Capra 2007: Where were we?

(or, if you like, Capra 2008: Where are we?)

Percise, yet useless answers:

2007: We were in Huntsville

2008: We are in Orléans

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UAHuntsville

- Analytical methods
  - Post Newtonian (EIH, Schäfer, Whiting)
  - Local neighborhood (Flanagan, Wiseman)
  - Adiabatic
    - Assume:

$$\tau_{\mathrm{RR}} \gg \tau_{\mathrm{orbital}}$$

$\langle \dot{C}_a \rangle$ describes orbital evolution

  - Two time-scale / post-adiabatic (Flanagan, Price, Poisson)
- Numerical methods
  - Weyl scalars
  - Metric perturbations
    - Frequency domain
    - Time domain

Metric Perturbations

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

$$T_{\alpha\beta} = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^4[x^\mu - x^\mu_{\rm p}(\tau)] u_\alpha u_\beta \, d\tau$$

$$\begin{aligned} \Box \bar{h}_{\alpha\beta} &= \underbrace{g^{\gamma\delta}\,\partial_\gamma\,\partial_\delta \bar{h}_{\alpha\beta} - g^{\gamma\delta}(\bar{h}_{\alpha\beta,\sigma}\Gamma^{\sigma}_{\delta\gamma}}_{\text{scalar operator } D^2} + 2\bar{h}_{\lambda\beta,\delta}\Gamma^{\lambda}_{\alpha\gamma} + 2\bar{h}_{\lambda\alpha,\delta}\Gamma^{\lambda}_{\beta\gamma}) + \bar{h}_{\lambda\alpha}\mathcal{G}^{\lambda}_{\beta} + \bar{h}_{\lambda\beta}\mathcal{G}^{\lambda}_{\alpha} + \bar{h}_{\lambda\sigma}\mathcal{F}^{\lambda\sigma}_{\alpha\beta} \\ &= D^2\bar{h}_{\alpha\beta} + 4g^{\gamma\delta}\Gamma^{\lambda}_{\gamma(\alpha}\bar{h}_{\beta)\lambda,\delta} + 2\bar{h}_{\lambda(\alpha}\mathcal{G}^{\lambda}_{\beta)} + \bar{h}_{\lambda\sigma}\mathcal{F}^{\lambda\sigma}_{(\alpha\beta)} \end{aligned}$$

$$\begin{aligned} \mathcal{G}^{\lambda}_{\alpha} &:= \left(\Gamma^{\lambda}_{\sigma\gamma}\Gamma^{\sigma}_{\alpha\delta} + \Gamma^{\lambda}_{\sigma\alpha}\Gamma^{\sigma}_{\gamma\delta} - \Gamma^{\lambda}_{\alpha\gamma,\delta}\right) g^{\gamma\delta} = g^{\gamma\delta}\left[2\Gamma^{\lambda}_{\sigma(\gamma}\Gamma^{\sigma}_{\alpha)\delta} - \Gamma^{\lambda}_{\alpha\gamma,\delta}\right] \\ \mathcal{F}^{\lambda\sigma}_{\alpha\beta} &:= \left(\Gamma^{\lambda}_{\beta\gamma}\Gamma^{\sigma}_{\alpha\delta} + \Gamma^{\lambda}_{\gamma\alpha}\Gamma^{\sigma}_{\beta\delta}\right) g^{\gamma\delta} = 2g^{\gamma\delta}\Gamma^{\lambda}_{\gamma(\alpha}\Gamma^{\sigma}_{\beta)\delta} \,. \end{aligned}$$

Adiabatic waveforms

- Waveforms built by invoking the two time scales

$$\tau_{\rm dyn} \sim M \qquad \tau_{\rm RR} \sim M \times \frac{M}{\mu} \qquad \tau_{\rm RR} \gg \tau_{\rm orbital}$$

- Assume $\langle \dot{C}_a \rangle$ describes orbital evolution
- Short-time motion is a geodesic of the background
- Can use fluxes (and average change of *Q*)
- All applications to date were pure dissipative, ignoring conservative effects

