Gravitational self-force calculations in the time domain in 2+1D: progress report

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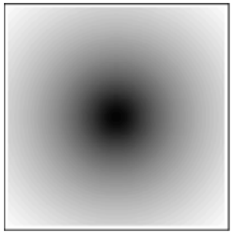
Approaches for SF calculations

- Frequency domain
  - ✓ Easy for simple orbits
  - o Hard to generalize to generic Kerr orbits

- Time domain
  - ✓ Natural extension to generic worldlines
  - ? Regularization method

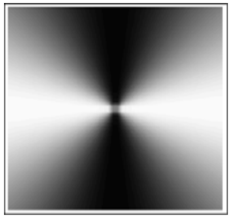
Our approach:

- ## Metric perturbations
  - no need to do metric reconstruction from Weyl scalars



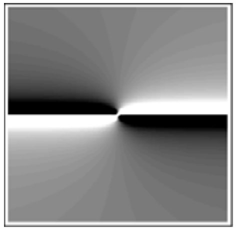
Lorenz gauge

- ## Lorenz Gauge
  - avoid need for gauge transformations
  - particle singularity is isotropic and isolated
  - manifest hyperbolicity



RW gauge

- ## Time domain in 2+1D
  - generic orbits are natural
  - much experience gained from the Teukolsky Equation
  - characteristic / first-order formulation / pseudospectral



Radiation gauge

- ## Regularization
  - *m* mode regularization / puncture method
  - Detweiler - Whiting regular part

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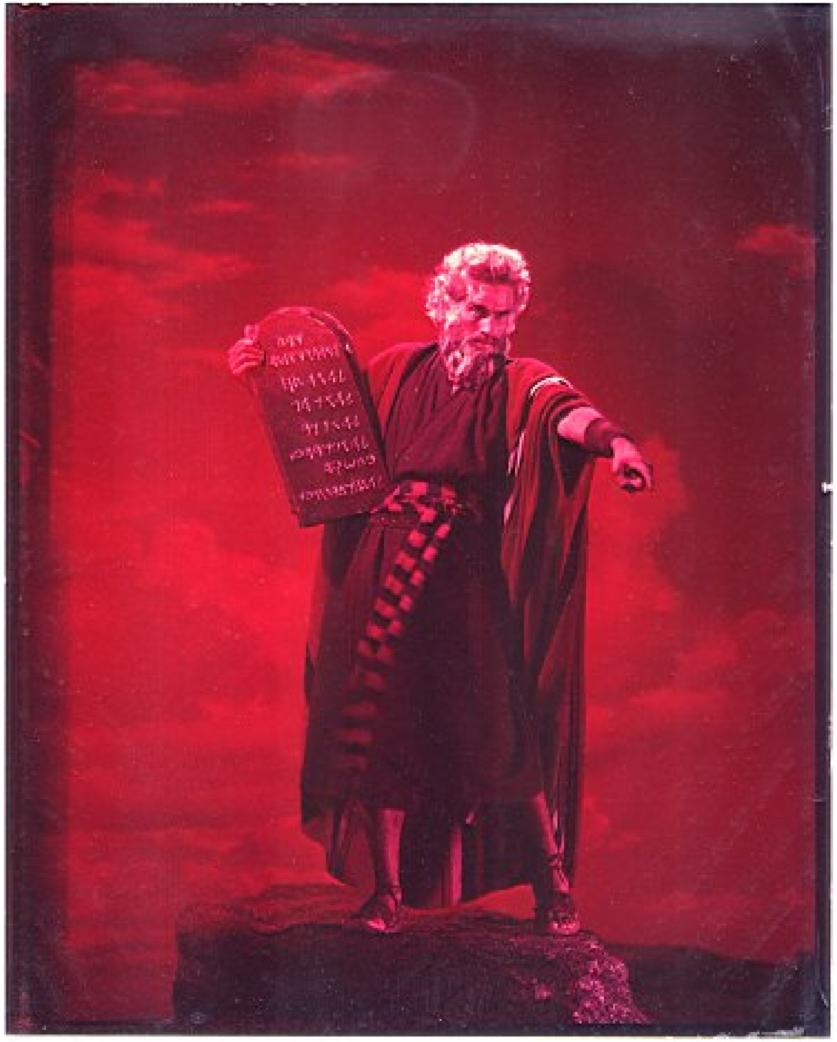
Thou shalt not mix thy polar and axial modes !



