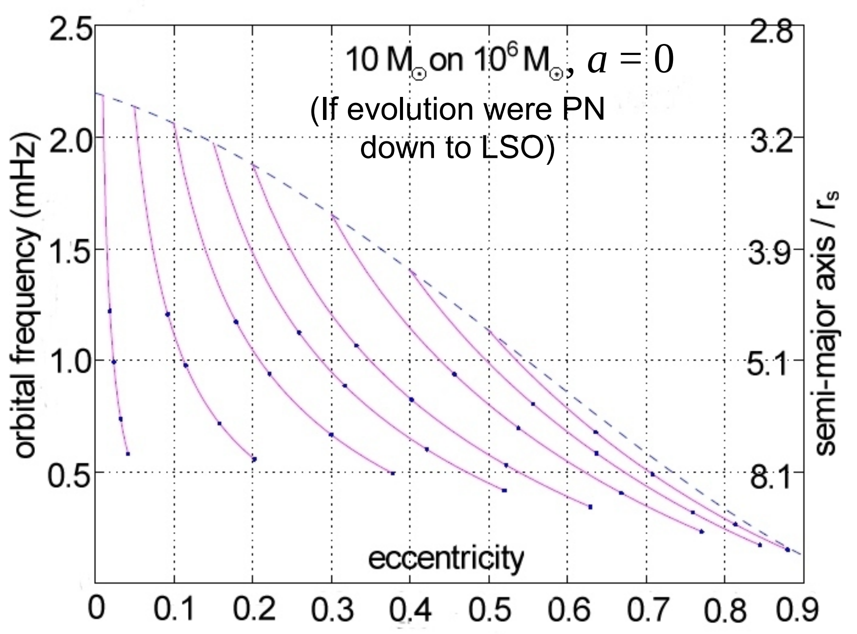
- LISA sees 10s-1000s inspirals, out to cosmological distances
- LISA sensitive to inspirals into massive black holes with $M \sim 10^6\ M_\odot$
- Detection rate dominated by inspirals of stellar holes with $\mu \sim 10\ M_\odot$
- $\sim 10^5$ wave cycles over last few years of inspiral
- Wealth of science encoded in waveforms (map of Kerr geometry, test of “no-hair” theorem)

LISA DA problem difficult!



theory challenge

- Accurate model for the orbital evolution and GW signature for
  - binaries with **extreme** mass-ratios ($m/M = 10^4$–10^7)
  - Kerr central hole
  - generic orbits: inclined, eccentric
  - strong field, down to LSO
- But assume
  - GR holds
  - vacuum
  - internal structure unimportant

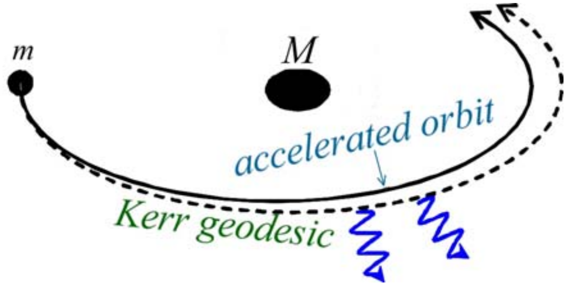


theory challenge (cont.)

- Inspiral timescales $(\text{for } m = 10M_\odot \text{ on } M = 10^6 M_\odot)$
  - $T_M \equiv M(G/c^3) \sim 5 \text{ sec}$
  - $T_{\rm orb} \sim 8 \text{ min}$
  - $T_{\rm rad\ react} \sim \omega/\dot{\omega} \sim T_M(M/m) \sim \textbf{months}$
  - $T_{\rm dephas} \sim \dot{\omega}^{-1/2} \sim T_M(M/m)^{1/2} \sim \textbf{hours}$
  - $T_{\rm obs} \sim \textbf{1–3 yrs}$
- So, to track the orbital phase over ~ $T_{\rm obs}$ we need

$$\delta\Phi(T_{\rm obs}) \sim \delta(\dot{\omega}) \times T_{\rm obs}^2 \lesssim 1 \quad \Rightarrow \quad \frac{\delta(\dot{\omega})}{\dot{\omega}} \lesssim (M/m)(T_M/T_{\rm obs})^2 \sim 10^{-6}$$

from self force to inspiral



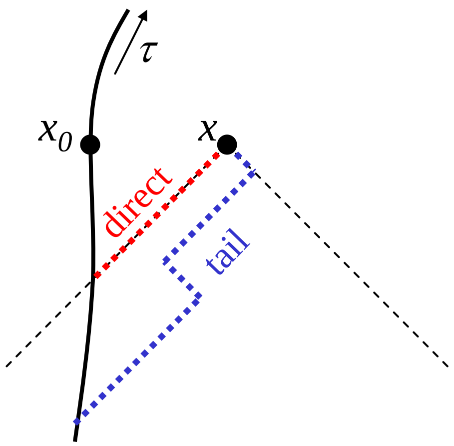
1. $\mathcal{L}[h_{\mu\nu}] = T_{\mu\nu}(\text{geodesic particle})$
2. $h_{\mu\nu} \to [\text{regularization scheme}] \to F^{\alpha}_{\text{self}}$
3. $m\,\ddot{x}^{\alpha} = F^{\alpha}_{\text{self}}$

- Note:
  - SF correction to waveform is same order as contribution from 2nd-order pert.!
  - Even 2nd order waveform goes out of phase after ~T_{dephase}
- Need a scheme for adiabatic evolution:
  - Osculating geodesics (Pound & Poisson 2007)
  - 2 timescale expansion (Hinderer & Flanagan 2008)
- This lecture focuses on obtaining the SF at a point along the orbit

the gravitational SF

- Two equivalent formulations:

$$
\begin{aligned}
F^{\alpha}_{\rm self} &= \lim_{x\to x_0} \nabla^{\alpha\mu\nu} h^{\rm tail}_{\mu\nu} = \lim_{x\to x_0} \nabla^{\alpha\mu\nu} \left(h^{\rm ret}_{\mu\nu} - h^{\rm dir}_{\mu\nu}\right) \\
&= \lim_{x\to x_0} \nabla^{\alpha\mu\nu} h^{\rm R}_{\mu\nu} = \lim_{x\to x_0} \nabla^{\alpha\mu\nu} \left(h^{\rm ret}_{\mu\nu} - h^{\rm S}_{\mu\nu}\right)
\end{aligned}
$$



- Gauge dependence:

$$
\begin{aligned}
x^{\alpha} &\to x^{\alpha} - \xi^{\alpha} (\propto m) \\
h_{\mu\nu} &\to h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \\
F^{\alpha}_{\rm self} &\to F^{\alpha}_{\rm self} - m \left[(g^{\alpha\lambda} + u^{\alpha}u^{\lambda})\ddot{\xi}_{\lambda} + R^{\alpha}{}_{\mu\lambda\nu} u^{\mu} \xi^{\lambda} u^{\nu} \right]
\end{aligned}
$$

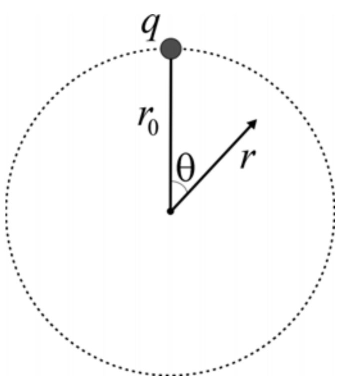
- Challenge (as of 1999): How to go about subtracting "ret - dir" (or "ret - S") in practice, particularly for orbits in BH spacetimes?

standard methods of black hole perturbation theory

Perturbation Eq.	Separation in time domain	Separation in frequency domain
Schwarzschild background		
$\mathcal{L}_{\text{Eins}}[h_{\mu\nu}] = T_h$	$h_{\mu\nu} = \sum_{l,m} \sum_{i=1}^{10} Y_{\mu\nu}^{lm(i)}(\theta,\varphi)\underline{h_{lm}^{(i)}(t,r)}$	$h_{\mu\nu} = \sum_{l,m} \sum_{i=1}^{10} Y_{\mu\nu}^{lm(i)} \int d\omega\, e^{-i\omega t} \underline{h_{lm\omega}^{(i)}(r)}$
Kerr background		
$\mathcal{L}_{\text{Teuk}}[\Psi] = T_\Psi$	$\Psi = \sum_m e^{im\varphi} \underline{\psi_m(t,r,\theta)}$	$\Psi = \sum_{l,m} \int d\omega\, S_{lm\omega}(\theta) e^{im\varphi} e^{-i\omega t} \underline{\psi_{lm\omega}(r)}$