Adiabatic waveforms: Frequency domain

- Frequency domain waveforms in Schwarzschild and Kerr (Hughes, Drasco)

  Mature approach, can in principle do any orbit:

  Solve the Teukolsky equation for Ψ_4 using mode decomposition

$$\Psi_4 = \frac{1}{(r - ia\cos\theta)^4} \int_{-\infty}^{\infty} d\Omega \sum_{\ell m} R_{\ell m\Omega}(r) \, {}_{-2}S^{a\Omega}_{\ell m}(\theta) \, e^{i(m\phi - \Omega t)}$$

$$\Omega_{mkn} := m\omega_\phi + k\omega_r + n\omega_\theta$$

next,

$$R_{\ell m\Omega} = Z^{\mathrm{H}}_{\ell m\Omega} R^{\infty}_{\ell m\Omega}(r) + Z^{\infty}_{\ell m\Omega} R^{\mathrm{H}}_{\ell m\Omega}(r)$$

$$h_+ - ih_\times = \sum_{\ell mk} \frac{Z^{\mathrm{H}}_{\ell mk}}{\Omega_{mk}} \, {}_{-2}S^{a\Omega_{mk}}_{\ell m}(\theta) \, e^{i(m\phi - \Omega_{mk} t)}$$

Adiabatic waveforms: Frequency domain

Hughes (2001):

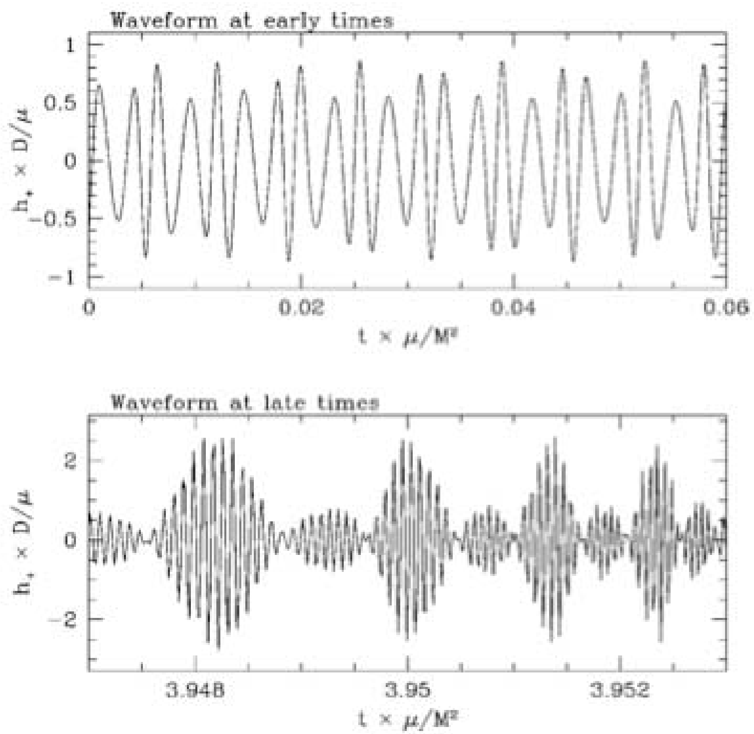
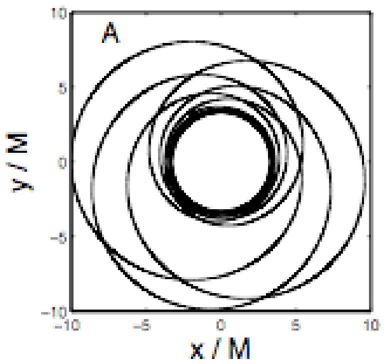