Field equations

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

$$\nabla^{\alpha}\bar{h}_{\alpha\beta} = 0$$

$$T_{\alpha\beta} = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^4[x^{\mu} - x^{\mu}_{\mathrm{p}}(\tau)] u_{\alpha} u_{\beta}\, d\tau$$

$$\begin{aligned}
\Box \bar{h}_{\alpha\beta} &= \underbrace{g^{\gamma\delta}\,\partial_{\gamma}\,\partial_{\delta}\bar{h}_{\alpha\beta} - g^{\gamma\delta}(\bar{h}_{\alpha\beta,\sigma}\Gamma^{\sigma}_{\delta\gamma}}_{\text{scalar operator } D^2} + 2\bar{h}_{\lambda\beta,\delta}\Gamma^{\lambda}_{\alpha\gamma} + 2\bar{h}_{\lambda\alpha,\delta}\Gamma^{\lambda}_{\beta\gamma}) + \bar{h}_{\lambda\alpha}\mathcal{G}^{\lambda}_{\beta} + \bar{h}_{\lambda\beta}\mathcal{G}^{\lambda}_{\alpha} + \bar{h}_{\lambda\sigma}\mathcal{F}^{\lambda\sigma}_{\alpha\beta} \\
&= D^2\bar{h}_{\alpha\beta} + 4g^{\gamma\delta}\Gamma^{\lambda}_{\gamma(\alpha}\bar{h}_{\beta)\lambda,\delta} + 2\bar{h}_{\lambda(\alpha}\mathcal{G}^{\lambda}_{\beta)} + \bar{h}_{\lambda\sigma}\mathcal{F}^{\lambda\sigma}_{(\alpha\beta)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{\lambda}_{\alpha} &:= \left(\Gamma^{\lambda}_{\sigma\gamma}\Gamma^{\sigma}_{\alpha\delta} + \Gamma^{\lambda}_{\sigma\alpha}\Gamma^{\sigma}_{\gamma\delta} - \Gamma^{\lambda}_{\alpha\gamma,\delta}\right) g^{\gamma\delta} = g^{\gamma\delta}\left[2\Gamma^{\lambda}_{\sigma(\gamma}\Gamma^{\sigma}_{\alpha)\delta} - \Gamma^{\lambda}_{\alpha\gamma,\delta}\right] \\
\mathcal{F}^{\lambda\sigma}_{\alpha\beta} &:= \left(\Gamma^{\lambda}_{\beta\gamma}\Gamma^{\sigma}_{\alpha\delta} + \Gamma^{\lambda}_{\gamma\alpha}\Gamma^{\sigma}_{\beta\delta}\right) g^{\gamma\delta} = 2g^{\gamma\delta}\Gamma^{\lambda}_{\gamma(\alpha}\Gamma^{\sigma}_{\beta)\delta}\,.
\end{aligned}$$

Field equations II

$$\bar{h}_{\alpha\beta} := \frac{K_{\alpha\beta}}{r}$$

$$K_{\alpha\beta} = \sum_{m=-\infty}^{\infty} \kappa_{\alpha\beta}^{(m)} e^{im\varphi}$$

$$T_{\alpha\beta} = \sum_{m=-\infty}^{\infty} \tau_{\alpha\beta}^{(m)} e^{im\varphi} .$$

Field equations III

Schwarzschild:

$$f := 1 - \frac{2M}{r}$$

10 equations of the form

$$\underbrace{\left[\frac{1}{r\,f}\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}\right)-\frac{2M}{r^4}+\frac{1}{r^3}\left(\frac{\partial^2}{\partial \theta^2}+\cot\theta\,\frac{\partial}{\partial\theta}\right)-\frac{m^2}{r^3\sin^2\theta}\right]}_{\text{scalar operator}}\kappa_{00}^{(m)}$$

$$-\frac{2M}{r^5}\left[\frac{2r^2}{f}\frac{\partial\kappa_{00}^{(m)}}{\partial r_*}-\frac{1}{f}(2r-3M)\,\kappa_{00}^{(m)}-2r^2\,\partial_0\kappa_{10}^{(m)}+f(2r-3M)\kappa_{11}^{(m)}-\frac{f}{r}\left(\kappa_{22}^{(m)}+\frac{\kappa_{33}^{(m)}}{\sin^2\theta}\right)\right]=-16\pi\,\tau_{00}^{(m)}$$

4 gauge conditions of the form:

$$r^3 f^2\,\partial_1\left(\frac{\kappa_{10}^{(m)}}{r}\right)+f\,\partial_2\kappa_{20}^{(m)}+im\,\frac{f}{\sin^2\theta}\kappa_{30}^{(m)}-r^2\,\partial_0\kappa_{00}^{(m)}+2(r-M)f\kappa_{10}^{(m)}+f\,\cot\theta\,\kappa_{20}^{(m)}=0$$