- Individual modes of retarded field obtained by solving ODEs (or PDEs in 1+1d or 2+1d) with suitable initial/boundary conditions
- "mode sum" approach to the self force uses these solutions as input; does the subtraction "ret - dir" mode by mode

an illustration:
static scalar charge in flat space (I)



- Use polar coordinates, set particle on "polar" axis (no loss of generality even in Schwarzschild)
- Write down field equation:

$$\nabla^2 \Phi = -4\pi q\, \delta^3(\vec{x} - \vec{x}_{\rm p})$$

- In this case field and "full" force obtained exactly:

$$\Phi = \frac{q}{|\vec{x} - \vec{x}_0|} \quad , \qquad F_x^{\rm full} \equiv q \nabla_x \Phi = q^2 \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|^3}$$

- Separate field into spherical harmonic "l-modes":

$$\Phi = \sum_{l=0}^{\infty} \tilde{\phi}^l(r)\, Y^{l,m=0}(\theta, \varphi) = \sum_{l=0}^{\infty} \phi^l(r)$$

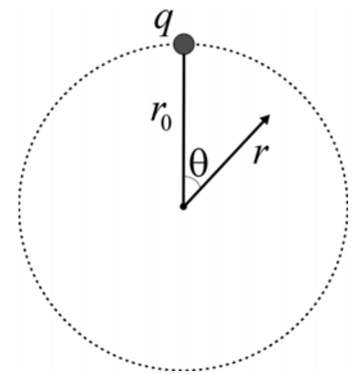
- Write down separated equations:

$$\tilde{\phi}^l_{,rr} + \frac{2}{r}\tilde{\phi}^l_{,r} - \frac{l(l+1)}{r^2}\,\tilde{\phi}^l = -\frac{\sqrt{4\pi(2l+1)}\, q}{r_0^2}\, \delta(r - r_0)$$

an illustration:
static scalar charge in flat space (II)

- Solve separated equations:

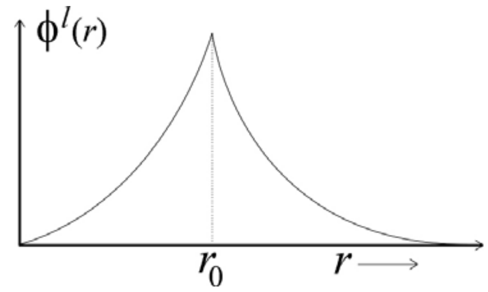
$$\phi^l(r,\theta) = \frac{q}{r_0} P_l(\cos\theta) \times \begin{cases} (r/r_0)^{-l-1}, & r \geq r_0 \\ (r/r_0)^l, & r \leq r_0 \end{cases}$$



- Obtain *l*-mode of full force:

$$F_r^{l,\mathrm{full}}(r_0^\pm) \equiv \nabla_{r^\pm}\phi^l\big|_{x_0} = \mp(l+1/2)\frac{q^2}{r_0^2} - \frac{q^2}{2r_0^2}$$

- Note:
  - each $F_r^{l,\mathrm{full}}$ is **finite**
  - $F_r^{l,\mathrm{full}} \propto l$ as $l \to \infty$



- Universal form:

$$\boxed{F_\alpha^{l,\mathrm{full}}(r_0^\pm) = \pm L A_\alpha + B_\alpha + O(L^{-2})} \quad \text{where} \quad L \equiv l + 1/2$$

"Regularization Parameters", depend on x^α and u^α at x_0

(l-)mode-sum method

$$\begin{aligned} F_{\rm self} &= \lim_{x\to x_0}\left[F_{\rm full}(x) - F_{\rm dir}(x)\right] \\ &= \lim_{x\to x_0}\sum_l \left[F^l_{\rm full}(x) - F^l_{\rm dir}(x)\right] \\ &= \lim_{x\to x_0}\left[\sum_l \left(F^l_{\rm full}(x) - LA - B\right) - \sum_l \left(F^l_{\rm dir}(x) - LA - B\right)\right] \end{aligned}$$

$$\begin{pmatrix} F_{\rm full} &=& m\nabla h_{\rm ret} \\ F_{\rm dir} &=& m\nabla h_{\rm dir} \end{pmatrix}$$

$$\boxed{F_{\rm self} = \sum_l \underbrace{\left(F^l_{\rm full}(x_0) - LA - B\right)}_{O(L^{-2})} - D}$$

$$D \equiv \sum_l \underbrace{\left(F^l_{\rm dir}(x) - LA - B\right)}_{O(L^{-2})}$$

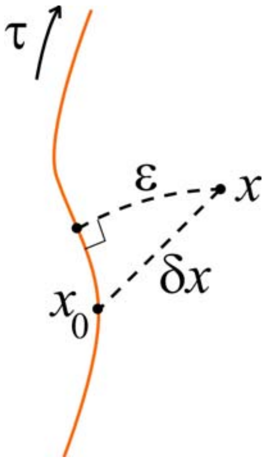
- Regularization parameters A, B, D derived analytically by analyzing the direct force near $x \to x_0$ and $l \to \infty$

derivation of the RP (scalar field)

$$\Phi^{\rm dir}(x) = \frac{q}{\epsilon}\left[1 + O(\delta x)^2\right]$$

$$F_\alpha^{\rm dir}(x) = q\nabla_\alpha \Phi^{\rm dir} = \frac{P_\alpha^{(1)}(\delta x)}{\epsilon_0^3} + \frac{P_\alpha^{(3)}(\delta x)}{\epsilon_0^5} + \frac{P_\alpha^{(7)}(\delta x)}{\epsilon_0^7} + O(\delta x)$$

where ϵ_0 is the $O(\delta x)$ term of ϵ and $P_\alpha^{(n)}$ is a polynomial of order n in δx.



$$\begin{aligned} F_\alpha^{l,\rm dir} &= \lim_{x \to x_0} \sum_m Y^{lm}(\Omega) \oint d\Omega'\, Y^{lm*}(\Omega')\, F_\alpha^{\rm dir}(x) \\ &= LA_\alpha + B_\alpha \end{aligned}$$

values of the RP
(scalar field, Schwarzschild, equatorial orbit)

$$A_{\pm r} = \mp \frac{q^2}{r^2} \frac{\text{☜}}{fV}, \qquad A_{\pm t} = \pm \frac{q^2}{r^2} \frac{\dot{r}}{V}, \qquad A_\varphi = 0$$

$$B_r = \frac{q^2}{r^2} \frac{(\dot{r}^2 - 2\text{☜}^2)\hat{K}(w) + (\dot{r}^2 + \text{☜}^2)\hat{E}(w)}{\pi f\,(1+\text{☹}^2/r^2)^{3/2}} \qquad B_\varphi = \frac{q^2}{r} \frac{\dot{r}\left[\hat{K}(w) - \hat{E}(w)\right]}{\pi(\text{☹}/r)V^{1/2}}$$

$$B_t = \frac{q^2}{r^2} \frac{\text{☜}\dot{r}\left[\hat{K}(w) - 2\hat{E}(w)\right]}{\pi V^{3/2}} \qquad D_\alpha = 0$$

$\text{☜} = -u_t$

$\text{☹} = u_\varphi$

$\dot{r} = u^r$

$f = 1 - 2M/r$

$w = \text{☹}^2/(\text{☹}^2 + r^2)$

$V = 1 + \text{☹}^2/r^2$

$\hat{E}, \hat{K}$ – elliptic intgs.