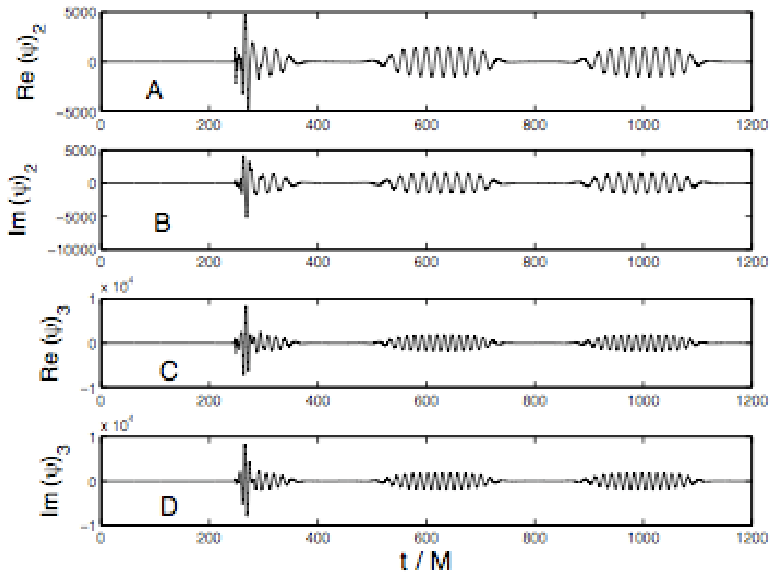
FIG. 8. The + polarization of the gravitational waveform for the inspiral trajectory that begins at $\iota = 40°$ about the $a = 0.998M$ black hole, viewed in the hole's equatorial plane. The upper panel is the waveform at very early times; the lower panel shows the waveform shortly before the inspiraling body plunges into the hole. Notice the very different time scales in the upper and lower panels. This is because of the "chirping" evolution of the frequencies Ω_ϕ and Ω_θ—at late times they are quite a bit larger than they are early on. Many more orbits per unit time are executed late in the inspiral than early.

Adiabatic waveforms: Time domain

- Traditionally suffering from poor accuracy
- Current accuracy comparable to frequency domain

- Equatorial orbits (LMB and Khanna, 2007):





$$\epsilon = 0.5$$

$$p = 5.0M$$

$$a = 0.5M$$

Adiabatic waveforms: Time domain

- Generic orbits (Sundrarajan, Khana, Hughes, Drasco, 2008):

p/M	e	$\theta_{\rm inc}$ (deg)	a/M	θ_d (deg)	h_+ corr.	$h_\times$ corr.
6	0.3	40	0.9	60	0.9978	0.9978
6	0.3	40	0.9	90	0.9976	0.9976
8	0.3	40	0.5	60	0.9898	0.9897
8	0.3	40	0.5	90	0.9910	0.9910
6	0.7	40	0.9	60	0.9898	0.9906
6	0.7	40	0.9	90	0.9889	0.9891
6	0.7	60	0.9	60	0.9905	0.9868
6	0.7	60	0.9	90	0.9895	0.9866
6	0.3	60	0.9	60	0.9961	0.9962
6	0.3	60	0.9	90	0.9950	0.9954
8	0.3	60	0.5	60	0.9906	0.9890
8	0.3	60	0.5	90	0.9884	0.9866

p/M	e	$\theta_{\rm inc}$ (deg)	a/M	θ_d (deg)	h_+ corr.	$h_\times$ corr.
6.472	0.3	0	0.3	30	0.9908	0.9909
6.472	0.3	0	0.3	60	0.9922	0.9922
6.472	0.3	0	0.3	90	0.9930	0.9931
5.768	0.3	0	0.7	30	0.9934	0.9935
5.768	0.3	0	0.7	60	0.9943	0.9944
5.768	0.3	0	0.7	90	0.9948	0.9948
6.472	0.7	0	0.3	30	0.9931	0.9931
6.472	0.7	0	0.3	60	0.9905	0.9906
6.472	0.7	0	0.3	90	0.9923	0.9923
5.768	0.7	0	0.7	30	0.9928	0.9929
5.768	0.7	0	0.7	60	0.9932	0.9930
5.768	0.7	0	0.7	90	0.9920	0.9921

