POST-NEWTONIAN EQUATION OF MOTION AND WAVEFORM OF COMPACT BINARIES

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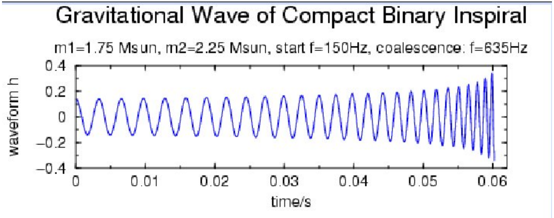
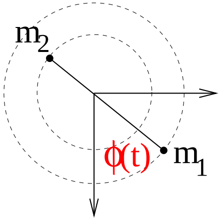
The inspiral (or chirp) of black hole binaries

- The most interesting known source of gravitational waves for the LIGO/VIRGO detectors and a very important one for LISA
- The dynamics of these systems is driven by gravitational radiation reaction effects or equivalently by the loss of energy by gravitational radiation
- Theoretical waveforms (templates) for detection and analysis of the signals should be very accurate in terms of a post-Newtonian expansion



Inspiralling black hole binaries: the PN theorist's paradise



The orbital phase $\phi(t)$ should be monitored in LIGO/VIRGO detectors with precision

$$\delta\phi \sim \pi$$

$$\phi(t) = \underbrace{\phi_0 - \frac{1}{\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\substack{\text{result of the quadrupole formalism} \\ \text{(sufficient for the binary pulsar)}}} \Big\{ 1 + \underbrace{\frac{1\mathrm{PN}}{c^2} + \frac{1.5\mathrm{PN}}{c^3} + \cdots + \frac{3\mathrm{PN}}{c^6} + \cdots}_{\text{needs to be computed with high PN precision}} \Big\}$$

Detailed data analysis (using the sensitivity noise curve of LIGO/VIRGO detectors) show that the required precision is at least 2PN for detection and 3PN for parameter estimation

Two problems in General Relativity

1. To obtain the equations of motion of the compact binary system (*i.e.* $m_1\, a_1^i = F_1^i$ and $m_2\, a_2^i = F_2^i$) at 3PN order beyond the Newtonian acceleration. From the conservative part of the equations of motion (deducible from a Lagrangian) we deduce

$$E = \text{the binary's center-of-mass energy}$$

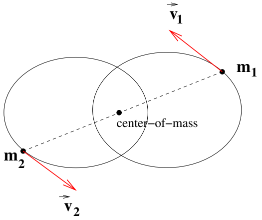
2. To compute the gravitational radiation field (*i.e.* $h^{\alpha\beta}$) of the compact binary by means of a gravitational wave generation formalism at 3PN order beyond the Einstein quadrupole formula. This yields

$$\mathcal{F} = \text{the binary's gravitational wave flux}$$

The orbital phase ϕ (a crucial observable for LIGO/VIRGO and LISA) follows from

$$\frac{dE}{dt} = -\mathcal{F} \quad \Longrightarrow \quad \phi \equiv \int \omega\, dt = -\int \frac{\omega\, dE}{\mathcal{F}}$$

Computing the 3PN equation of motion



The equations of motion are written in Newtonian-like form (with $t = x^0/c$ playing the role of Newton's "absolute time")

$$\frac{d\mathbf{v}_1}{dt} = \mathbf{A}_1^{\rm N} + \frac{1}{c^2}\mathbf{A}_1^{\rm 1PN} + \frac{1}{c^4}\mathbf{A}_1^{\rm 2PN} + \underbrace{\frac{1}{c^5}\mathbf{A}_1^{\rm 2.5PN}}_{\text{radiation reaction}} + \overbrace{\frac{1}{c^6}\mathbf{A}_1^{\rm 3PN}}^{\text{difficult term to compute}} + \underbrace{\frac{1}{c^7}\mathbf{A}_1^{\rm 3.5PN}}_{\text{1PN rad. react.}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

The EOM are solutions of general relativity so they

1. Reduce in the test-mass limit $m_2 \to 0$ to the geodesic equations of Schwarzschild metric
2. Are derivable from a Lagrangian/Hamiltonian formalism (when the gravitational radiation reaction is neglected)
3. Are invariant under a global Lorentz-Poincaré transformation