- Values obtained also for gravitational case, generic orbits in Kerr
- Work to obtain next order terms in the $1/L$ series, to improve convergence of mode sum

mode-sum prescription summarized

① For a given geodesic source, solve the perturbation equations (mode by mode), and obtain the retarded Metric Perturbation (MP);

② Construct the "full force" modes ("grad MP") at the particle;

③ Apply the mode-sum formula:

$$\boxed{F_{\text{self}} = \sum_{l} \left(F^{l}_{\text{full}}(x_0) - LA - B \right)}$$

implementation issues:
choice of gauge (I)

$$F_{\text{self}}^{[G]} = \lim_{x\to x_0} [\nabla h_{\text{ret}}^{[G]} - \nabla h_{\text{dir}}^{[G]}]$$

- Subtraction formulated in Lorenz gauge:

$$\bar{h}_{\mu\nu}{}^{;\nu} = 0 \qquad (\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h)$$

- Note: particle singularity looks "isotropic" in Lorenz gauge,

$$\bar{h}_{\mu\nu}^{\text{dir}} \sim \frac{m u_\mu u_\nu}{\epsilon}$$

  but not necessarily in other gauges!

- Two strategies:
  - Obtain h_{ret} in Lorenz gauge, or
  - Reformulate subtraction in other gauge, but make sure SF still makes sense!

$$(\text{recall } \delta_\xi F_{\text{self}}^\alpha = -m[(g^{\alpha\lambda} + u^\alpha u^\lambda)\ddot{\xi}_\lambda + R^\alpha{}_{\mu\lambda\nu} u^\mu \xi^\lambda u^\nu])$$

implementation issues: choice of gauge (II)

- Popular gauges:
  - **Regge-Wheeler gauge** (Schwarzschild)

    $$h_{\theta\varphi} = h_{\theta\theta} - \sin^{-2}\theta\, h_{\varphi\varphi} = h_{t[\theta,\varphi]} = h_{r[\theta,\varphi]} = 0$$

    [even parity]
  - **Radiation gauge** (Kerr)

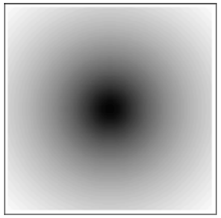
    $$h_{\mu\nu} l^{\nu} = 0 \quad \text{or} \quad h_{\mu\nu} n^{\nu} = 0$$
  - **Lorenz gauge** (Kerr)

    $$\bar{h}_{\mu\nu}{}^{;\nu} = 0$$

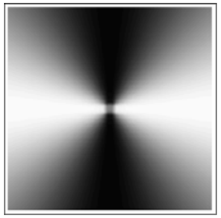
Regge-Wheeler and radiation gauges: conform with symmetry/light-cone structure of background

Lorenz gauge: conforms with isotropic form of particle singularity

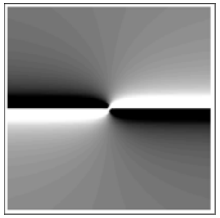
- Form of particle singularity (in particle frame):



Lorenz



RW



Radiation

Lorenz-gauge implementation:
i. perturbation equations

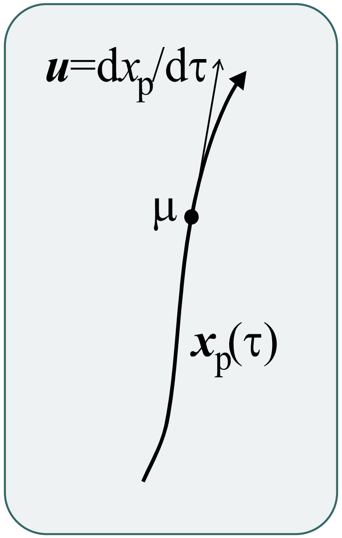
$$\Box\bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} + g_{\alpha\beta}\bar{h}^{\mu\nu}{}_{;\mu\nu} - 2g^{\mu\nu}\bar{h}_{\mu(\alpha;\nu\beta)}$$
$$= -16\pi\mu\int_{-\infty}^{\infty}(-g)^{-1/2}\,\delta^4[x^\mu - x_p^\mu(\tau)]u_\alpha u_\beta\,d\tau \equiv S_{\alpha\beta},$$

Impose Lorenz gauge condition,

$$g^{\beta\gamma}\bar{h}_{\alpha\beta;\gamma} = 0.$$

Get

$$\boxed{\Box\bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = S_{\alpha\beta}}$$



Lorenz-gauge implementation:
ii. Tensor-harmonic decomposition (S. Case)

$$\bar{h}_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} \bar{h}^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi)$$

$$S_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} S^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi)$$

Numerical variables are 10 "scalar" fields,

$$\bar{h}^{(1)},\ \bar{h}^{(2)},\ \bar{h}^{(3)},\ \bar{h}^{(4)},\ \bar{h}^{(5)},\ \bar{h}^{(6)},\ \bar{h}^{(7)},\ \bar{h}^{(8)},\ \bar{h}^{(9)},\ \bar{h}^{(10)}$$

"scalar" sector

Even parity (polar)

"vector" sector

"tensor" sector

Odd parity (axial)

Lorenz-gauge implementation:
iii. mode-separated equations

$$\boxed{\underbrace{\bar{h}^{(i)lm}_{,tt} - \bar{h}^{(i)lm}_{,r_*r_*}}_{\text{Principal part: no coupling between } l\text{'s}} + \underbrace{\mathcal{M}^{(i)}_{(j)}\bar{h}^{(j)lm}}_{\text{Coupling terms: 1st derivatives at most}} = S^{(i)lm}}$$

10 equations

Source terms:

$$S^{(i)lm} = f(x_{\rm p}(t))\,\delta(r - r_{\rm p}(t))$$

Lorenz-gauge implementation:
iv. gauge conditions in separated form

$$i\bar{h}^{(1)}_{,t} + i\bar{h}^{(3)}_{,t} + f\left(\bar{h}^{(2)}_{,r} + \bar{h}^{(2)}/r - \bar{h}^{(4)}/r\right) = 0$$

$$i\bar{h}^{(2)}_{,t} - f\left(\bar{h}^{(1)}_{,r} - \bar{h}^{(3)}_{,r}\right) + (1 - 4M/r)\bar{h}^{(3)}/r - (f/r)\left(\bar{h}^{(1)} - \bar{h}^{(5)} - 2f\bar{h}^{(6)}\right) = 0$$

$$i\bar{h}^{(4)}_{,t} - f\left(\bar{h}^{(5)}_{,r} + 2\bar{h}^{(5)}/r + l(l+1)\,\bar{h}^{(6)}/r - \bar{h}^{(7)}/r\right) = 0$$

$$i\bar{h}^{(8)}_{,t} + f\left(\bar{h}^{(9)}_{,r} + 2\bar{h}^{(9)}/r - \bar{h}^{(10)}/r\right) = 0$$

Lorenz-gauge implementation:
v. "div damping" in time evolution

- Define $Z_\alpha \equiv \nabla^\beta \bar{h}_{\alpha\beta}$. Then $\Box Z_\alpha = 0$ (since $\nabla^\beta T_{\alpha\beta} = 0$)
- If Z_α satisfies a well-posed initial-value problem, with $Z_\alpha = 0$ on the initial surfaces, then $Z_\alpha \equiv 0$ everywhere.
- This may fail in practice because numerical error from discretization and from imperfect initial data may grow during the evolution
- To make sure Z_α is damped over the evolution, add to pert. Eqs a term

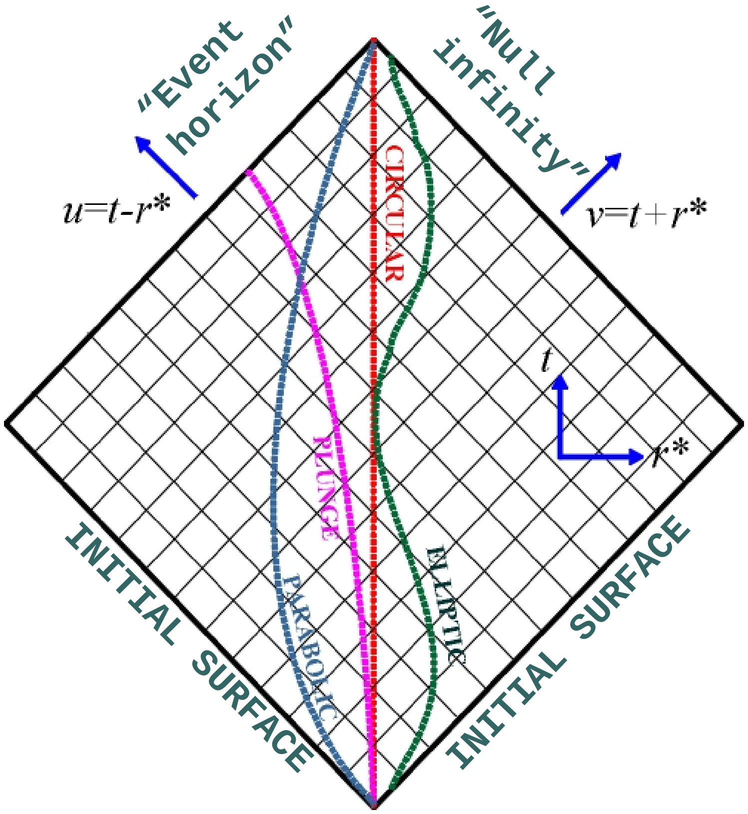
$$-\kappa(t_\alpha Z_\beta + t_\beta Z_\alpha)$$

- Then $\Box Z_\alpha - \kappa \dot{Z}_\alpha + \ldots = 0$ and Z_α damps over timescale κ^{-1}.