Adiabatic waveforms: Time domain

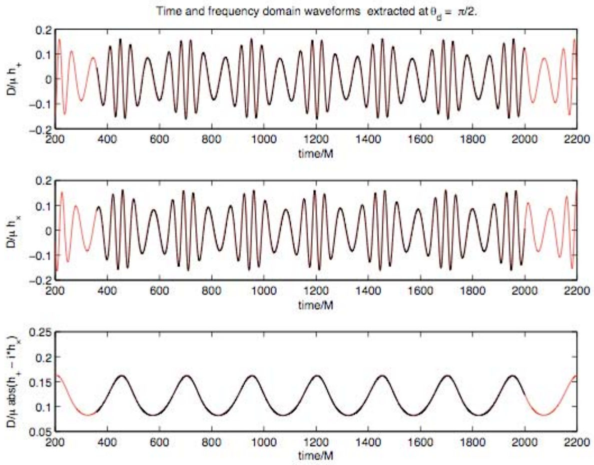


FIG. 1: Comparison of time- and frequency-domain waveforms. We show waves for the $m = 2$ mode from a point particl orbital parameters $p = 6.472M$, $e = 0.3$ and $\theta_{\rm inc} = 0$ orbiting a black hole with spin $a/M = 0.3$. The angle between th axis of the black hole and the line of sight is $\theta_d = \pi/2$. Time-domain results are in black, frequency-domain results in red panel: "plus" polarizations in dimensionless units. Middle: "cross" polarizations. Bottom: Comparison of $|h_+ - ih_\times|$. last quantity gives a good visual measure of the level of agreement between the two waveforms. The correlations betwee two waveforms are 0.9974 (plus) and 0.9975 (cross).

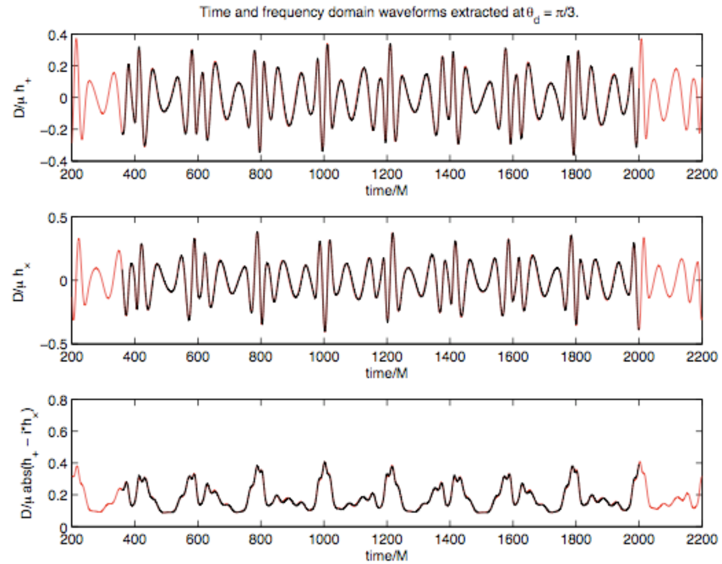


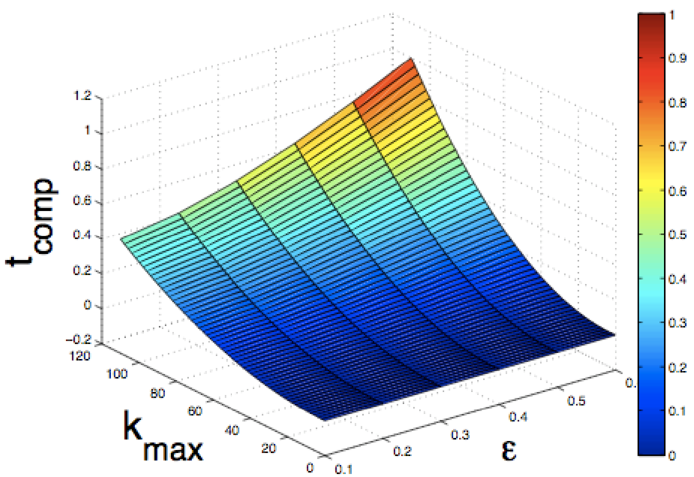
FIG. 2: Comparison of time- and frequency-domain waveforms. Here, we show waves for the $m = 2$ mode for a geodesic with $p = 6M$, $e = 0.3$ and $\theta_{\rm inc} = \pi/3$ about a black hole with spin $a/M = 0.9$; black is time-domain results, red is frequency domain. The correlations in this case are 0.9961 (plus) and 0.9962 (cross).