Computing the 3PN radiation field

1. The multipole moment needed with the highest post-Newtonian precision is the mass quadrupole

$$I_{ij} = I_{ij}^{\mathrm{N}} + \frac{1}{c^2} I_{ij}^{1\mathrm{PN}} + \frac{1}{c^4} I_{ij}^{2\mathrm{PN}} + \frac{1}{c^5} I_{ij}^{2.5\mathrm{PN}} + \overbrace{\frac{1}{c^6} I_{ij}^{3\mathrm{PN}}}^{\text{difficult term to compute}} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

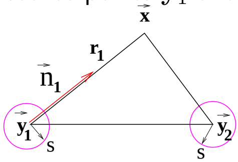
   Higher-order multipole moments need less post-Newtonian precision is the mass quadrupole

$$\begin{aligned} I_{ijk} &= I_{ijk}^{\mathrm{N}} + \frac{1}{c^2} I_{ijk}^{1\mathrm{PN}} + \frac{1}{c^4} I_{ijk}^{2\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^5}\right) \\ J_{ij} &= J_{ij}^{\mathrm{N}} + \frac{1}{c^2} J_{ij}^{1\mathrm{PN}} + \frac{1}{c^4} J_{ij}^{2\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^5}\right) \\ &\cdots \end{aligned}$$

2. All tail and tail-of-tail effects up to 3PN are computed for compact binaries and inserted into the expressions of the radiative moments U_L and V_L which parametrize the asymptotic waveform

Problem of self-field regularization: the PN headache

Hadamard's regularization is used as a basic tool. Let $F(\mathbf{x})$ be a singular function at source points $\mathbf{y}_1$ and $\mathbf{y}_2$ such that when $r_1 \to 0$



$$F(\mathbf{x}) = \sum_{a_0 \leq a \leq N} r_1^a \underset{1}{f}_a(\mathbf{n}_1) + o(r_1^N)$$

Hadamard partie finie of the singular function F at point 1

$$(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} \underset{1}{f}_0(\mathbf{n}_1)$$

Hadamard partie finie of the divergent integral $\int d^3\mathbf{x}\, F$

$$\mathrm{Pf} \int d^3\mathbf{x}\, F(\mathbf{x}) \equiv \lim_{s\to 0} \left\{ \int_{\mathbb{R}^3 \setminus B_1(s) \cup B_2(s)} d^3\mathbf{x} F(\mathbf{x}) - \mathrm{div}(s) \right\}$$

Self-field regularization ambiguities at 3PN order

Hadamard's regularization is non-distributive, in the sense that the regularization of a product differs from the product of regularizations

$$(F\,G)_1 \neq (F)_1\,(G)_1$$

As a result the computation cannot be complete. A few regularization ambiguities appear at the 3PN order, but Hadamard's regularization works extremely well up to 2PN order and can be implemented to compute most of the terms at 3PN order

All terms but for a few regularization ambiguities have been computed with Hadamard's regularization

- λ in the equations of motion [Jaranowski & Schäfer 1999; Blanchet & Faye 2000, 2001]
- ξ, κ and ζ in the radiation field [Blanchet, Iyer & Joguet 2002; Blanchet & Iyer 2004]

The regularization ambiguities do not have a direct physical meaning. They are due to a mathematical deficiency of Hadamard's regularization

Computation of the regularization ambiguity parameters

- Following dimensional regularization one solves Einstein's field equations in

$$d = 3 + \varepsilon \quad \text{spatial dimensions}$$

with source terms involving Dirac's d-dimensional function $\delta^{(d)}(\mathbf{x} - \mathbf{y}_A)$

- One does not compute the result in d dimensions from scratch, but only the *difference* between dimensional regularization and Hadamard's one, say

$$\mathcal{D}F(\mathbf{y}_1) \equiv \underbrace{F^{(d)}(\mathbf{y}_1)}_{\text{value in } d \text{ dimensions}} - \underbrace{(F)_1}_{\text{Hadamard's partie finie}}$$