Numerical setup

Krivan, Laguna, Papadopoulos and Andersson (1997):

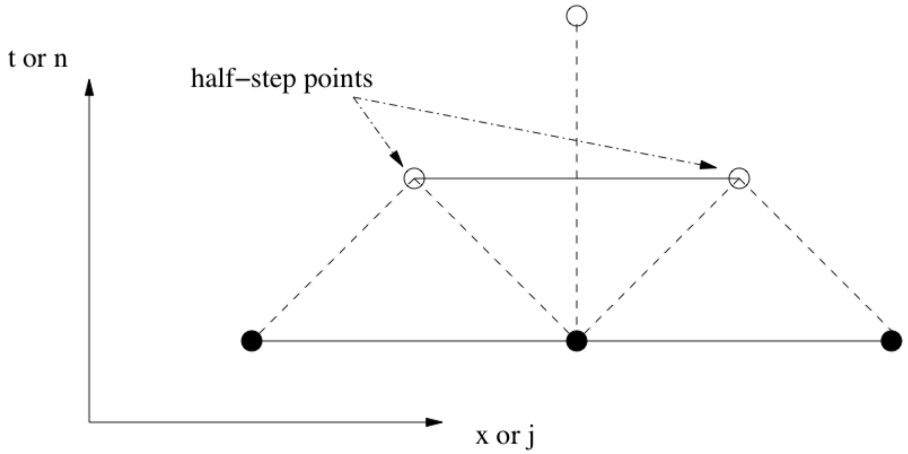
$$\psi := e^{im\tilde{\phi}} r^3 \Phi$$

$$\Pi := \partial_t \Phi + b\,\partial_{r^*} \Phi \qquad b := \frac{r^2 + a^2}{\Sigma}$$

$$\mathbf{u} := (\Phi_R, \Phi_I, \Pi_R, \Pi_I)$$

$$\partial_t \mathbf{u} + \mathbf{D}\,\partial_{r^*} \mathbf{u} = \mathbf{S}$$

Solution is then obtained by the Lax-Wendroff 2-step scheme



Numerical setup II

$$\underbrace{\left[\frac{1}{rf}\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}\right)-\frac{2M}{r^4}+\frac{1}{r^3}\left(\frac{\partial^2}{\partial \theta^2}+\cot\theta\,\frac{\partial}{\partial\theta}\right)-\frac{m^2}{r^3\sin^2\theta}\right]\kappa_{00}^{(m)}}_{\text{scalar operator}}$$

$$-\frac{2M}{r^5}\left[\frac{2r^2}{f}\frac{\partial\kappa_{00}^{(m)}}{\partial r_*}-\frac{1}{f}(2r-3M)\,\kappa_{00}^{(m)}-2r^2\,\partial_0\kappa_{10}^{(m)}+f(2r-3M)\kappa_{11}^{(m)}-\frac{f}{r}\left(\kappa_{22}^{(m)}+\frac{\kappa_{33}^{(m)}}{\sin^2\theta}\right)\right]=-16\pi\,\tau_{00}^{(m)}$$

$$\Pi_{ij}=\partial_0\kappa_{ij}+\partial_{r_*}\kappa_{ij} \qquad (-\partial_0^2+\partial_{r*}^2)\kappa_{ij}=(\partial_{r*}-\partial_0)\Pi_{ij}$$

$$\begin{aligned}
\mathbf{u}^{\mathrm{T}} &:= \{\overbrace{\kappa_{00}\ \ \kappa_{10}\ \ \kappa_{20}\ \cdots\ \kappa_{33}}^{10\ \text{terms}}\ \overbrace{\Pi_{00}\ \ \Pi_{10}\ \ \Pi_{20}\ \cdots\ \Pi_{33}}^{10\ \text{terms}}\}\\
\mathbf{T}\,\partial_0\mathbf{u} &+ \tilde{\mathbf{D}}\,\partial_{r^*}\mathbf{u}+\tilde{\mathbf{A}}\,\mathbf{u}+\tilde{\mathbf{L}}\,\mathbf{u}=\tilde{\mathbf{S}}\\
\mathbf{T}^{-1}\mathbf{T}\,\partial_0\mathbf{u} &+ \underbrace{\mathbf{T}^{-1}\tilde{\mathbf{D}}}_{\mathbf{D}}\,\partial_{r^*}\mathbf{u}+\underbrace{\mathbf{T}^{-1}\tilde{\mathbf{A}}}_{\mathbf{A}}\,\mathbf{u}+\underbrace{\mathbf{T}^{-1}\tilde{\mathbf{L}}}_{\mathbf{L}}\,\mathbf{u}=\underbrace{\mathbf{T}^{-1}\tilde{\mathbf{S}}}_{\mathbf{S}}\\
\partial_0\mathbf{u} &+ \mathbf{D}\,\partial_{r^*}\mathbf{u}+\mathbf{A}\,\mathbf{u}+\mathbf{L}\,\mathbf{u}=\mathbf{S}
\end{aligned}$$