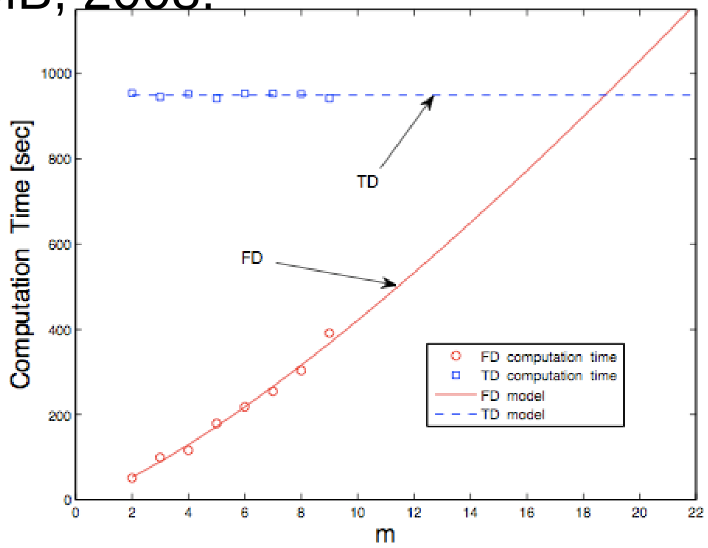
Adiabatic waveforms: Frequency vs. Time domain

- Inclusion of dissipation of the orbit in the time domain is still work in progress

- Which method should EMRI waveforms be calculated with?

Barton, Lazar, Kennefick, Khanna, LMB, 2008:





Metric Perturbations

- Frequency domain: using Mode-Sum Regularization
  - Important test cases (useful for comparisons)
  - Schwarzschild scalar charge circular orbits (LMB; Detweiler, Messaritaki & Whiting)
  - Schwarzschild gravitational SF circular orbits (Barack & Sago)
- Time domain: "*m* regularization"
  - Schwarzschild scalar charge circular orbit (Barack & Goulburn)
  - Schwarzschild gravitational SF circular orbit (Lackeos, Barack, Khanna & LMB, in progress)

Gauge problem

Given the SF, how is it to be used?

$$u^\alpha(\tau) = u^\alpha|_{\rm bg}(\tau) + \frac{1}{\mu}\int_{-\infty}^{\tau} f(\tau')\,d\tau$$

In what gauge is the SF to be written?

Detweiler (2008): Even in Newtonian gravity the SF is ambiguous

$$r_1\Omega^2 = \frac{Gm_2}{(r_1+r_2)^2} \qquad m_1 r_1 = m_2 r_2$$

$$\lim_{m_1\to 0} r_1 + r_2 = r_1.$$

$$r_1\Omega^2 = \frac{Gm_2}{(r_1+r_2)^2} = \frac{Gm_2}{r_1^2}\left(1 - \frac{2m_1}{m_2} + \dots\right)$$

Gauge problem

- Attempts to backreact on the orbit with the SF were made in some very simple cases:
  - Quasi-circular Schwarzschild orbits w/ scalar field SF (LMB, 2003)
  - Quasi-circular Schwarzschild orbits w/ PN gravitational SF and spin-orbit coupling (LMB, 2004): In what gauge is the orbit calculated?