- That difference will typically be of the form (when $\varepsilon \to 0$)

$$\mathcal{D}F(\mathbf{y}_1) = \frac{a_{-1}}{\varepsilon} + a_0 + \mathcal{O}(\varepsilon)$$

- The ambiguities in Hadamard's regularization are precisely associated with the occurence of poles $\propto 1/\varepsilon$ at 3PN order in dimensional regularization

Dimensional regularization of the equations of motion

- The 3PN acceleration in Hadamard's regularization is

$$\mathbf{a}_1[\lambda] = \mathbf{a}_1^{\text{pure Hadamard Schwartz}} + \underbrace{\Delta\mathbf{a}_1[\lambda]}_{\text{ambiguity part}}$$

- while in dimensional regularization ($d = 3 + \varepsilon$) it reads

$$\mathbf{a}_1^{(d)} = \mathbf{a}_1^{\text{pure Hadamard Schwartz}} + \mathcal{D}\mathbf{a}_1$$

- We require physical equivalence between dimensional and Hadamard regularizations. We look for a shift $\mathbf{y}_A \rightarrow \mathbf{y}_A + \boldsymbol{\eta}_A$ such that

$$\mathbf{a}_1[\lambda] = \lim_{\varepsilon\rightarrow 0}\left[\mathbf{a}_1^{(d)} + \delta_{\boldsymbol{\eta}}\,\mathbf{a}_1\right]$$

- This requirement determines the particle's worldlines shift $\boldsymbol{\eta}_A = \mathcal{O}(1/\varepsilon)$ and fixes uniquely [Blanchet, Damour & Esposito-Farèse 2003]

$$\boxed{\lambda = -\frac{1987}{3080}}$$

(result equivalent to [Damour, Jaranowski & Schäfer 2001] and [Itoh & Futamase 2003])

Dimensional regularization of the radiation field

- The 3PN quadrupole moment in Hadamard's regularization reads

$$I_{ij}[\xi,\,\kappa,\,\zeta] = I_{ij}^{\text{pure Hadamard Schwartz}} + \underbrace{\Delta I_{ij}[\xi,\,\kappa,\,\zeta]}_{\text{ambiguity part}}$$

- while in dimensional regularization it reads

$$I_{ij}^{(d)} = I_{ij}^{\text{pure Hadamard Schwartz}} + \mathcal{D}I_{ij}$$

- Physical equivalence between the dimensional and Hadamard results means

$$I_{ij}[\xi,\,\kappa,\,\zeta] = \lim_{\varepsilon\to 0}\left[I_{ij}^{(d)} + \delta_{\boldsymbol{\eta}} I_{ij}\right]$$

  with *the same shift* $\boldsymbol{\eta}_A$ as for the equations of motion
- This uniquely determines [Blanchet, Damour, Esposito-Farèse & Iyer 2004, 2005]

$$\boxed{\xi = -\frac{9871}{9240},\quad \kappa = 0,\quad \zeta = -\frac{7}{33}}$$

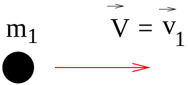
Independent computations of the ambiguity parameters

1. $\lambda = -\frac{1987}{3080}$ has been obtained by three different approaches
   - ADM Hamiltonian and dimensional regularization
   - Harmonic-coordinates equations of motion and dimensional regularization
   - Surface-integral approach à la Einstein-Infeld-Hoffmann [Itoh & Futamase 2003]
2. $\xi + \kappa = -\frac{9871}{9240}$ follows from imposing that

$$\underbrace{I_i}_{\text{binary's dipole moment}} \equiv \underbrace{C_i}_{\text{center-of-mass position}}$$

3. $\kappa = 0$ can be deduced from an argument based on space-time diagrams
4. $\zeta = -\frac{7}{33}$ is the unique value for which the formalism is globally Poincaré invariant