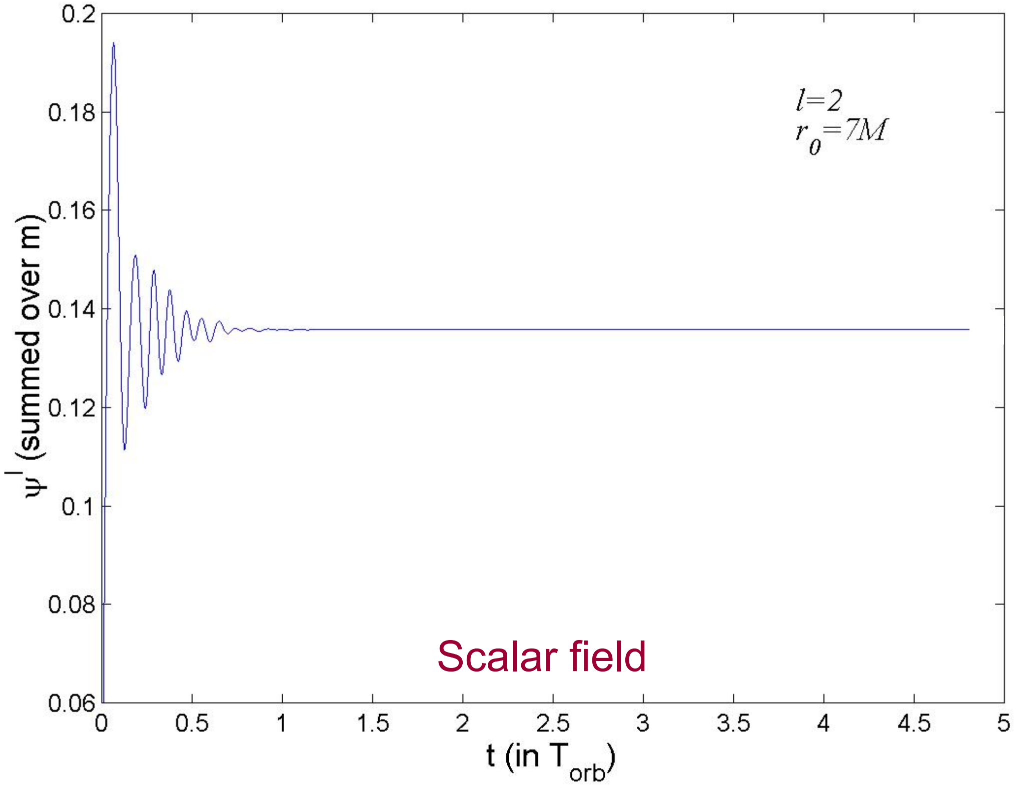
Lorenz-gauge implementation:
vi. Time-domain numerical evolution

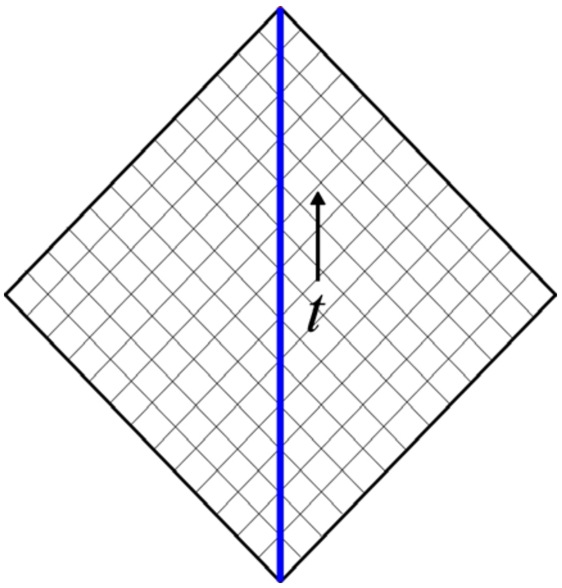
- Time evolution code in 1+1D, using a 4th-order finite-differences scheme based on double-null coordinates
  [LB & Lousto (2005) , LB & Sago (2007)]

  - Set fields to zero initially; evolve for 2-3 orbits until initial spurious waves die off
  - Read field and gradients along particle's trajectory
  - Use as input for mode-sum formula



Lorenz-gauge implementation:
vii. sample results for circular geodesics





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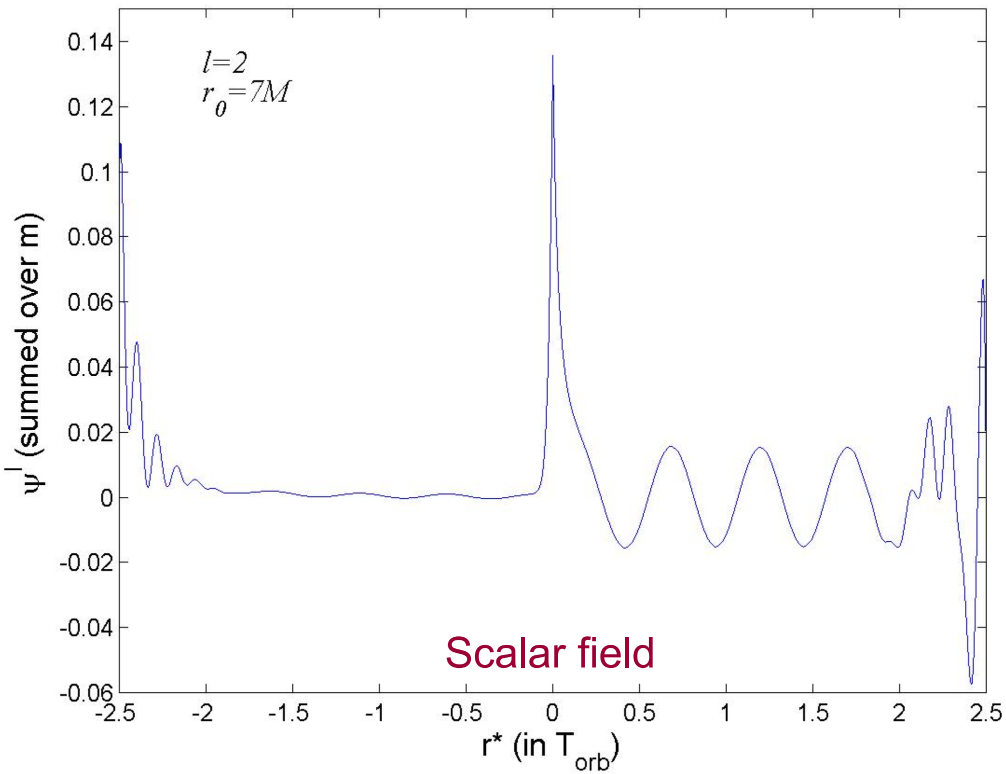

