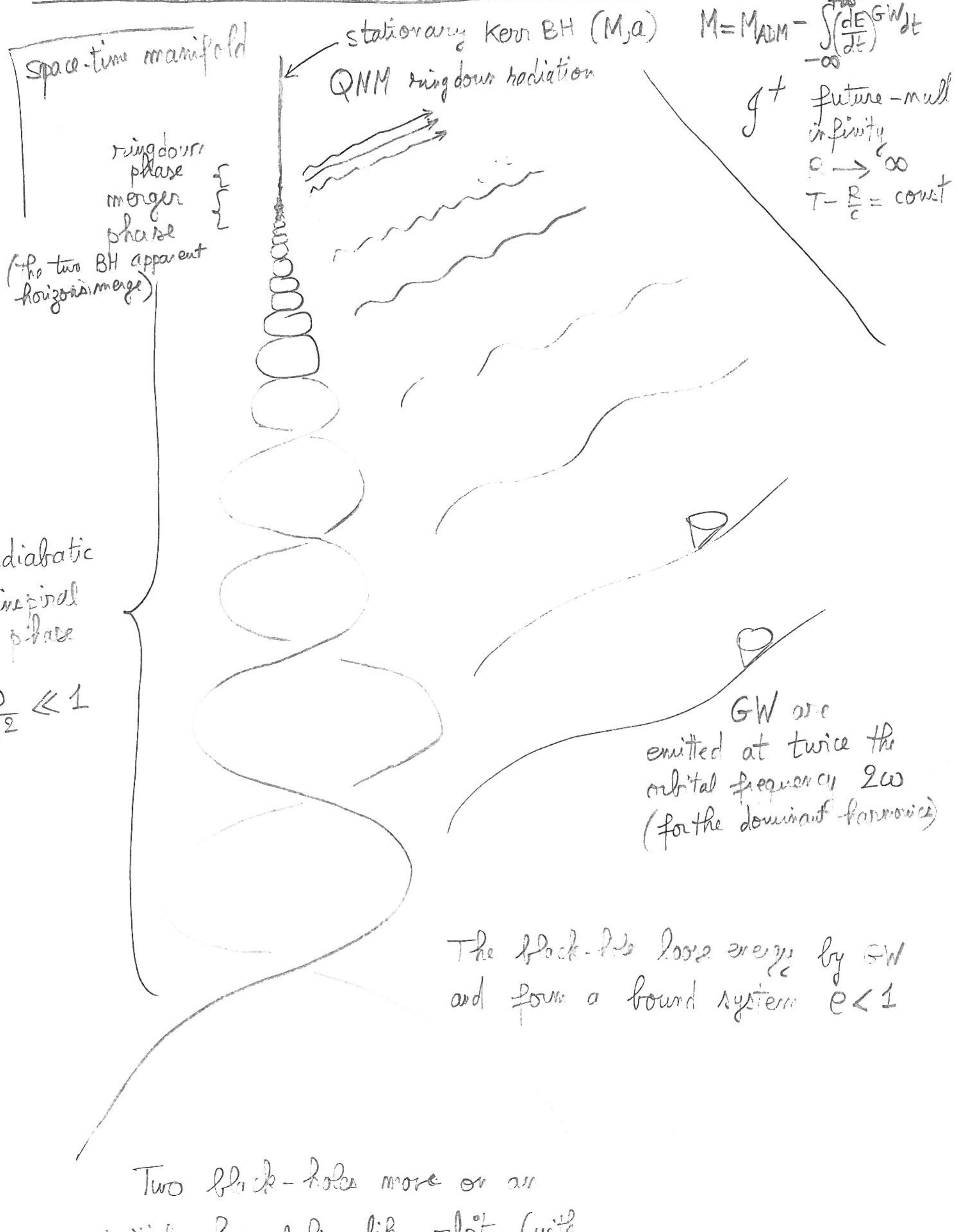
# THE TWO-BODY PROBLEM IN GENERAL RELATIVITY

Luc BLANCHET
Institut d'Astrophysique de Paris

# GENERAL ASPECTS OF THE TWO-BODY PROBLEM



$M = M_{ADM} - \int_{-\infty}^{+\infty} \left(\frac{dE}{dt}\right)^{GW} dt$

$R \longrightarrow \infty$

$T - \frac{R}{c} = \text{const}$

$\frac{\dot{\omega}}{\omega^2} \ll 1$

GW are emitted at twice the orbital frequency $2\omega$ (for the dominant harmonics)

The black-hole loose energy by GW and form a bound system $e < 1$

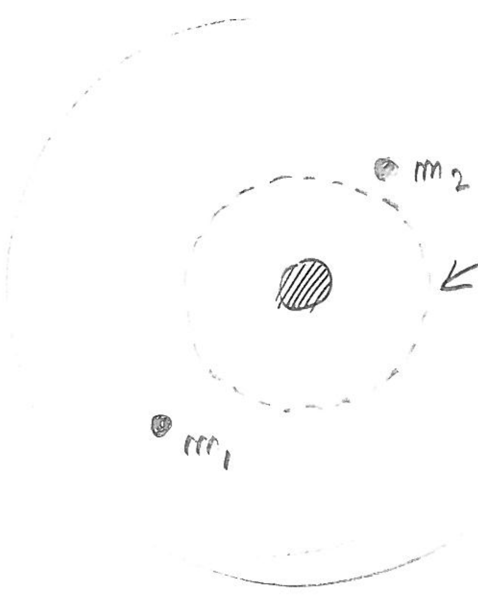
Two black-holes move on an initial hyperbolic-like orbit (with a very small eccentricity $e \gtrsim 1$)

Total ADM mass $M_{ADM} = m_1 + m_2$

The orbit is <u>circularized</u> by gravitational radiation reaction

For Hulse-Taylor binary pulsar

now $\begin{cases} \omega_0 \approx 10^{-4} Hz \\ e_0 \approx 0.617 \end{cases}$ when observable by LIGO/VIRGO $\begin{cases} \omega \approx 10\ Hz \\ e \approx 10^{-6} \end{cases}$



Innermost circular orbit (I.C.O.)

defined by the minimum of gravitational binding energy for circular orbits $\left(\frac{dE}{dt}\right)_{ICO} = 0$

<u>Quadrupole moment formalism</u>: gives the inspiral and GW emission at lowest-order (where the quadrupole is merely Newtonian, and where one can neglect higher-order multipoles)