Poincaré invariance of the 3PN radiation field

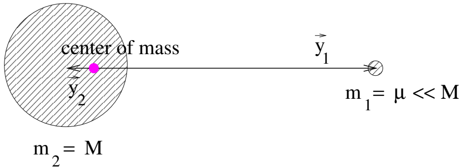


- The ambiguity ζ can be obtained by comparing the result of two-body calculations in Hadamard's regularization in the limit $m_2 \to 0$ with the case of a boosted Schwarzschild black hole (BBH)
- Computing the 3PN quadrupole of the BBH we get

$$\zeta = -\frac{7}{33}$$

  in perfect agreement with the result of dimensional regularization
- This constitutes a direct check of the global Poincaré invariance of the 3PN wave generation formalism

Computing the self-force from the PN equations of motion



1. Define the center-of-mass frame by imposing the nullity of the center-of-mass integral G^i (which varies linearly with time as a consequence of the equations of motion) at 3PN order,

$$G^i \equiv m_1 y_1^i + m_2 y_2^i + \frac{1}{c^2} G^i_{1\text{PN}} + \frac{1}{c^4} G^i_{2\text{PN}} + \frac{1}{c^6} G^i_{3\text{PN}} + \mathcal{O}\left(\frac{1}{c^8}\right) = 0$$

2. Use the center-of-mass variables y_1^i and y_2^i which are given by the previous relation as functions of the relative position and velocity y^i and v^i up to 3PN
3. Expand at first order when $\mu \ll M$ to get

$$\underbrace{\frac{dv^i}{dt} + \left(\Gamma^i_{\alpha\beta} - \frac{v^i}{c}\Gamma^0_{\alpha\beta}\right) v^\alpha v^\beta}_{\text{geodesic of Schwarzschild}} = \mu\, F^i_{\text{self}} + \mathcal{O}\left(\mu^2\right)$$

3PN self-force for general orbits in harmonic coordinates

$$
\begin{aligned}
F^i_{\mathsf{self}} &= \frac{G}{r^2}\Bigg\{-n^i + \frac{1}{c^2}\left[\left(-4v^2+\frac{3}{2}(nv)^2+10\frac{GM}{r}\right)n^i + 2(nv)v^i\right] \\
&+ \frac{1}{c^4}\Bigg[\left(-3v^4-\frac{15}{8}(nv)^4+\frac{13}{2}\frac{GM}{r}v^2+29\frac{GM}{r}(nv)^2-\frac{195}{4}\frac{G^2M^2}{r^2}\right)n^i \\
&\qquad + \left(\frac{15}{2}v^2-\frac{9}{2}(nv)^2-\frac{49}{2}\frac{GM}{r}\right)(nv)v^i\Bigg] \\
&+ \frac{1}{c^5}\left[\left(\frac{24}{5}v^2+\frac{136}{15}\frac{GM}{r}\right)\frac{GM}{r}(nv)n^i + \frac{GM}{r}\left(\cdots\right)v^i\right] \\
&+ \frac{1}{c^6}\Bigg[\left(-\frac{11}{4}v^6+\cdots+\frac{GM}{r}\left[-\frac{75}{4}v^4+\cdots\right]+\frac{G^2M^2}{r^2}\left[\cdots\right]+\cdots\right)n^i \\
&\qquad + \left(\cdots+\frac{GM}{r}\left[\cdots\right]+\frac{G^2M^2}{r^2}\left[\cdots\right]\right)(nv)v^i\Bigg] + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}
\end{aligned}
$$

From F^i_{self} (computed for general orbits) one deduces the self-force contributions in the particle's conserved energy, angular momentum, linear momentum, ...

Reducing the 3PN self-force to circular orbits

$$F^i_{\text{self}} = \frac{G}{r^2}\Bigg\{\Bigg(-1+\overbrace{\frac{GM}{c^2 r}}^{\text{1PN}}-\overbrace{\frac{113}{4}\frac{G^2M^2}{c^4r^2}}^{\text{2PN}} + \underbrace{\left[\frac{101359}{840}-\frac{44}{3}{\color{red}\lambda}-\frac{41}{64}\pi^2-\ln\left(\frac{r}{r_0}\right)\right]\frac{G^3M^3}{r^3c^6}}_{\text{The 3PN term contains the ambiguity parameter } {\color{red}\lambda}}\Bigg)n^i - \frac{G^2M^2}{r^2c^5}v^i\Bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

