Matched Expansion Method for the Calculation of the Self-Force

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Motivation

- Calculate the motion of Extreme Mass Ratio Inspirals
- Computing the gravitational self-force on a particle orbiting a black hole
So far looking at scalar SF for simplicity - method works for gravitational case with few changes.

Start with Quinn expression for scalar SF

\[ f^\mu(x) = q^2 \left( \frac{1}{3} (\dot{a}^\mu - a^2 u^\mu) + \frac{1}{6} (R^{\mu\beta} u_\beta + R_{\beta\gamma} u^\beta u^\gamma u^\mu) - \frac{1}{12} (1 - 6\xi) R \right) u^a + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^\mu G_{ret}(x, x') d\tau \]

Mainly interested in calculating the tail integral of the derivative of the retarded Green's function over the past world-line of the particle
Matched Expansion

- Anderson & Wiseman (CQG 22 (2005))
- Select point $\Delta \tau$ along the world-line
- Separate tail integral into two regimes:
  - 1. Quasilocal region from the recent past
  - 2. Contribution from “distant” past
Matched Expansion

- Calculate Green’s function in each region separately
- Analytically in quasilocal region
- Numerically in “distant” past region
- Match them up at the point $\Delta \tau$

\[
q^2 \int_{\tau - \Delta \tau}^{\tau} \nabla^a G_{\text{ret}} d\tau'
\]

Quasilocal integral back $\Delta \tau$ along the world-line

\[
q^2 \int_{-\infty}^{\tau - \Delta \tau} \nabla^a G_{\text{ret}} d\tau'
\]

Integral outside quasilocal region

Current location of the particle - $x(\tau)$

Matching point - $x'(\Delta \tau)$

Boundary of causal domain where Hadamard form is valid
Quasilocal Contribution
Hadamard form

- Provided \( x \) and \( x' \) are sufficiently "close" together, the Hadamard Form of the Green’s function can be used

\[
G_{ret}(x, x') = \theta_-(x, x') \{ U(x, x') \delta(\sigma(x, x')) - V(x, x')\theta(-\sigma(x, x')) \}
\]

- Only part with \( V(x, x') \) contributes to the self-force

\[
f^a_{QL} = -q^2 \int_{\tau - \Delta \tau}^{\tau} \nabla^a V(x, x') d\tau'
\]

- The problem is now to calculate \( V(x, x') \)
Calculating $V(x,x')$

- Since $x$ and $x'$ close, write $V(x,x')$ as a series in $\sigma$:

$$V(x,x') = \sum_{n=0}^{\infty} V_n(x,x') \sigma^n(x,x')$$

- The coefficients $V_n(x,x')$ are related by a set of recursion relations

$$(n + 1) (2n + 4) V_{n+1} + 2 (n + 1) V_{n+1;\mu} \sigma^{;\mu}$$

$$- 2 (n + 1) V_{n+1} \Delta^{-1/2} \Delta^{1/2} \sigma^{;\mu} + \Box x V_n = 0$$

- Along with the boundary condition

$$2V_0 + 2V_0;\mu \sigma^{;\mu} - 2V_0 \Delta^{-1/2} \Delta^{1/2} \sigma^{;\mu} = -\Box x \Delta^{1/2}$$
Calculating $V(x, x')$

- Write $V_n(x, x')$ as a (covariant) series expansion about the particle’s position:

$$V_n(x, x') = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} v_{n\alpha_1...\alpha_p}(x) \sigma;^{\alpha_1}(x, x') \ldots \sigma;^{\alpha_p}(x, x')$$

- Also need to compute series expansions of the other terms that appear in the recursion relations

$$\Delta^{1/2}, \Box \Delta^{1/2}, \Delta^{-1/2} \Delta^{1/2}_{\mu} \sigma;^{\mu}, \sigma;^{\mu\nu}, g_{\alpha\beta';\gamma}$$

- Match up powers of $\sigma;^{\alpha}$ to get the expression for the coefficients in the expansion of $V_n(x, x')$. 
Calculating $V(x, x')$

- Coefficients are geometric quantities at $x$ (i.e., polynomials in $R_{abcd}$, $R_{ab}$, $R$ and their derivatives).