Numerical setup III

$$\begin{aligned}
\partial_0 \Pi^{(m)}_{00} \quad & + \quad \frac{4M}{r^2}\,\partial_{r^*}\kappa^{(m)}_{00} + \frac{4M}{r^3}(r-2M)\,\partial_{r^*}\kappa^{(m)}_{10} - \partial_{r^*}\Pi^{(m)}_{00} \\
& + \quad \left[-\frac{8M}{r^4}(r-M) + \frac{m^2(r-2M)}{r^3\,\sin^2\theta}\right]\,\kappa^{(m)}_{00} \\
& + \quad \frac{2M}{r^6}(r-2M)^2(2r-3M)\,\kappa^{(m)}_{11} - \frac{2M}{r^7}(r-2M)^2\,\kappa^{(m)}_{22} \\
& - \quad \frac{2M}{\sin^2\theta\,r^7}(r-2M)^2\,\kappa^{(m)}_{33} - \frac{4M}{r^3}(r-2M)\,\Pi^{(m)}_{00} \\
& - \quad \frac{r-2M}{r^3}(\,\partial_{22} + \cot\theta\,\partial_2)\,\kappa^{(m)}_{00} = 16\pi\,(r-2M)\,\tau^{(m)}_{00}
\end{aligned}$$

$$_{11}\mathbf{D} = \left\{ \underbrace{\frac{4M}{r^2}\,,\ \frac{4M(r-2M)}{r^2}\,,0\,,0\,,0\,,0\,,0\,,0\,,0\,,0\,,}_{\kappa_{ij}} \overbrace{-1\,,0\,,0\,,0\,,0\,,0\,,0\,,0\,,0\,,0}^{\Pi_{ij}} \right\}$$

m-mode regularization

2+1D perturbation diverges logarithmically at worldline

Barack & Golbourn (2007):

The asymptotic divergence has an analytical and known form:

subtract it out and evolve the residual regular field

$$h = h_R + h_P$$

$h_R{}^{(m)}$ are finite and continuous on the worldline

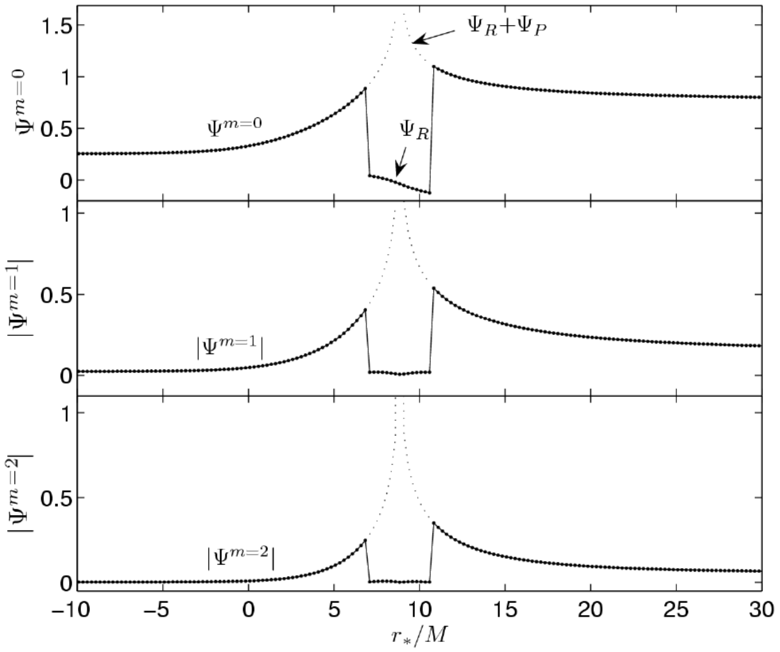
h_P are regular everywhere except on the worldline: *meshugah* of the field