$\varepsilon = \frac{v}{c}$ <u>small PN parameter</u>

<u>Asymptotic wave-form</u>

$$h_{ij}^{TT} = \frac{2G}{c^4 R}\left[\ddot{Q}_{ij}\left(T - \frac{R}{c}\right) + O(\varepsilon)\right]^{TT} + O\left(\frac{1}{R^2}\right)$$

<u>Total energy flux in GW emission</u>

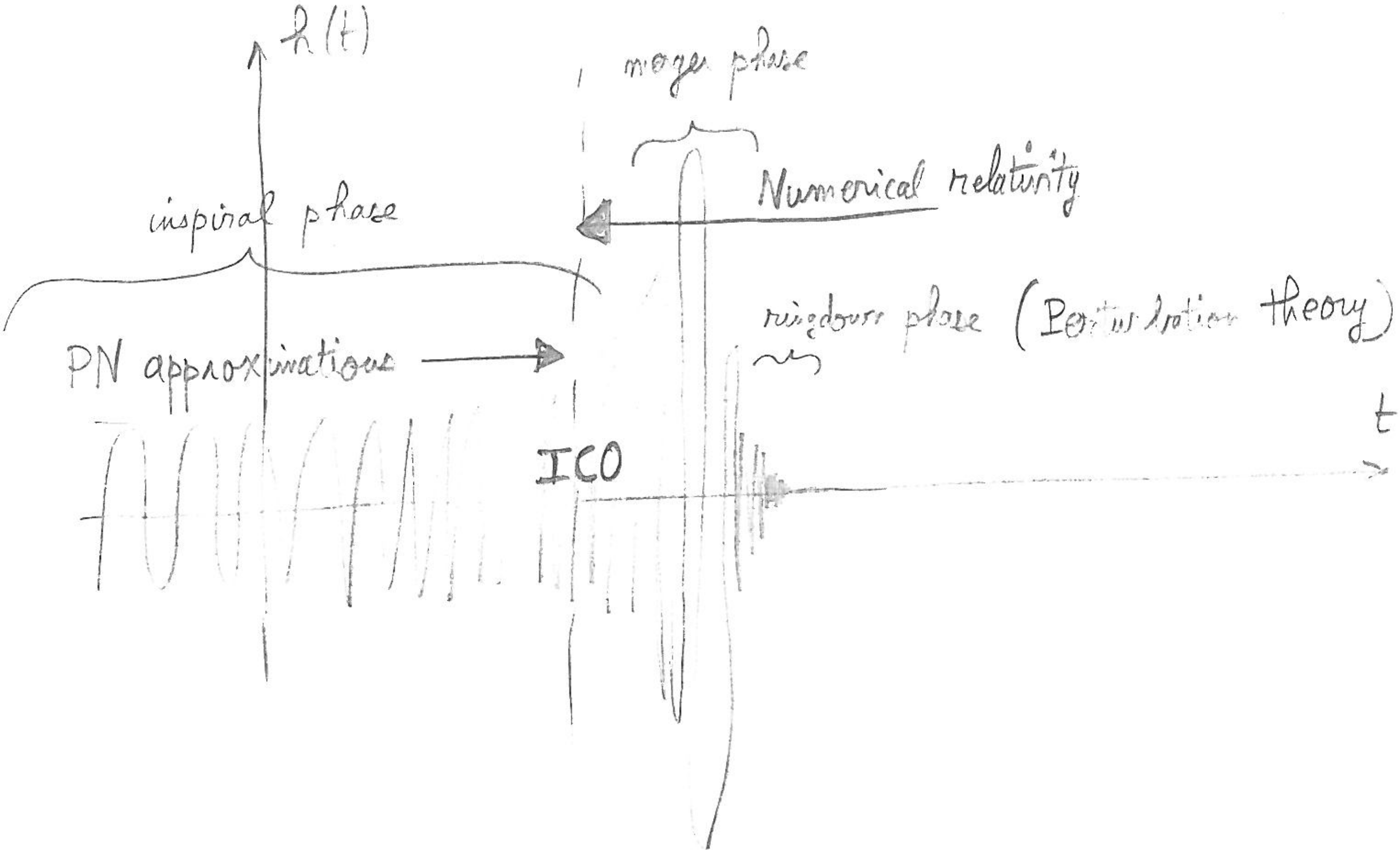
$$\mathcal{F}^{GW} = \left(\frac{dE}{dt}\right)^{GW} = \frac{G}{5c^5}\left(\dddot{Q}_{ij}\dddot{Q}_{ij} + O(\varepsilon^2)\right)$$

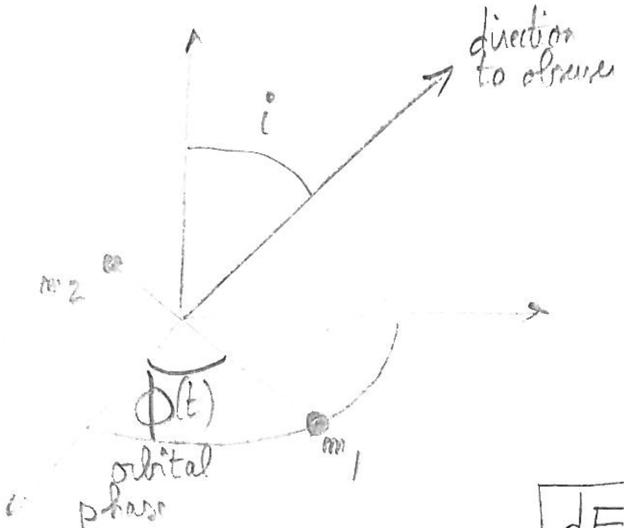
<u>Radiation-reaction force in equations of motion</u>

$$\frac{1}{c^5}F^i_{rad.react} = \underbrace{\frac{2G}{5c^5}\rho\, x^j \overset{\cdot\cdot\cdot\cdot\cdot}{Q}_{ij}}_{\text{a small 2.5PN} \sim \frac{1}{c^5} \text{ correction in the eqs. of motion}} + O(\varepsilon^7)$$

$$\rho\frac{dv^i}{dt} = \rho\partial_i U - \partial_i p + \overbrace{\frac{1}{c^2}(\cdots) + \frac{1}{c^4}(\cdots) + \frac{1}{c^6}(\cdots)}^{\text{conservative part of the dynamics}} + \cdots$$

$$+ \underbrace{\frac{1}{c^5}(\cdots) + \frac{1}{c^7}(\cdots)}_{\text{radiation reaction part of dynamics}}$$