- PN theory predicts $\lambda = -\frac{1987}{3080}$ (resulting from subtle issues associated with self-field regularization) which should be recovered exactly by direct self-force calculations
- Higher-order terms (4PN ...) in the limit $\mu \ll M$ may be easier to get by direct self-force calculations rather than by PN theory

Gravitational wave flux from BH linear perturbation theory

The GW flux from a particle orbiting a circular geodesic of Schwarzschild (neglecting the self-force) is known to 5.5PN order beyond the quadrupole formula [Poisson 1993; Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\mathcal{F}_{5.5\text{PN}} = \mathcal{F}_{\text{N}}\Big\{1 - \frac{1247}{336}x + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}} - \frac{44711}{9072}x^2 - \frac{8191}{672}\pi x^{5/2} + \underbrace{\left(\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{856}{105}\ln 16\,x\right)x^3}_{\text{3PN tail-of-tail}} - \frac{16285}{504}\pi x^{7/2} + (\cdots)\,x^4 + (\cdots)\,x^{9/2} + (\cdots)\,x^5 + (\cdots)\,x^{11/2} + \mathcal{O}\left(\frac{\mu}{M}\right)\Big\}$$

PN theory adds all μ/M corrections up to 3.5PN order

- Second-order BH perturbation should yield the μ/M correction due to self-force effects on the particle's motion
- Such self-force contributions are known from PN theory up to 3.5PN order

$$\frac{\mathcal{F}_{3.5\mathrm{PN}}}{\mathcal{F}_{\mathrm{N}}} = \text{geodesic motion contribution}$$

$$+ \frac{\mu}{M}\Bigg(-\frac{35}{12}x + \frac{9271}{504}x^2 - \overbrace{\frac{583}{24}\pi x^{5/2}}^{\text{2.5PN tail term}}$$

$$+ \overbrace{\left[-\frac{11497453}{272160} + \frac{41}{48}\pi^2 + \frac{176}{9}\lambda - \frac{88}{3}\theta\right] x^3}^{\text{The 3PN term contains the ambiguity parameters } \lambda \text{ and } \theta} + \underbrace{\frac{214745}{1728}\pi x^{7/2}}_{\text{3.5PN tail term}}\Bigg)$$

$$+ \mathcal{O}\left(\frac{\mu}{M}\right)^2$$

PN theory predicts $\theta = \xi + 2\kappa + \zeta = -\frac{11831}{9240}$

The 3PN asymptotic waveform

The asymptotic GW is decomposed in spin-weighted spherical harmonics

$$h_+ - i h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}\, Y_{-2}^{\ell m}(\Theta, \Phi)$$

All the modes (ℓ, m) are known with 3PN accuracy [Blanchet, Faye, Iyer & Sinha 2008]

$$\begin{aligned}\hat{h}^{22}_{\text{3PN}} &= 1 + x\,(\cdots) + x^{3/2}\,(\cdots) + x^2\,(\cdots) + x^{5/2}\,(\cdots)\\ &+x^3\left(\frac{27027409}{646800} - \frac{856C}{105} + \frac{428 i\pi}{105} + \frac{2\pi^2}{3}\right.\\ &\left.+\underbrace{\left[-\frac{278185}{33264} + \frac{41\pi^2}{96}\right]}_{\text{self-field regularization ambiguity parameters are there}}\nu - \frac{20261\nu^2}{2772} + \frac{114635\nu^3}{99792} - \frac{428}{105}\ln(16x)\right)\end{aligned}$$

The difficult self-field regularization terms at 3PN order could be checked by investigating effects of the self-force

Strategies to compare self-force and PN calculations

A difficulty in relating self-force calculations to post-Newtonian ones is the different gauges employed in the two approaches. Two solutions are possible

- Compare only gauge-invariant quantities such as the energy $E(\omega)$ or energy flux $\mathcal{F}(\omega)$ as a function of the orbital frequency
- Do not attempt to derive gauge-invariant results but

1. Expand directly the gauge-dependent self-force for general orbits (not only circular) in a post-Newtonian way
2. Compare the result with the PN calculation which is done in harmonic coordinates of ADM coordinates
3. Look for a coordinate transformation between the two results in the form of a PN expansion and determine iteratively the PN coefficients