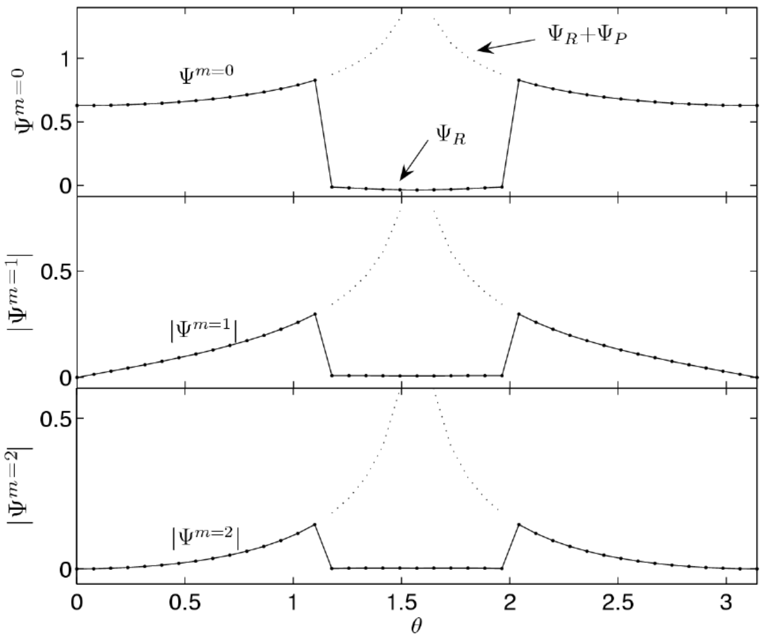
$h_P{}^{(m)}$ may be found analytically

Near worldline evolve $h_R{}^{(m)}$
At a "safe distance" switch back to vacuum variable $h^{(m)}$

m-mode regularization

Barack and Golbourn (2007): scalar field





Puncture function

$$
\begin{aligned}
S &= S_0 + S_1 + \cdots \\
S_0 &= (g_{\alpha\beta} + u_\alpha u_\beta)\,\delta x^\alpha\,\delta x^\beta \\
S_1 &= (g_{\alpha\beta} + u_\alpha u_\beta)\Gamma^\beta_{\rho\sigma}\,\delta x^\alpha\,\delta x^\rho\,\delta x^\sigma
\end{aligned}
$$

Barack and Ori, 2003:

$$\bar{h}^{\mathrm{dir}}_{\beta\gamma} = 4\mu\, S^{-1/2}\hat{u}_\beta\hat{u}_\gamma + S^{-1/2}\tilde{P}^{(2)}_{\beta\gamma}(x, z_0)$$

Schwarzschild circular orbits:

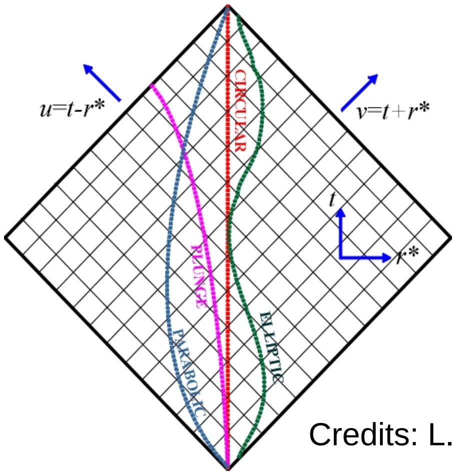
$$S_0 = \frac{1-2M/r}{1-3M/r}\left(\frac{M}{r}\,\delta x^0\,\delta x^0 + r^2\,\delta x^3\,\delta x^3 - 2\sqrt{Mr}\,\delta x^0\,\delta x^3\right) + \frac{1}{1-2M/r}\,\delta x^1\,\delta x^1 + r^2\,\delta x^2\,\delta x^2$$

$$S_1 = \cdots$$

What are we doing now?

- Still coding the vacuum equations…
  - An interesting application: vacuum tails
  - Schwarzschild, but not making use of spherical symmetry, preparing the infrastructure for Kerr
  - Even in Schwarzschild, we can do many interesting cases
  - Finite differencing / characteristic pseudospectral collocation
- And next, Kerr…

  (Not that hard! Well.. Maybe…)



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$$\kappa_{20}^{(m)} = \frac{1}{4}\, C(r)\, \frac{r^2}{f} \left(1 - \frac{\sin 3\theta}{3 \sin\theta}\right)$$

$$\kappa_{00}^{(m)} = t\, C(r) \sin\theta \, \cos\theta$$

$$\tau_{10}^{(0)} = \frac{1}{16\pi r^2} C(r)\, \frac{3f - 1}{f^2}\, \sin\theta \, \cos\theta$$