From quadrupole formalism

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \frac{2G\mu}{c^2 R}\left(\frac{GM\omega}{c^3}\right)^{2/3}\begin{pmatrix}(1+\cos^2 i)\cos 2\phi \\ (2\cos i)\sin 2\phi\end{pmatrix}$$

Evolution of orbital frequency is computed by <u>energy balance equation</u>

$$\boxed{\frac{dE}{dt} = -\mathcal{F}^{GW}} \longleftarrow \text{given by Einstein quadrupole formula at lowest order}$$

$$\text{hence} \quad \dot{\omega}\,\frac{dE}{d\omega} = -\mathcal{F}^{GW} \Longrightarrow \phi(t) = \int \omega\, dt = -\int \frac{\omega\, dE}{\mathcal{F}^{GW}}$$

At lowest order (from quadrupole formula) in the adiabatic inspiral phase

$$h(t) \propto (t_c - t)^{-1/4}$$

$$\omega(t) \propto (\underset{\uparrow}{t_c} - t)^{-3/8}$$

coalescence instant $t_c$ at which $\omega \to \infty$
(of course the approximation is no longer valid after the ICO)

# TWO PROBLEMS IN GENERAL RELATIVITY

Data analysis in LIGO/VIRGO and LISA detectors show that to detect and analyse <u>inspiralling compact binaries</u> (ICBs) one must know the wave form $h(t)$ and phase-evolution $\phi(t)$ to order $3PN \sim \varepsilon^6 = \left(\frac{v}{c}\right)^6$ beyond the quadrupole moment formalism.

$$\phi(t) = \phi_c - \underbrace{\frac{1}{\nu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\substack{\text{quadrupole formalism}\\ \text{(sufficient for Hulse-Taylor}\\ \text{binary pulsar)}}}\left\{1 + \frac{1PN}{c^2} + \frac{1.5PN}{c^3} + \cdots + \frac{3PN}{c^6} + \cdots\right\}$$

Using balance equation argument this yields two problems in GR

$$\frac{dE^{3PN}}{dt} = -\mathcal{F}^{3PN}$$

## PROBLEM OF MOTION

Compute the acceleration of two bodies to 3PN order beyond Newtonian acceleration. Then $E^{3PN}$ is deduced from the conservative part of the dynamics

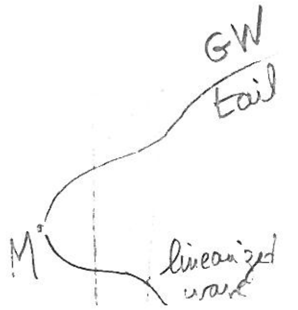
## PROBLEM OF RADIATION

Compute the radiation to 3PN order beyond the Einstein quadrupole formalism. One needs a <u>wave generation</u> formalism to relate the asymptotic wave form $h_{ij}^{TT}$ to the source

For instance we shall need to consistently include <u>tails</u>

a 1.5PN effect in wave form

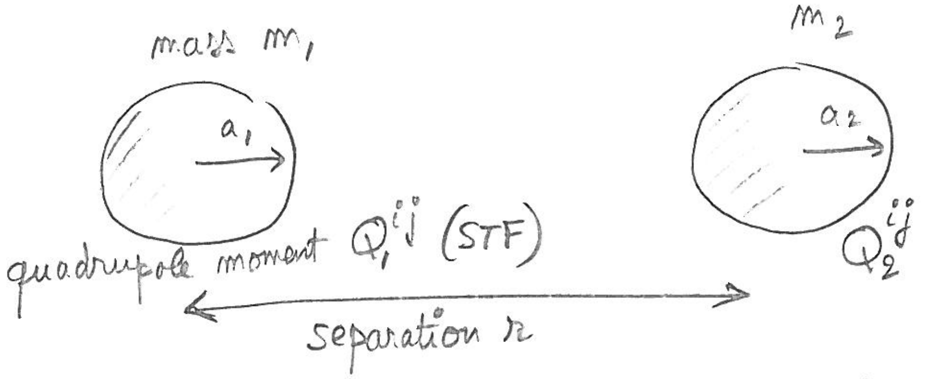
$$h_{ij}^{TT} \simeq \frac{1}{R}\left[\ddot{Q}_{ij} + \frac{2GM}{c^3}\int_{-\infty}^{T-R} \ddddot{Q}_{ij}(t)\ln(T-R-t) + \cdots\right]^{TT}$$



$$F^{i}_{\text{rad. react}} = \frac{G}{c^5}\rho x^j\left[\overset{.....}{Q}_{ij} + \frac{4GM}{c^3}\int_{-\infty}^{T-R} Q \ln + \cdots\right]$$

# MODELLING COMPACT OBJECTS IN GR

Influence of internal structure on eqs of motion and radiation



Eq. of relative motion in Newtonian theory at quadrupolar approximation — means order $\alpha^2$ where

$$\boxed{\alpha = \frac{a}{r} \ll 1}$$

$$\boxed{\frac{d^2 x^i}{dt^2} = M \partial_i \frac{1}{r} + \frac{1}{2} Q^{jk} \partial_i \partial_j \partial_k \frac{1}{r} + \underset{\substack{\uparrow \\ \text{neglect higher} \\ \text{multipoles}}}{\mathcal{O}(\alpha^3)}}$$

where $M = m_1 + m_2$

$$Q_{ij} = \frac{M}{m_1} Q_1^{ij} + \frac{M}{m_2} Q_2^{ij}$$

For quadrupoles induced by <u>tidal field</u>

$$Q_1^{ij} = \underset{\substack{\uparrow \\ \underline{\text{Love number}}}}{k_1} a_1^5 \underbrace{\partial_{<i} \partial_{j>} \frac{m_2}{r}}_{\substack{\text{Newtonian tidal field produced} \\ \text{by body 2}}}$$

which characterizes the response of body 1 to the tidal field.