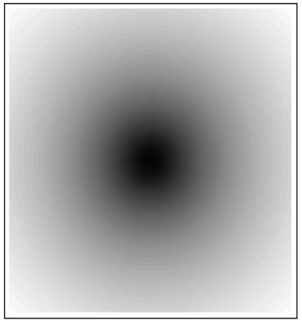
  Focus on gauge-independent quantities: the waveform
- Barack and Ori (2001): "The meaningful description of the gravitational self force must include both f^{α}_{SF} and the metric perturbations $h_{\alpha\beta}$."
- Detweiler (2008): "The value in calculating the self force, in any particular gauge, is to apply it to a question whose answer is related to some physical observable. And a physical observable ought to be independent of the gauge choice."

Gauge problem

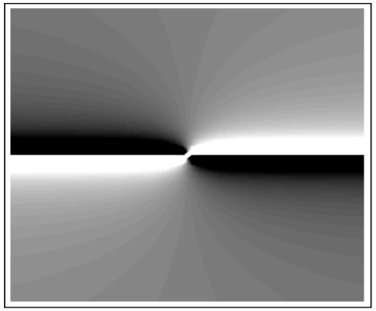
A way to avoid this problem: Instead (or in addition) of a self force, find geodesic motion in a perturbed metric

$$g_{\alpha\beta} = g_{\alpha\beta}|_{BG} + h_{\alpha\beta}$$

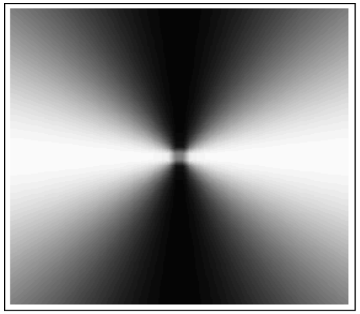
One still needs to calculate $h_{\alpha\beta}$ in some gauge. Which one?



Lorenz gauge



Radiation gauge



RW gauge

Credit: L. Barack

Post-adiabatic expansion: test case (Quasi-circular Schwarzschild orbits)

The EOM are described in full by $\Omega^2 := \frac{M}{r^3} + \sigma(r)$ and $V(r) := \frac{dr}{d\tau}$

We write the EOM with unknown $V(r), \sigma(r)$:

$$VV' - \frac{3MV^2}{r(r-2M)} - (r-2M)\sigma - \frac{1}{\mu(u^t)^2}\left[\left(1-\frac{2M}{r}\right)f_r^{\rm SF} + \frac{V}{1-2M/r}f_t^{\rm SF}\right] = 0$$

$$V\sigma' - \frac{3MV}{r^4} + 2\frac{M/r^3+\sigma}{1-2M/r}\left[\frac{2}{r}V\left(1-\frac{3M}{r}\right) - \frac{f_t^{\rm SF}}{\mu(u^t)^2}\right] - \frac{2(M/r^3+\sigma)^{1/2}}{\mu(u^t)^2 r^2}f_\phi^{\rm SF} = 0$$

$$(u^t)^2 = 1/\left[1-3M/r - r^2\sigma - V^2/(1-2M/r)\right]$$

which we can solve perturbatively

Need for second-order self forces

- Talks by Poisson and by Pound; Rosenthal; Flanagan
- For quasi-circular Schwarzschild orbits (LMB, 2003):

$$\omega^2 = M/r^3 + \sigma(r)$$

$$\sigma_{(1)} = -\frac{r-3M}{\mu r^2} f_r^{(1)}$$

$$\Delta \frac{dN_{\rm cyc}}{d(\ln f)} = -\frac{2}{(2)3\pi}\sqrt{\frac{M}{r}}\left[\frac{3}{2}\frac{r^3\sigma_{(1)}}{MV_{(1)}} + \frac{1}{3}\frac{r^4\sigma'_{(1)}}{MV_{(1)}} - \frac{V_{(2)}}{V_{(1)}^2}\right]$$

$$V_{(1)} = \frac{2r}{\mu M}\frac{r-3M}{r-6M}\left[\left(\frac{M}{r}\right)^{\frac{1}{2}}\left(1-\frac{2M}{r}\right) f_\phi^{(1)} + M f_t^{(1)}\right]$$