- Traditionally, recursive methods of DeWitt are used to calculate these expansions - painful after the first couple of orders (Christensen, MathTensor).
Calculating $V(x,x')$

- There is a much better way to compute these expansions
- Based on the non-recursive algorithm of Avramidi
- Implemented (by hand) for a scalar field by Décanini and Folacci (Phys. Rev. D 73, 044027) to calculate $V(x,x')$ to 4th order for scalar case.
- Can be extended one order (relatively) easily by symmetry of Green’s function
Calculating $V(x,x')$

- We’ve modified the Avramidi approach to a recursive form and implemented it in Mathematica.
- Also able to expand the approach to calculate expansions of other fundamental bitensors such as $g_{\alpha\beta';\gamma}$.
- Works well - easily calculates up to (and beyond) 20th order in $\sigma^{;\alpha}$ without too much difficulty.
- Useful beyond self-force calculations.
Calculating $V(x, x')$

- $V(x, x')$ now looks like:

$$V(x, x') = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} u_{\alpha_1...\alpha_p}(x) \sigma^{;\alpha_1}(x, x') \ldots \sigma^{;\alpha_p}(x, x')$$

- For time-like geodesics,

$$\sigma^{;a} = - (\tau - \tau') u^a$$

- Coefficients of expansion are all local quantities at $x$
Quasilocal Self-Force

Coefficients of expansion are all local quantities at \( x \), integral is at \( x' \)

\[
f^a_{\text{QL}} = -q^2 \int_{\tau - \Delta \tau}^{\tau} \nabla^a V(x, x') d\tau'
\]

So the integration is trivial and we get the result

\[
f^a_{\text{QL}} = -q^2 \left( A^a \Delta \tau^1 + \frac{1}{2} B^a \Delta \tau^2 + \frac{1}{3} C^a \Delta \tau^3 + \frac{1}{4} D^a \Delta \tau^4 + \frac{1}{5} E^a \Delta \tau^5 + O(\Delta \tau^6) \right)
\]
Quasilocal Self-Force

As an example (see PRD 77 104002 (2008)) for more detail) of the expressions we get, in vacuum spacetime (up to third order is identically 0),

\[
D^\mu = \left( -\frac{2}{525} C^\rho_{(a|\sigma|b|} \square C^\sigma_{c|\rho|d)} - \frac{2}{105} C^\rho_{\sigma\tau (a \big| \rho \sigma \tau \big| b ; c d)} - \frac{1}{280} C^\rho_{(a|\sigma|b|} \big| \tau \big| C^\sigma_{c|\rho|d); \tau} \\
- \frac{1}{56} C^\rho_{\sigma\tau (a;b|\rho \sigma \tau | c; d) - \frac{2}{1575} C^\rho_{\sigma\tau \kappa C_\rho(a|\tau|b|C_\sigma|\kappa|d) - \frac{2}{525} C^\rho_{\rho\tau \kappa} (a|C_\rho|\sigma|b|C_\sigma|\kappa|d)} \right) g^{\mu a} u_b u_c u_d

For e.g. circular orbit in Schwarzschild

\[
f_{QL}^r = -\frac{3q^2 M^2 (r - 2M)(53M^3 + 54r M^2 - 81Mr^2 + 20r^3)}{11200 (r - 3M)^2 r^{11}} \Delta \tau^5
\]
\[
f_{QL}^\phi = \frac{9q^2 M^2 (r - 2M)(3r - 5M)}{2240r^{10}(r - 3M)} \sqrt{\frac{M}{r - 3M}} \Delta \tau^4
\]
\[
f_{QL}^t = -\frac{3q^2 M^2 (r - 2M)(5r - M)}{2240r^{9}(r - 3M)} \sqrt{\frac{r}{r - 3}} \Delta \tau^4
\]
Where can we put $\Delta \tau$?

- Hadamard form of Green’s function only valid within a convex normal neighborhood, i.e. $x$ and $x'$ must be in a domain where they are separated by a unique geodesic within the domain.

- This puts an upper limit on how big $\Delta \tau$ can be.
Convex Normal Neighborhood

- Van Vleck determinant might give a good idea of where the region of validity of the Hadamard form ends.