$k_1$ depends on the body's internal structure

Hence the Newtonian interaction is modified

$$\frac{d^2x^i}{dt^2} = - m \frac{n^i}{r^2}\left[ 1 + \underbrace{k\left(\frac{a}{r}\right)^5}_{\substack{\text{influence of}\\ \text{internal structure}\\ \text{on the motion}}} \right]$$

For compact objects the compacity parameter

$$K = \frac{2Gm}{c^2 a} \approx 1$$

Hence a scales like $\frac{1}{c^2}$ in a formal PN counting. The effect of internal structure is thus

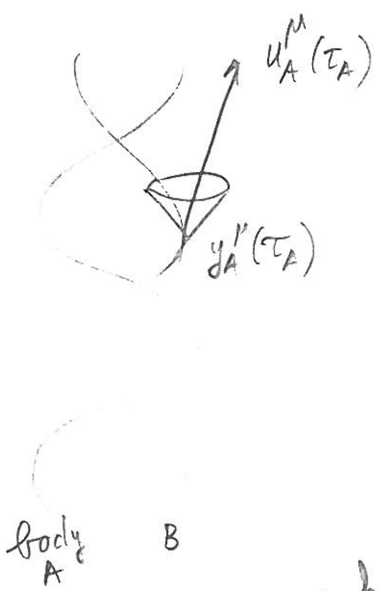
$$\boxed{\frac{Q}{m r^2} \approx k \left(\frac{Gm}{c^2 r}\right)^5 \sim \frac{1}{c^{10}} \sim 5PN}$$

For spinning bodies there will be a quadrupole moment induced by the spins and the effect of internal structure is at 2PN order.

We shall therefore model compact objects by a point mass, possibly with a spin.

Note that in PN approximations it will be perfectly legitimate to use delta functions to model compact objects. Even point masses constitute the correct model to describe BHs in PN approximations. E.g. Schwarzschild metric will be recovered by starting from a point-mass source $\propto m\,\delta(\vec{x})$ and iterating the field equations in a PN way.

# STRESS-ENERGY TENSOR OF SPINNING PARTICLES



$$d\tau_A = \sqrt{-\,g_{\mu\nu}(y_A)\,dy_A^\mu\,dy_A^\nu}$$

$$\boxed{T_M^{\mu\nu}(x) = \sum_A \int_{-\infty}^{+\infty} d\tau_A\, p_A^{(\mu} u_A^{\nu)}\, \frac{\delta_4(x-y_A)}{\sqrt{-g}}}$$

"monopolar" part of stress-energy tensor

where $\underbrace{p_A^\mu = m_A u_A^\mu + (\text{spin effects})}_{\text{linear momentum of particle A}}$

"Dipolar" part includes the effects of spins

$$T_S^{\mu\nu}(x) = -\sum_A \nabla_\rho \left[ \int_{-\infty}^{+\infty} d\tau_A\, S_A^{\rho(\mu} u_A^{\nu)}\, \frac{\delta_4(x-y_A)}{\sqrt{-g}} \right]$$

covariant derivative w.r.t. field point $x^\rho$

anti-symmetric spin tensor of part. A

The eq. of evolution of spin is imposed

$$\frac{DS_A^{\mu\nu}}{d\tau_A} = p_A^\mu u_A^\nu - p_A^\nu u_A^\mu$$

The matter equation $\nabla_\nu T^{\mu\nu} = 0$ where $T^{\mu\nu} = T_M^{\mu\nu} + T_S^{\mu\nu}$ (when integrated on a small volume surrounding A) gives the Papapetrou eq. of motion

$$\boxed{\frac{Dp_\mu^A}{d\tau_A} = -\frac{1}{2} S_A^{\rho\sigma} u_A^\nu R^{(A)}_{\mu\nu\rho\sigma}}$$

coupling to curvature

One must impose a <u>spin suplementary condition</u> (SSC) which corresponds to a choice (in the case of extended bodies) of a center-of-mass worldline w.r.t. which is defined the spin

$$S_A^{\mu\nu} p_\nu^A = 0 \iff S_A^{\mu\nu} = -\frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} \frac{p_\rho^A}{m_A} S_\sigma^A$$

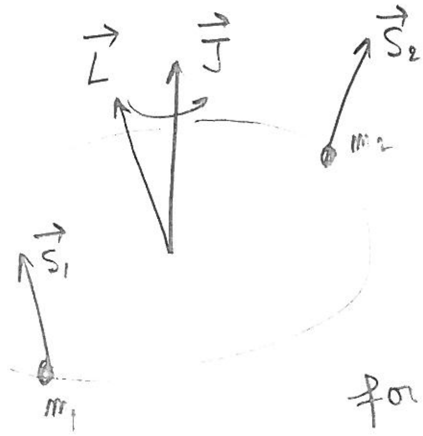
and $S_\mu^A u_A^\mu = 0$ (spin is purely spatial in particle's rest frame).

Then one can show that $p_A^\mu = m_A u_A^\mu + \mathcal{O}(S^2)$ and

$$\boxed{\frac{DS_\mu^A}{d\tau_A} = \mathcal{O}(S^2)}$$

eq. of parallel transport for the spin
(gives the precession equation of the spin vector)

## Contributions of spins in the E.O.M. and radiation



$$\vec{J} = \vec{L} + \underbrace{\vec{S}_1 + \vec{S}_2}_{\text{spins}} \quad \text{is constant}$$

orbital angular momentum

$$S \sim m\, a\, v_{spin}$$

for
- compact objects $a \sim \frac{Gm}{c^2}$
- maximally rotating $v_{spin} \sim c$

$$\boxed{S \sim \chi \frac{Gm}{c}}$$

where $\chi \leq 1$

Relativistic spin-orbit (SO) $\frac{1}{c^2} S \sim \frac{Gm}{c^3}$ ⟵ 1.5PN order

spin-spin (SS) $\frac{1}{c^2} SS \sim \frac{G^2 m^2}{c^4}$

$$\frac{d^2x}{dt^2} = N + \frac{1PN}{c^2} + \frac{1}{c^3}(SO) + \frac{1}{c^4}\left[2PN + SS\right] + \frac{1}{c^5}\left[\underset{\substack{\uparrow \\ \text{R.R.} \\ \text{term}}}{2.5PN} + \begin{pmatrix}\text{1PN} \\ \text{correction} \\ \text{to SO}\end{pmatrix}\right] + \dots$$

# EINSTEIN FIELD EQUATIONS

They derive from the action

$$S = \underbrace{\frac{c^3}{16\pi G}\int d^4x \sqrt{-g}\, R}_{\text{Einstein-Hilbert action}} + \underbrace{S_m[g, \Psi_m]}_{\text{matter action}}$$

Varying w.r.t. the metric

$$\underbrace{\boxed{G^{\mu\nu}[g, \partial g, \partial^2 g]}}_{\text{Einstein tensor}} = \frac{8\pi G}{c^4} \underbrace{T^{\mu\nu}}_{\text{stress-energy tensor}} \qquad T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

Assume small deviation from the Minkowski metric

$$h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu} \leftarrow \text{deviation from "gothic" metric}$$