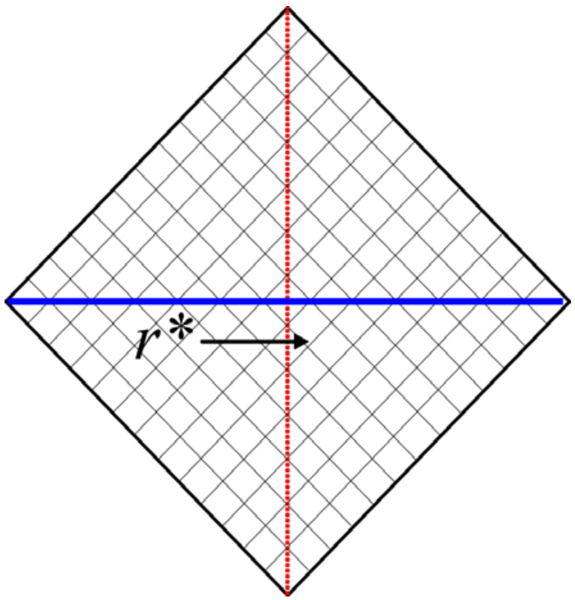

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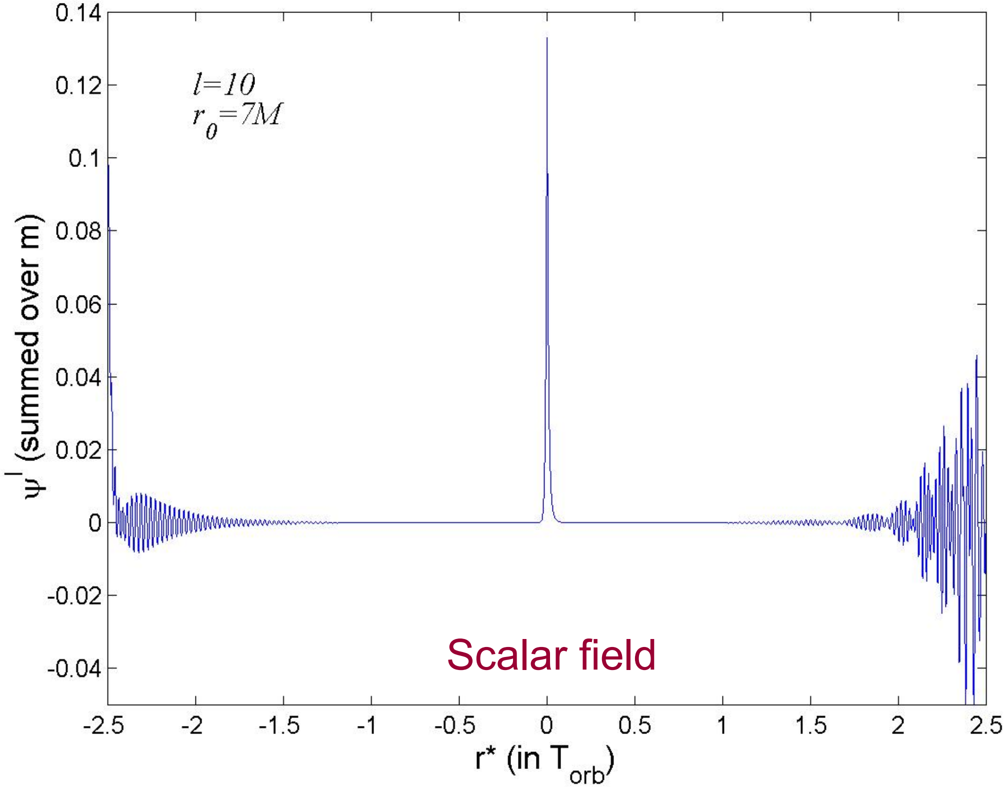
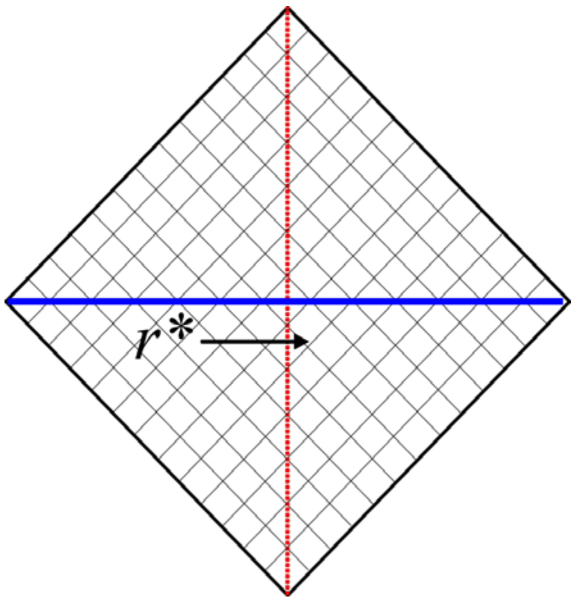
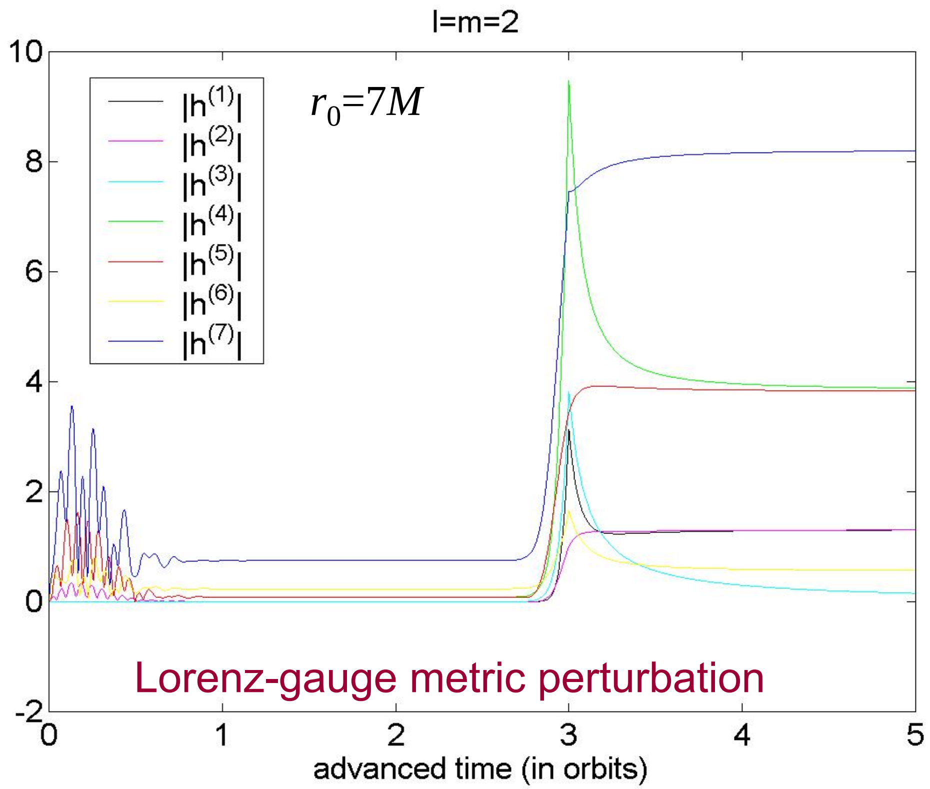
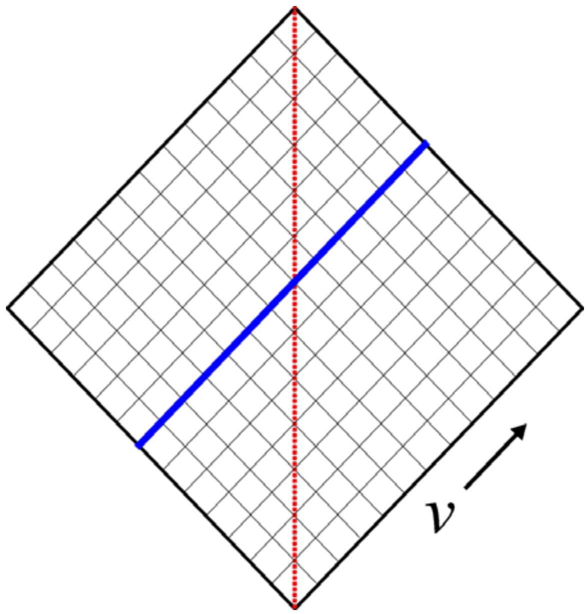