- It blows up when neighboring geodesics from a point converge back to a point.
Convex Normal Neighborhood

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Convex Normal Neighborhood

- Suggestion by Anderson & Wiseman
- Consider a time-like circular geodesic intersecting a null geodesic coming from $x'$
- Calculate the proper time when they re-intersect at $x$
Truncation Error

- We are truncating our series, so would like to estimate the truncation error as it affects how far we can push $\Delta \tau$

- Best we can do is local truncation error

\[
\epsilon \equiv \frac{f_{QL}^a [n]}{\sum_{i=0}^{n} f_{QL}^a [i]}
\]

Local truncation error at $O(\Delta \tau^7)$ for circular geodesic motion in Schwarzschild at $r=6M, 10M, 20M, 100M$. 
Conclusions, Future

- Quasilocal contribution to scalar self-force to 5th order in expansion in $\Delta \tau$ (PRD 77 104002 (2008))

- Recently been able to take many of the necessary expansions to extremely high order (20th order in $\sigma; ^{\alpha}$ without too much difficulty) - hopefully this will give good accuracy a long way out towards the boundary of the normal neighborhood

- Gravitational case shouldn’t pose many problems

- What’s a good measure of the domain of validity of Hadamard form? Van Vleck? Intersecting null and time-like geodesics?
Contribution From “Distant” Past
“Distant” contribution

- How can we calculate the distant part of the retarded Green’s function?
- What is the (singularity-) structure of the tail part?
“Distant” contribution

Teukolsky(1973): Separation of variables of spin-field perturbations in Kerr

\[
\left[ \frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \right) + (a\omega)^2 (x^2 - 1) - \frac{(m + hx)^2}{1 - x^2} + \lambda + 2a\omega (m - hx) + h \right] h S_{lm\omega} = 0
\]

\[
\left[ \Delta^{-h} \frac{d}{dr} \left( \Delta^{h+1} \frac{d}{dr} \right) + \frac{K^2 - 2ih(r - M)K}{\Delta} + 4i\omega r - \lambda + a\omega (2m - a\omega) \right] h R_{lm\omega} = 0
\]

where \( x \equiv \cos \theta \), \( \Delta \equiv r^2 - 2Mr + a^2 \), \( K \equiv (r^2 + a^2)\omega - am \) and \( \lambda \):eigenvalue.

h: helicity
“Distant” contribution

In the scalar (h=0) case:

\[
G^{\text{ret}}(x, x') = \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} i \frac{\omega}{|\omega|} \sum_{lm} e^{+im(\phi-\phi')} h S_{lm\omega}(\theta) h S_{lm\omega}(\theta') h R_{lm\omega}^{\text{up}}(r>) h R_{lm\omega}^{\text{in}}(r<) lm\omega
\]

where b. c. :

\[
h R_{lm\omega}^{\text{up}}(r) \sim e^{i\omega r_*} \text{ at } r \sim +\infty, \quad h R_{lm\omega}^{\text{in}}(r) \sim e^{-i\omega r_*} \text{ at } r \sim r_+.
\]

\[
\frac{dr_*}{dr} = \frac{(r^2 + a^2)}{\Delta}
\]

Use of open Simpson’s rule for ω-integration
Results

Scalar case, $a=0$, $\Phi=\Phi'$, $\theta=\theta'={\pi}/2$, $r'\sim10.3M$

Movie 1

Director: M.C.
Actors $G^{\text{ret}}(x,x')$ at $r=r'\sim10.3M$ as function of $t-t'>0$
Title: You think it’s dead and then...
Green’s function as a function of $t$ and $r^*$
Movie 2

Director: Kirill Ignatiev

Actors:

Title:
Check

- **Analytic - Wiseman PRD61(2000):**

\[
G_{\omega=0}(x, x') = \left( 1 - \frac{M^2}{(r' - M + \sqrt{r'^2 - 2r'M})^2} \right) \\
\left( (r - M)^2 - 2(r - M)(r' - M) \cos \gamma + (r' - M)^2 - M^2 \sin^2 \gamma \right)^{-1/2}
\]

- **Numerical: two integrations using open Simpsons rule:**

\[
G_{\omega=0}(x, x') = \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} G_{\omega}(\vec{x}, \vec{x}')
\]
Questions & Improvements

* Q1: Determine the convergence in $l$ and in $\omega$

* I1: Use WKB (and/or Green-Liouville method) for finding high-$\omega$, high-$l$ asymptotics in order to improve convergence.....it kind of defeats the purpose though!

* Q2: Green function has a branch point at $\omega=0$, value which we have interpolated - what integration contour in the complex $\omega$-plane are we choosing?

* I2: Alternatively, use of Leaver method of integration over the complex $\omega$-plane.

* Three distinct contributions:

* (1) QNMs, (2) along branch cut, (3) large complex $\omega$. 
Questions & Improvements

Q3: What determines the scale of the oscillations of $G(x,x')$ in-between the wavefronts?

I3: Look for an analytic approximation to the value of $G(x,x')$ at the appearance of the 2nd (and following) wavefront(s), i.e., value of $U(x,x')$. 