Choice of harmonic (or de Donder) coordinates $\boxed{\partial_\nu h^{\mu\nu} = 0}$

$$\boxed{\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}}$$

$\Box$: ordinary flat d'Alembertian

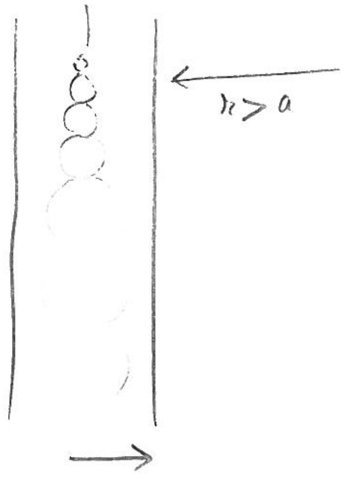
$$\tau^{\mu\nu} = |g|\, T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

Harmonic coordinate condition is equivalent to matter equation

$$\boxed{\partial_\nu h^{\mu\nu} = 0 \iff \partial_\nu \tau^{\mu\nu} = 0 \iff \nabla_\nu T^{\mu\nu} = 0}$$

$\tau^{\mu\nu}$ = pseudo stress energy tensor of matter and gravitational fields

# NON-LINEARITY EXPANSION FOR THE FIELD EXTERIOR TO SOURCE



radius of time-like world-tube enclosing source

In external region we have to solve

$$\begin{cases} \Box h^{\mu\nu}_{ext} = \Lambda^{\mu\nu}(h_{ext}) \\ \partial_\nu h^{\mu\nu}_{ext} = 0 \end{cases}$$

of order $h_{ext}^2$

Suppose we solve these eqs by means of a non-linearity or post-Minkowskian expansion

$$h^{\mu\nu}_{ext} = G h^{\mu\nu}_{(1)} + G^2 h^{\mu\nu}_{(2)} + \dots + G^n h^{\mu\nu}_{(n)}$$

At <u>linearized order</u> we have to solve

$$\begin{cases} \Box h_{(1)} = 0 \\ \partial_\nu h_{(1)} = 0 \end{cases}$$

Most general "monopolar" retarded solution of $\Box h_{(1)} = 0$

$$h^{Mono}_{(1)} = \frac{R(t-r)}{r}$$

Applying $\partial_i \equiv \frac{\partial}{\partial x^i}$ one gets the most general "dipolar" solution

$$h^{Dip}_{(1)} = \partial_i\left(\frac{R_i(t-r)}{r}\right)$$

General multipolar solution is

$$\boxed{h^{\mu\nu}_{(1)} = \sum_{l=0}^{+\infty} \hat{\partial}_L\left(\frac{R^{\mu\nu}_L(t-r)}{r}\right)}$$

one can remove the traces, i.e. assume the $R^{\mu\nu}_L$'s are trace-free

Imposing $\partial h_{(1)} = 0$ reduces the number of independent components of $R^{\mu\nu}_L$ from 10 to 6. So there is in general 6 types of source multipole moments.

On can write the general solution as

$$h^{\mu\nu}_{(1)} = \underset{\substack{\uparrow \\ \text{Thorne's} \\ \text{canonical} \\ \text{expression} \\ \text{parametrized by } I_L,\, J_L}}{k^{\mu\nu}_{(1)}} + \underbrace{\partial^\mu \varphi^\nu_{(1)} + \partial^\nu \varphi^\mu_{(1)} - \eta^{\mu\nu}\partial_\rho \varphi^\rho_{(1)}}_{\substack{\text{gauge transformation} \\ \text{parametrized by } \underbrace{W_L, X_L, Y_L, Z_L}_{\text{gauge moments}}}}$$

$$\boxed{k_{(1)} = \sum \partial_L\left(\frac{1}{r}\underset{\substack{\uparrow \\ \text{mass-moment}}}{I_L(t-r)}\right) + \varepsilon\,\partial\left(\frac{1}{r}\underset{\substack{\uparrow \\ \text{current-moment}}}{J_L(t-r)}\right)}$$

### Non-linear iteration

Insert $h_{(1)}$ into RHS of quadratic-order eqs

$$\Box h_{(2)} = \overset{\downarrow}{\Lambda_{(2)}}(h_{(1)}) \qquad \text{and} \quad \partial h_{(2)} = 0$$

$$\Box h_{(2)} = \sum \partial\left(\frac{R(t-r)}{r}\right)\partial\left(\frac{S(t-r)}{r}\right) \underset{\substack{\uparrow \\ \text{expand all} \\ \text{derivatives}}}{=} \sum \frac{\hat{n}_L}{r^k} F(t-r)$$

where $\hat{n}_L$ is STF product of $n_i$ equivalent to sph. harm. $Y_{lm}(\theta,\varphi)$

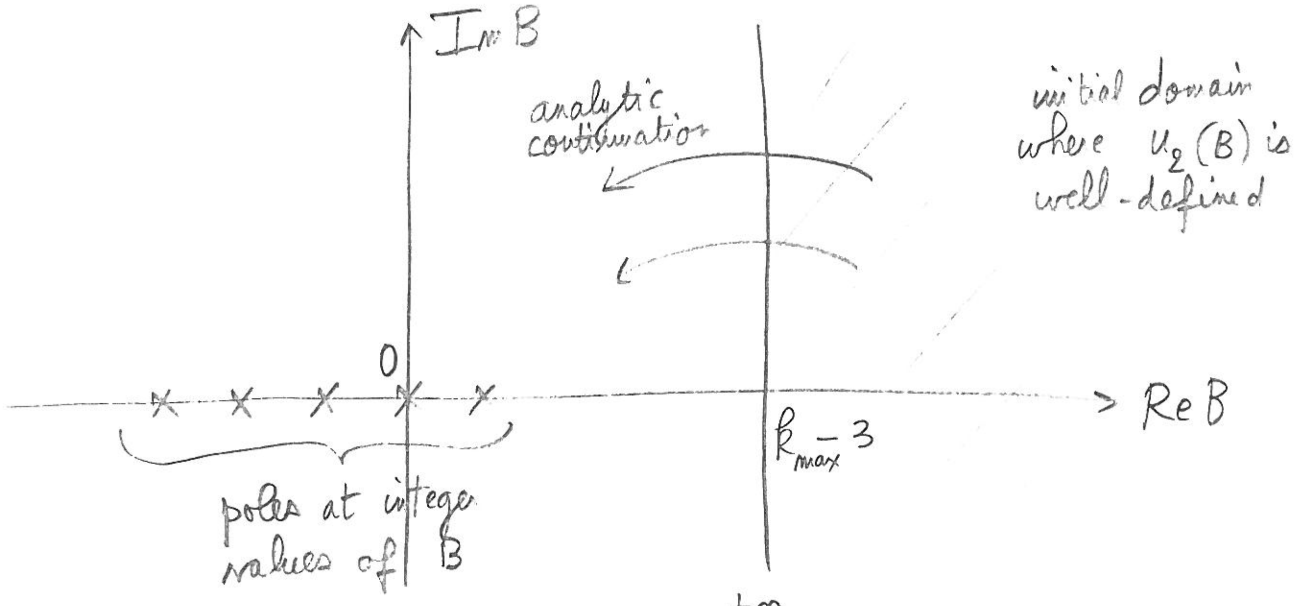
$$\begin{cases} \hat{n}_L = \sum_{m=-l}^{l} \alpha_L^{lm} Y_{lm} \\ \alpha_L^{lm} = \int d\Omega\, \hat{n}_L Y^{*}_{lm} \end{cases}$$