$$\begin{aligned} V^{(2)} &= \frac{r(r-3M)}{\mu^2 M^2 (r-6M)^2}\left[2\left(\frac{M}{r}\right)^{\frac{1}{2}} f_\phi^{(1)} f_r^{(1)\prime} r(r-2M)^2 \right. \\ &\times (r-3M) + \left(\frac{M}{r}\right)^{\frac{1}{2}} f_\phi^{(1)} f_r^{(1)} (5r-6M)(r-2M) \\ &\times (r-3M) + 2M f_t^{(1)} f_r^{(1)\prime} r^2 (r-2M)(r-3M) \\ &+ 4M f_t^{(1)} f_r^{(1)} r^2 (r-3M) + 2\mu M^2 f_t^{(2)} (r-6M) \\ &+ \left. 2\mu \left(\frac{M}{r}\right)^{\frac{3}{2}} f_\phi^{(2)} (r-2M)(r-6M)\right] . \end{aligned}$$

$$\begin{aligned} \sigma^{(2)} &= -\frac{r-3M}{\mu r^2}\left[f_r^{(2)} + r^2 \frac{V^{(1)} f_t^{(1)}}{(r-2M)^2}\right] \\ &+ \frac{r}{\mu}\sigma^{(1)} f_r^{(1)} - \frac{3M {V^{(1)}}^2}{r(r-2M)^2} + \frac{V^{(1)} V^{(1)\prime}}{r-2M} \end{aligned}$$

Two timescale / post-adiabatic expansion

- Use the two different time scales: $\frac{\tau_{\rm dyn}}{\tau_{\rm RR}} \sim \frac{\mu}{M} := \epsilon \ll 1$
- Expand the equation of motion:

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = \varepsilon a^{(1)\,\nu} + \varepsilon^2 a^{(2)\,\nu} + O(\varepsilon^3)$$

$$a = \underbrace{a^{(0)}}_{=0} + [\underbrace{a_D^{(1)}}_{\substack{\text{phase to}\\ O(\epsilon^{-1})}} + \underbrace{a_C^{(1)}]\,\epsilon + [a_D^{(2)}}_{\text{phase to } O(\epsilon^0)} + \underbrace{a_C^{(2)}]\,\epsilon^2 + \cdots}_{\text{phase to } O(\epsilon)}$$

- Action-angle variables and adiabatic invariants
- Hinderer and Flanagan (2008): applied for Kerr orbits

Two timescale analysis of extreme mass ratio inspirals in Kerr. I. Orbital Motion

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(Dated: draft of June 12, 2008; printed June 17, 2008 at 21:35)

Inspirals of stellar mass compact objects into massive black holes are an important source for future gravitational wave detectors such as Advanced LIGO and LISA. Detection of these sources and extracting information from the signal relies on accurate theoretical models of the binary dynamics. We cast the equations describing binary inspiral in the extreme mass ratio limit in terms of action angle variables, and derive properties of general solutions using a two-timescale expansion. This provides a rigorous derivation of the prescription for computing the leading order orbital motion. As shown by Mino, this leading order or adiabatic motion requires only knowledge of the orbit-averaged, dissipative piece of the self force. The two timescale method also gives a framework for calculating the post-adiabatic corrections. For circular and for equatorial orbits, the leading order corrections are suppressed by one power of the mass ratio, and give rise to phase errors of order unity over a complete inspiral through the relativistic regime. These post-1-adiabatic corrections are generated by the fluctuating, dissipative piece of the first order self force, by the conservative piece of the first order self force, and by the orbit-averaged, dissipative piece of the second order self force. We also sketch a two-timescale expansion of the Einstein equation, and deduce an analytic formula for the leading order, adiabatic gravitational waveforms generated by an inspiral.

Weyl scalars

- Metric reconstruction
- Treatment of the source: Detweiler-Whiting approach
- Non-radiative modes
- Kinnersley tetrad