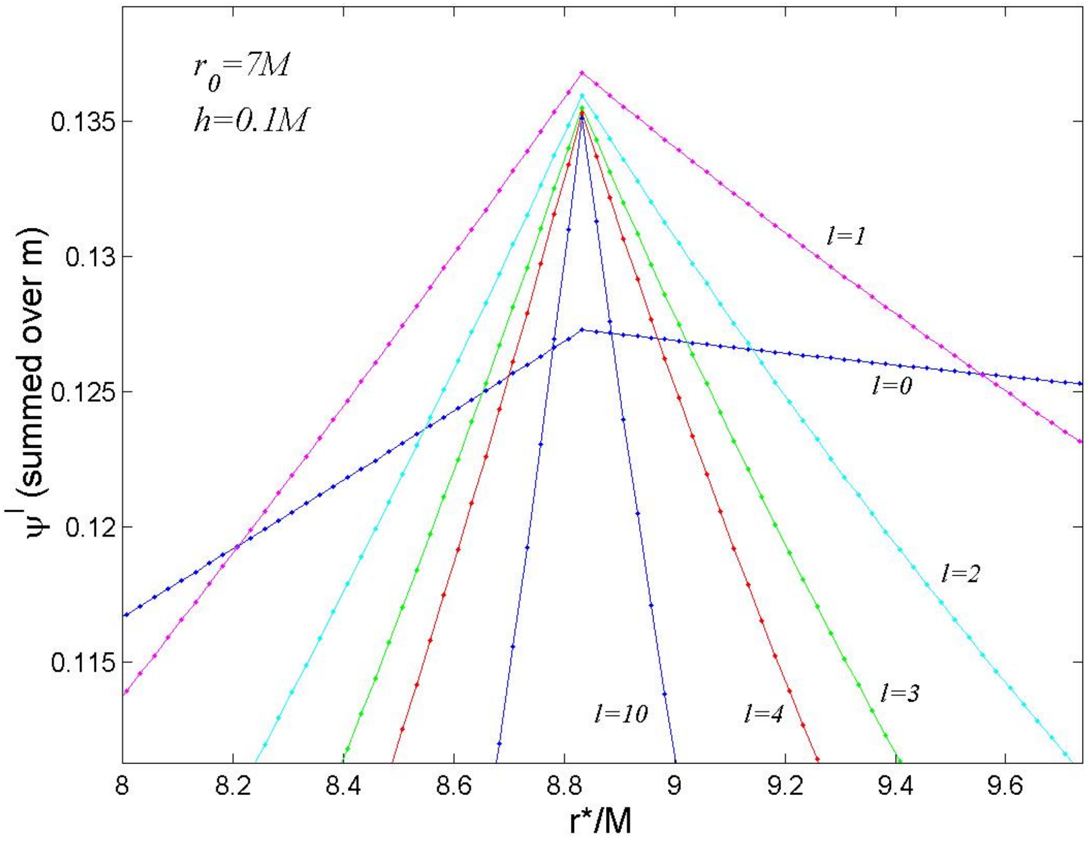


Lorenz-gauge implementation:
vii. sample results for circular geodesics





Lorenz-gauge implementation:
vii. sample results for circular geodesics



Scalar field

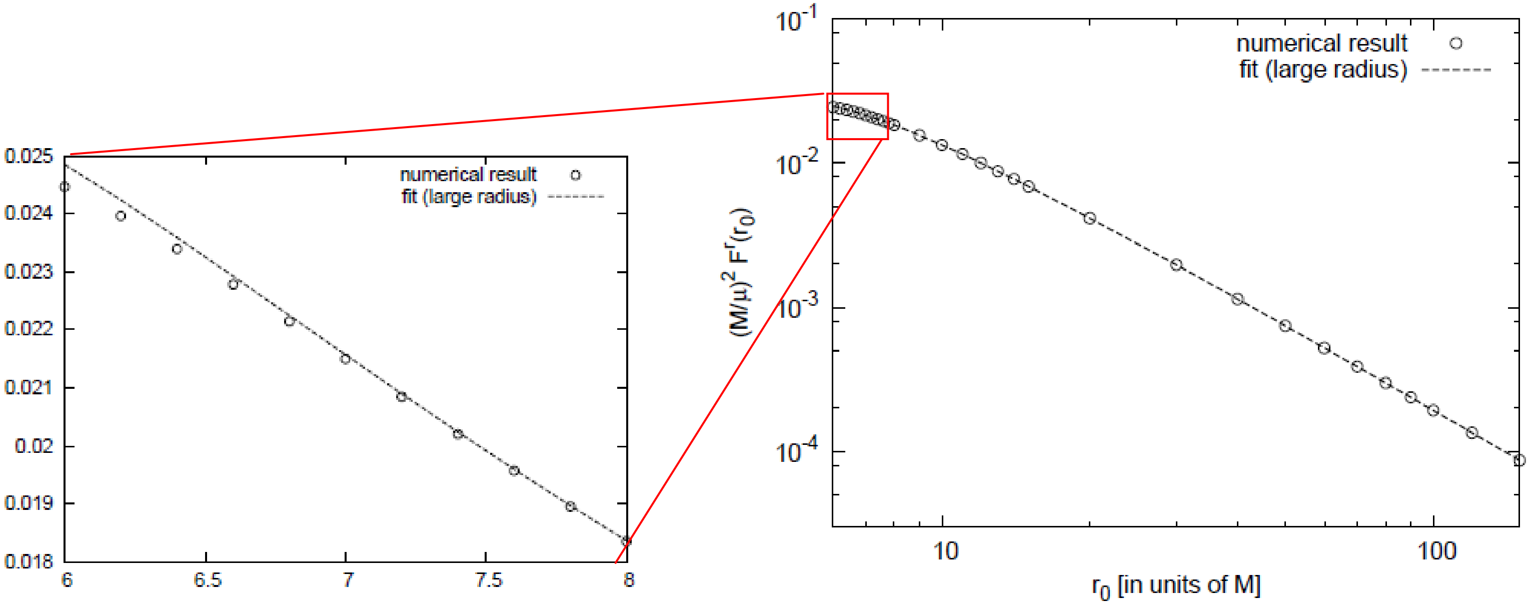
Lorenz-gauge implementation:
viii. dissipative piece of the self force

Work done by self force on particle = energy radiated to infinity and down the hole:

$$F_t / u_0^t = \dot{E}_\infty + \dot{E}_{\mathrm{EH}} \equiv \dot{E}_{\mathrm{total}}$$

r_0/M	$(M/\mu)^2 F_t/u_0^t$	$(M/\mu)^2 \dot{E}_{\mathrm{total}}$	rel. diff.
6.0	9.40338×10^{-4}	9.40190×10^{-4}	1.6×10^{-4}
10.0	6.15158×10^{-5}	6.15047×10^{-5}	1.8×10^{-4}
20.0	1.87151×10^{-6}	1.87111×10^{-6}	2.2×10^{-4}
50.0	1.96249×10^{-8}	1.96203×10^{-8}	2.3×10^{-4}
100.0	6.23806×10^{-10}	6.23628×10^{-10}	2.9×10^{-4}
150.0	8.27475×10^{-11}	8.27279×10^{-11}	2.4×10^{-4}

Lorenz-gauge implementation:
ix. conservative piece of the self force



Best-fit formula at large radius:

$$F^r \cong \frac{\mu^2}{r_0^2}\left[a_0 + a_1\frac{M}{r_0} + a_2\left(\frac{M}{r_0}\right)^2 + a_3\left(\frac{M}{r_0}\right)^3\right] \quad \text{with} \quad \begin{cases} a_0 = 1.999991 \\ a_1 = -6.9969 \\ a_2 = 6.29 \\ a_3 = -24.6 \end{cases}$$