The source is divergent when $r \to 0$. Define

$$u_{(2)}(B) \equiv \Box^{-1}_{\text{Ret}}\left[r^B \Lambda_{(2)}(h_{(1)})\right]$$

for $B \in \mathbb{C}$

Suppose $h_{(1)}$ is made of a finite number of moments, up to some $l_{max}$. Then there is a maximal value of divergences $\boxed{k \leq k_{max}}$



Laurent expansion when $B \to 0$ $$u_{(2)}(B) = \sum_{p=p_0}^{+\infty} \lambda_p B^p \qquad (p \in \mathbb{Z})$$

Applying $\Box$ $$r^B \Lambda_{(2)} = \sum (\Box \lambda_p) B^p$$

$$p_0 \leq p \leq -1 \implies \Box \lambda_p = 0$$

$$p \geqslant 0 \implies \Box \lambda_p = \frac{(\ln r)^p}{p!} \Lambda_{(2)}$$

Hence $\lambda_0$ satisfies the eq. we wanted. Pose $u_{(2)} \equiv \lambda_0$

$$\boxed{u^{\mu\nu}_{(2)} = \underset{B \to 0}{\text{Finite Part}}\ \Box^{-1}_{\text{Ret}} \left[ r^B \Lambda_{(2)} \right]}$$

But $w^{\mu}_{(2)} \equiv \partial_\nu u^{\mu\nu}_{(2)} = \text{FP}\ \Box^{-1}_{\text{Ret}} \left[ B r^{B-1} m^i \Lambda^{\mu i}_{(2)} \right]$ is non-zero (a priori).

$$= \sum \partial_L \left( \frac{T^{\mu}_{L}(t-r)}{r} \right)$$

$\exists$ $v_2^{\mu\nu}$ such that $\partial_\nu v^{\mu\nu}_{(2)} = -w^{\mu}_{(2)}$ and $\Box v^{\mu\nu}_{(2)} = 0$. Hence

$$\boxed{h^{\mu\nu}_{(2)} = u^{\mu\nu}_{(2)} + v^{\mu\nu}_{(2)}}$$

is solution of the quadratic-order equations. This process can be generalized to any non-linear order $n$.

$$h^{gen}_{(n)} = h_{(n)}[I_L \cdots Z_L] + \underbrace{h_{(1)}[\delta I_L \cdots \delta Z_L]}_{\substack{\text{can be re-absorbed into} \\ h_{(1)}[I_L \cdots Z_L] \text{ by posing}}}$$

$$\begin{cases} I_L^{new} = I_L + G^{n-1}\delta I_L \\ \vdots \\ Z_L^{new} = Z_L + G^{n-1}\delta Z_L \end{cases}$$

Hence the previous construction represents the <u>most general</u> solution of Einstein's field eqs. outside the source

Resulting metric $$\boxed{g^{ext}_{\mu\nu}(x;\ \underbrace{I_L\, J_L\, \overbrace{W_L\, X_L\, Y_L\, Z_L}^{\text{4 gauge moments}}}_{\text{6 source moments}})}$$

One can define by coord. transformation $x \longrightarrow x'$ a "canonical" metric which depends only on 2 moments $M_L\ S_L$.

Thus $$\boxed{g^{can}_{\mu\nu}(x';\ \underbrace{M_L\, S_L}_{\text{2 canonical moments}})}$$

is <u>isometric</u> to $g^{ext}_{\mu\nu}$ i.e. $g^{can}_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g^{ext}_{\rho\sigma}(x)$ where

$$\boxed{x'^\mu = x^\mu + G\, \underbrace{\varphi^\mu_{(1)}(x;\ W_L X_L Y_L Z_L)}_{\substack{\text{gauge vector in the} \\ \text{general linear solution}}} + \underset{\substack{\uparrow \\ \text{crucial} \\ \text{non-linear} \\ \text{corrections}}}{\mathcal{O}(G^2)}}$$

# THE MATCHING EQUATION

We have contructed the exterior field (physically valid when $r > a$) of any isolated source

$$h_{ext} = \sum_{n=1}^{+\infty} G^n h_{(n)}\Big[\underbrace{I_L\; J_L\; W_L \dots Z_L}_{\text{source moments (for the moment arbitrary)}}\Big]$$

We suppose that $h_{ext}$ comes from the multipole expansion of $h$ defined everywhere inside and outside the source $\left(\begin{smallmatrix}\text{for}\\ \text{any } r\end{smallmatrix}\right)$

$$\boxed{h_{ext} = \underset{\substack{\uparrow \\ \text{operation of taking} \\ \text{the multipole expansion}}}{\mathcal{M}}(h)}$$

Note that $\mathcal{M}(h)$ is defined of any $r > 0$ but agrees with the "true" field $h$ only when $r > a$

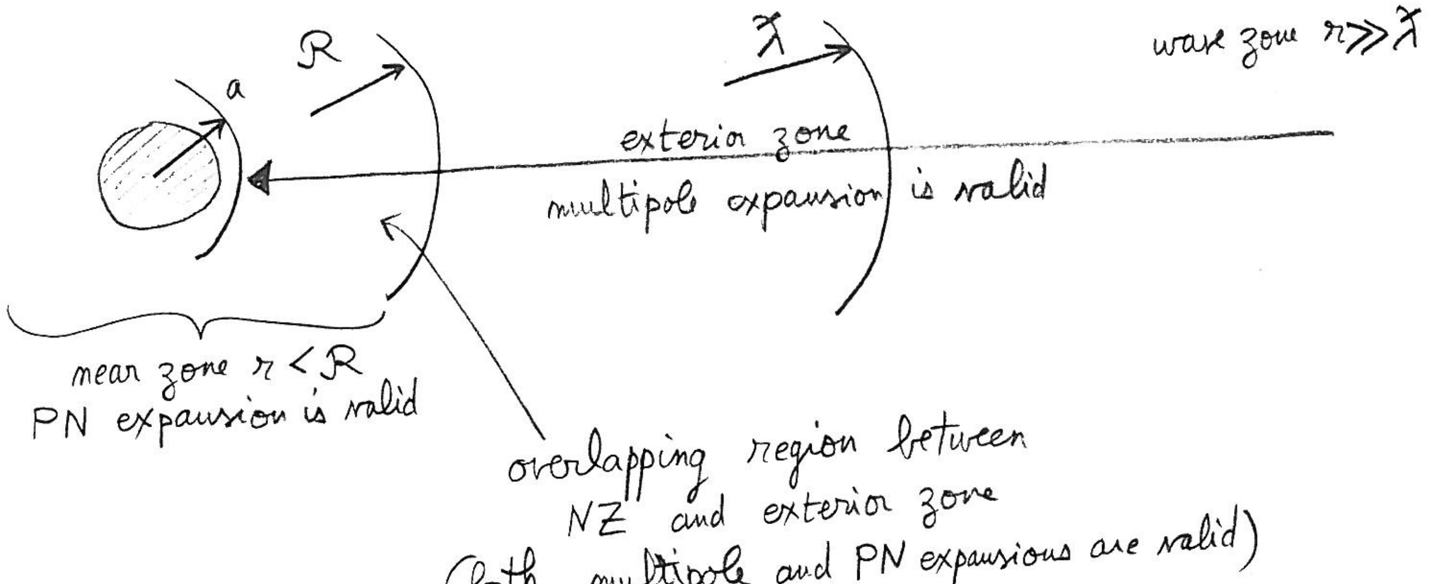
$$\boxed{r > a \quad \Rightarrow \quad \mathcal{M}(h) = h \qquad (\text{numerically})}$$

But when $r \longrightarrow 0$ $\mathcal{M}(h)$ diverges while $h$ is a perfectly smooth solution of Einstein field eqs. inside the matter (of the extended source).

Suppose the source is post-Newtonian (existence of the PN parameter $\varepsilon = \frac{v}{c} \ll 1$). We know that the near zone $r < \mathcal{R}$ where $\mathcal{R} \ll \bar{\lambda}$ encloses totally the PN source ($\mathcal{R} > a$).

In the NZ the field $h$ can be expanded as a PN expansion ($\bar{h} = \sum c^{-p} (\ln c)^q$)

$$\boxed{r < \mathcal{R} \quad \Rightarrow \quad h = \bar{h} \quad \text{(numerically)}}$$



$$\boxed{a < r < \mathcal{R} \quad \Rightarrow \quad \mathcal{M}(h) = \bar{h} \quad \text{(numerically)}}$$

The matching equation follows from transforming the latter numerical equality in a functional identity (valid $\forall (\vec{x}, t)$ in $\mathbb{R}^3_* \times \mathbb{R}$) between two formal asymptotic series

Matching equation: $\boxed{\overline{\mathcal{M}(h)} \equiv \mathcal{M}(\bar{h})}$

NZ expansion $\left(\frac{r}{c}\to 0\right)$ of each multipolar coeff. of $\mathcal{M}(h)$

multipole expansion of each PN coefficient of $\bar{h}$

We assume (as part of our fundamental assumptions) that the matching eq. is correct (in the sense of formal series)

$$\boxed{\underset{\frac{r}{c}\to 0}{\text{NZ expansion}} \begin{pmatrix} \text{multipolar} \\ \text{expansion} \\ \frac{a}{r}\to 0 \end{pmatrix} \equiv \underset{r\to\infty}{\text{FZ expansion}} \begin{pmatrix} \text{PN series} \\ c\to\infty \end{pmatrix}}$$

The NZ expansion $\frac{r}{c}\to 0$ is "equivalent" to the PN expansion $c\to+\infty$ for fixed $r$

The multipole expansion $\frac{a}{r}\to 0$ is "equivalent" to the FZ expansion $r\to+\infty$ for a given source (fixed $a$)

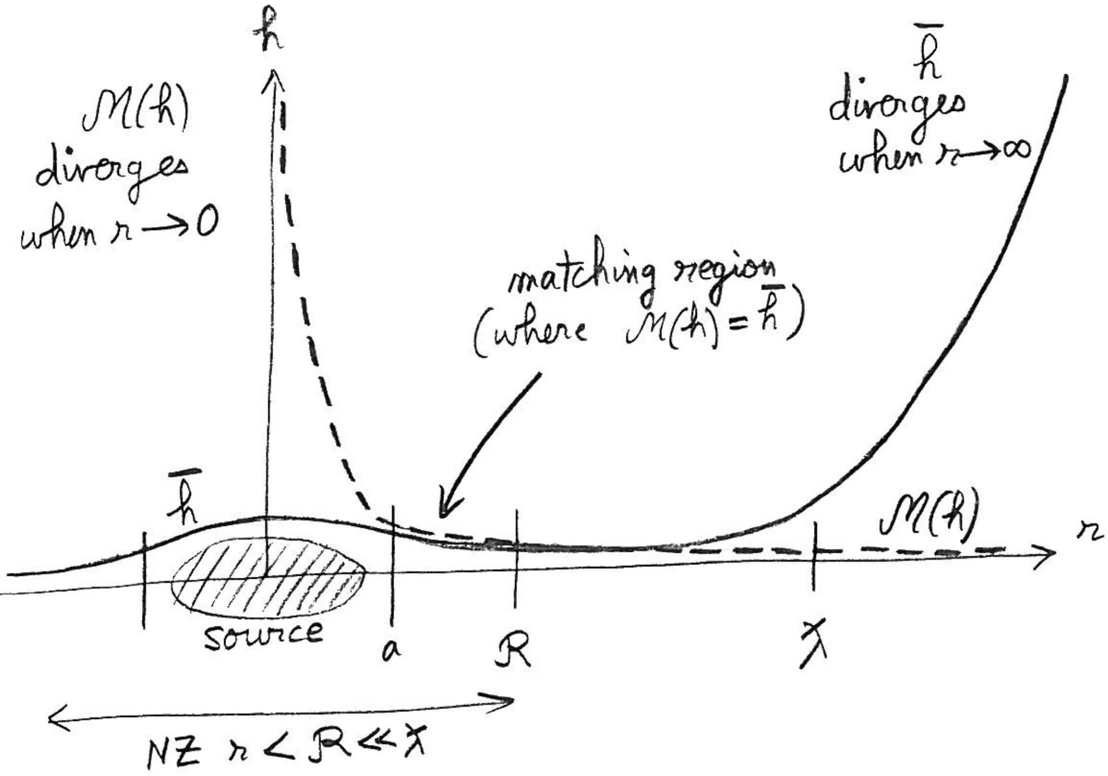
The matching equation says basically the NZ and multipole expansions can be commuted.