Lorenz-gauge implementation:
ix. conservative piece of the self force

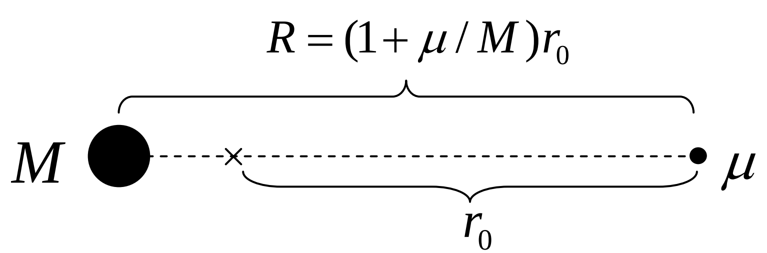
$$\Omega^2 = \Omega_0^2\left[1 - \frac{r_0^2(r_0 - 3M)}{M\mu(r_0 - 2M)} F^r\right] \qquad \left(\Omega_0^2 = M/r_0^3\right)$$

$$= \Omega_0^2\left\{1 + \frac{\mu}{M}\left[-a_0 + (3a_0 - a_1)\frac{M}{r_0} + \cdots\right]\right\}$$

Newtonian SF ($a_0 \approx 2$)

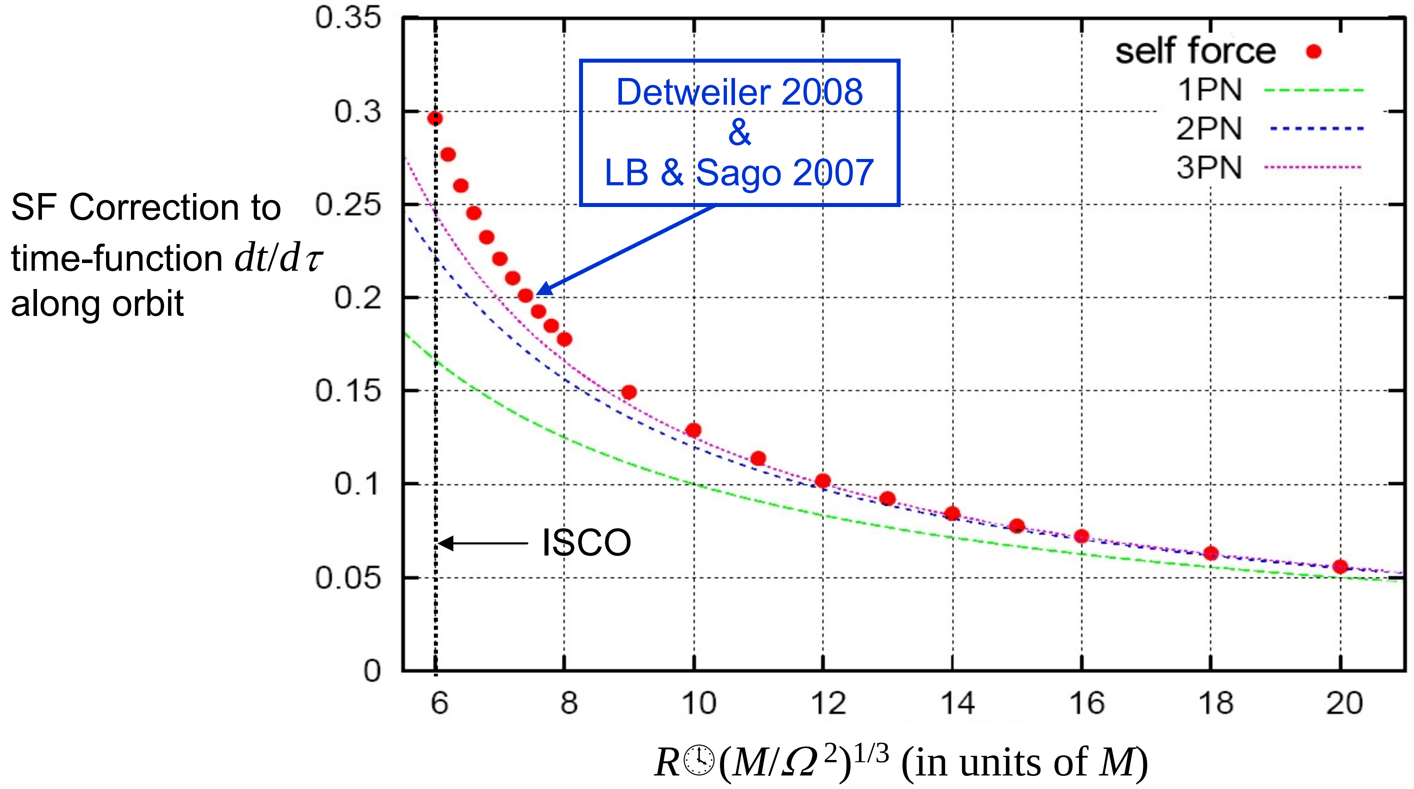
Post-Newtonian SF ($3a_0$-$a_1 \approx 13$)

(Newtonian)



$$\Omega^2_{\text{Newton}} = \frac{M + \mu}{R^3} \cong \frac{M}{r_0^3}\left(1 - 2\frac{\mu}{M}\right)$$

Lorenz-gauge implementation:
x. gauge-invariant conservative effects



Case of Kerr

- Perturbation separable either in the frequency domain or in 2+1D, but not in 1+1D.

$$\Psi = \sum_m e^{im\varphi}\psi_m(t,r,\theta) \qquad\qquad \Psi = \sum_{l,m}\int d\omega\, S_{lm\omega}(\theta)e^{im\varphi}e^{-i\omega t}\psi_{lm\omega}(r)$$

- Mode sum implemented most naturally using 1+1D decomposition
- Desired: formulation of the subtraction in 2+1D — **"m mode scheme"**
- Existing platform (numerical codes) for evolution in 2+1D
- **Difficulty:** each m-mode diverges (logarithmically) at worldline

m-mode diverges at the particle
(demonstrated for a scalar field)

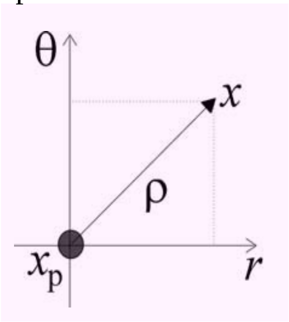
- Decompose $\Phi = \sum_{m=-\infty}^{\infty} e^{im\varphi}\Phi^m(t,r,\theta)$ $\quad\Phi^m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi e^{-im\varphi}d\varphi$
- Near particle $\Phi(x) \simeq \frac{q}{\epsilon}$

With $\epsilon^2 \simeq P_{\alpha\beta}\delta x^\alpha \delta x^\beta$ $\quad P_{\alpha\beta} = g_{\alpha\beta}(x_\mathrm{p}) + u_\alpha(x_\mathrm{p})u_\beta(x_p)$

- Pick a worldline point x_p. Look at field near x_p on surface $t=t_\mathrm{p}$:

$$\Phi^m(\rho) \simeq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-im\varphi}}{\left(\rho^2 + P_{\varphi\varphi}\varphi^2\right)^{1/2}}d\varphi$$

$$\rho^2 = P_{rr}\delta r^2 + P_{\theta\theta}\delta\theta^2$$



m-mode diverges at the particle
(demonstrated for a scalar field) - cont'd

- Split integral as

$$\int_{-\pi}^{\pi} \frac{e^{-im\varphi}}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi = \int_{-\pi}^{\pi} \frac{e^{-im\varphi} - 1}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi + \int_{-\pi}^{\pi} \frac{1}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi$$

- Bound 1st integral by a constant:

$$\left| \int_{-\pi}^{\pi} \frac{e^{-im\varphi} - 1}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi \right| \le \int_{-\pi}^{\pi} \frac{m|\varphi|}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi \le \int_{-\pi}^{\pi} \frac{m}{P_{\varphi\varphi}^{1/2}}\, d\varphi = \frac{2\pi m}{P_{\varphi\varphi}^{1/2}}$$

- Evaluate 2nd (m-independent) integral explicitly:

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{d\varphi}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}} &= \frac{\pi}{\rho_0} \ln \left[\frac{(\rho_0^2 + \rho^2)^{1/2} + \rho_0}{(\rho_0^2 + \rho^2)^{1/2} - \rho_0} \right] \\ &= -\frac{2\pi}{\rho_0} \ln \left(\frac{\rho}{2\rho_0} \right) + O(\rho^2), \end{aligned}$$

$\Rightarrow$

$$\Phi^m(\rho \to 0) = -\rho_0^{-1} \ln \left(\frac{\rho}{2\rho_0} \right)$$

$$\rho_0 \equiv \pi P_{\varphi\varphi}^{1/2}$$

Puncture scheme
(demonstrated for a scalar field)

- Main idea: The asymptotic divergence has a simple form; subtract it out and evolve the residual regular field

Write
$$\Phi = \Phi_{\rm rem} + \Phi_{\rm punc},$$
where the "**puncture function**" $\Phi_{\rm punc}$, given analytically, is such that

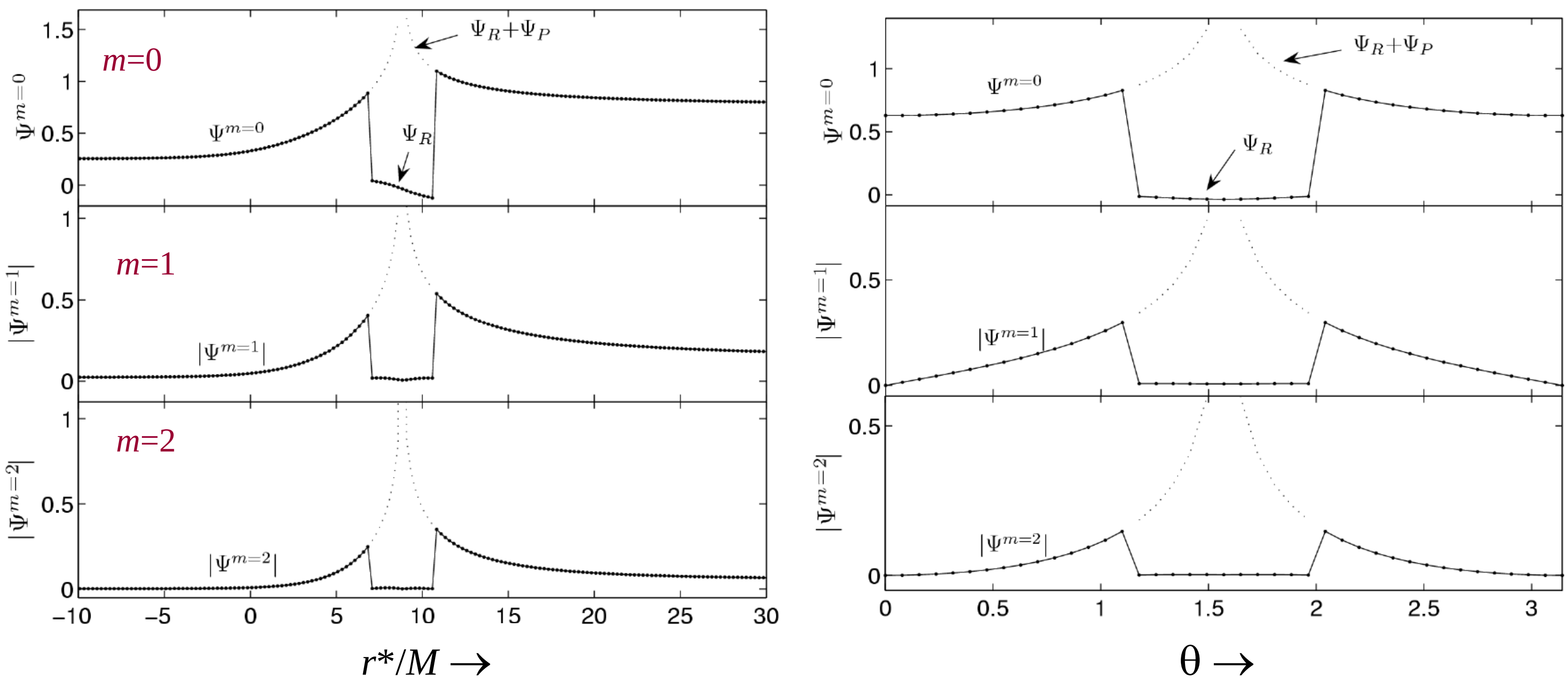
- $(\Phi_{\rm rem})^m$ are at least C^1 at the particle,
- $\Phi_{\rm punc}$ and $\Box\Phi_{\rm punc}$ are easily decomposable into m modes.

$$\Box(\Phi_{\rm rem} + \Phi_{\rm punc}) = S$$

$$\Box\Phi_{\rm rem} = S - \Box\Phi_{\rm punc} \equiv S_{\rm rem}$$

$$\boxed{\Box\Phi_{\rm rem} = S^m_{\rm rem}}$$

sample results from puncture evolution in 2+1 (scalar field)



m-mode scheme for the self force

- A method for obtaining the SF directly from the numerical solutions $(\Phi_{\mathrm{R}})^m$
- It can be shown that

$$F^{\alpha}_{self} = q\sum_m \nabla^{\alpha}\Phi^m_R$$

  with no further "regularization" required!

- Sum over modes converges like ~$(\sin m)/m$
- Similar result in gravitational case
- Has not been implemented so far

Main ideas summarized

- Long-term phase evolution of GW from EMRIs offers a sensitive microscope onto spacetime structure near BH, but for that same reason, full exploitation of waveforms requires accurate theoretical templates
- "Mode sum" is not a new regularization method, but a practical implementation of MiSaTaQuWa
- Technology for SF calculations has now matured enough that we can extract meaningful physics and compare with other methods (PN)
- Move toward time-domain methods
- Move from l-mode to m-mode, puncture schemes.

Basis of tensor harmonics (in t, r, θ, φ coordinates): I. Even parity

$$Y_{lm}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm}, \qquad Y_{lm}^{(2)} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm}, \qquad Y_{lm}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y_{lm}^{(4)} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & \partial_\theta & \partial_\varphi \\ 0 & 0 & 0 & 0 \\ \partial_\theta & 0 & 0 & 0 \\ \partial_\varphi & 0 & 0 & 0 \end{pmatrix} Y_{lm}, \qquad Y_{lm}^{(5)} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta & \partial_\varphi \\ 0 & \partial_\theta & 0 & 0 \\ 0 & \partial_\varphi & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y_{lm}^{(6)} = \frac{r^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s^2 \end{pmatrix} Y_{lm}, \qquad Y_{lm}^{(7)} = \frac{r^2}{2\sqrt{\lambda l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{L}_2 & \mathbb{L}_1 \\ 0 & 0 & \mathbb{L}_1 & -s^2\mathbb{L}_2 \end{pmatrix} Y_{lm},$$

$$\mathbb{L}_1 \equiv 2(\partial_{\theta\varphi} - \cot\theta\,\partial_\varphi), \quad , \mathbb{L}_2 \equiv \partial_{\theta\theta} - \cot\theta\,\partial_\theta - s^{-2}\partial_{\varphi\varphi}, \quad s \equiv \sin\theta, \quad \lambda(l-1)(l+2)/2$$

Basis of tensor harmonics (in t, r, θ, φ coordinates): II. Odd parity

$$Y_{lm}^{(8)} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & s^{-1}\partial_\varphi & -s\,\partial_\theta \\ 0 & 0 & 0 & \\ s^{-1}\partial_\varphi & 0 & 0 & 0 \\ -s\,\partial_\theta & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y_{lm}^{(9)} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s^{-1}\partial_\varphi & -s\,\partial_\theta \\ 0 & s^{-1}\partial_\varphi & 0 & 0 \\ 0 & -s\,\partial_\theta & 0 & 0 \end{pmatrix} Y^{lm},$$

$$Y_{lm}^{(10)} = \frac{-ir^2}{2\sqrt{\lambda l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -s^{-1}\mathbb{L}_1 & s\,\mathbb{L}_2 \\ 0 & 0 & s\,\mathbb{L}_2 & s\,\mathbb{L}_1 \end{pmatrix} Y_{lm},$$

10 Basis tensors are orthonormal:

$$\int d\Omega\, \eta^{\alpha\mu}\eta^{\beta\nu}[Y_{\mu\nu}^{(i)lm}]^* Y_{\alpha\beta}^{(j)l'm'} = \delta_{ij}\delta_{ll'}\delta_{mm'} \qquad [\text{where } \eta^{\alpha\beta} = \mathrm{diag}(-1, 1, r^{-2}, r^{-2}s^{-2})]$$