Thus there is a common structure for the formal NZ and FZ expansions

$$\overline{\mathcal{M}(h)} \equiv \underbrace{\sum \hat{n}_L\, r^p (\ln r)^q F(t)} \equiv \mathcal{M}(\bar{h})$$

can be interpreted either as

- NZ singular expansion when $r \to 0$
- FZ ——————— $r \to \infty$



## GENERAL EXPRESSION OF THE MULTIPOLE MOMENTS

$h$ is the sol. of Einstein eqs (in harmonic coord. $\partial h = 0$) valid everywhere inside and outside the source

$$h = \frac{16\pi G}{c^4} \Box^{-1}_{\mathrm{Ret}} \tau \qquad \begin{pmatrix}\text{suppress} \\ \text{indices } \mu\nu\end{pmatrix}$$

where $\tau = |g|\, T + \underbrace{\frac{c^4}{16\pi G} \Lambda}_{\substack{\text{gravitational source-term} \\ \text{(non-linearities in } h)}}$

Define $\boxed{\Delta \equiv h - \mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\mathcal{M}(\Lambda)}$

where $\mathcal{M}(\Lambda) = \Lambda[\mathcal{M}(\Lambda)] = \Lambda_{\mathrm{ext}}$ and FP is the finite part when $B \to 0$ (plays a crucial role because $\Lambda_{\mathrm{ext}}$ diverges when $r \to 0$)

$$\Delta = \underbrace{\frac{16\pi G}{c^4}\,\Box^{-1}_{\mathrm{Ret}}\,\tau}_{\substack{\text{no FP here} \\ \text{since } T \text{ is regular } (C^\infty)}} - \mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\,\mathcal{M}(\Lambda)$$

However we can add FP on the first term (do not change the value because it converges). Using also $\mathcal{M}(T) = 0$ since $T$ has a compact support

$$\Delta = \frac{16\pi G}{c^4}\,\mathrm{FP}\,\Box^{-1}_{\mathrm{Ret}}\left[\tau - \mathcal{M}(\tau)\right]$$

Hence $\Delta$ appears as the retarded integral of a source with compact support. Indeed

$$\tau = \mathcal{M}(\tau) \quad \text{when } r > a$$

$$\boxed{\mathcal{M}(\Delta) = -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\,\partial_L\left(\frac{1}{r}\,\mathcal{H}_L(u)\right)}$$

This is standard expression of multipolar expansion outside a compact-support source. Here the moments are

$$\mathcal{H}_L = \mathrm{FP}\int d^3x\; x_L \underbrace{\left[\tau - \mathcal{M}(\tau)\right]}_{\substack{\text{since this has compact support}\\ (r<a\text{, inside the NZ) we can}\\ \text{replace by the NZ or PN expansion}}}$$

$$\mathcal{H}_L = \mathrm{FP}\int d^3x\; x_L \left[\overline{\tau} - \overline{\mathcal{M}(\tau)}\right]$$

But we know the structure $\overline{\mathcal{M}(\tau)} = \sum \hat{n}_L r^p (\ln r)^q F(t)$ which is sufficient to prove that the second term is <u>zero</u> by analytic continuation

$$\mathrm{FP}\int d^3x\; x_L \overline{\mathcal{M}(\tau)} = \sum \mathrm{FP}\int d^3x\; x_L \hat{n}_Q r^p (\ln r)^q$$

$$\overset{\text{integrate over angles}}{=} \sum \underset{B\to 0}{\text{Finite Part}} \int dr\, r^{B+S} (\ln r)^p$$

$$= \sum \underset{B\to 0}{\mathrm{FP}} \left(\frac{d}{dB}\right)^p \int_0^{+\infty} dr\, r^{B+S}$$

$$\int_0^{+\infty} dr\, r^{B+S} = \underbrace{\int_0^{\mathcal{R}} dr\, r^{B+S}}_{\substack{\text{computed}\\ \text{when } \mathcal{R}e\, B > -S-1}} + \underbrace{\int_{\mathcal{R}}^{+\infty} dr\, r^{B+S}}_{\substack{\text{computed}\\ \text{when } \mathcal{R}e\, B < -S-1}}$$

$$= \underbrace{\frac{\mathcal{R}^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}} \qquad = -\underbrace{\frac{\mathcal{R}^{B+S+1}}{B+S+1}}_{\text{by analytic continuation}}$$

$$\boxed{\text{Analytic Continuation} \qquad \int_0^{+\infty} dr\, r^{B+s} (\ln r)^p = 0 \qquad \forall B \in \mathbb{C}}$$

The general multipole expansion outside the domain of a PN isolated source reads (Blanchet 1995, 1998)

$$\boxed{\mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left( \frac{1}{r} \mathcal{H}_L(u) \right)}$$

where

$$\boxed{\mathcal{H}_L(u) = \mathrm{FP} \int d^3x \; x_L \, \underset{\uparrow}{\overline{\tau}} (\vec{x}, u)}$$

PN expansion crucial here
(this is where the formalism applies only to PN sources)

Same result but in STF guise

$$\boxed{\begin{gathered} \mathcal{M}(h) = \mathrm{FP}\, \Box^{-1}_{\mathrm{Ret}} \mathcal{M}(\Lambda) - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left( \frac{1}{r} \mathcal{F}_L(u) \right) \\ \text{where} \\ \mathcal{F}_L(u) = \mathrm{FP} \int d^3x \; \hat{x}_L \int_{-1}^{1} dz \, \delta_l(z) \, \overline{\tau}\left(\vec{x}, u + z|\vec{x}|/c\right) \\ \delta_l(z) = \frac{(2l+1)!!}{2^{l+1}\, l!} (1-z^2)^l \quad \text{such that} \quad \begin{array}{l} \int_{-1}^{1} dz \, \delta_l(z) = 1 \\ \lim_{l \to +\infty} \delta_l(z) = \delta(z) \end{array} \end{gathered}}$$