Matched Expansion Method for the Calculation of the Self-Force

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Motivation

- Calculate the motion of Extreme Mass Ratio Inspirals
- Computing the gravitational self-force on a particle orbiting a black hole

MiSaTaQuWa Self-Force

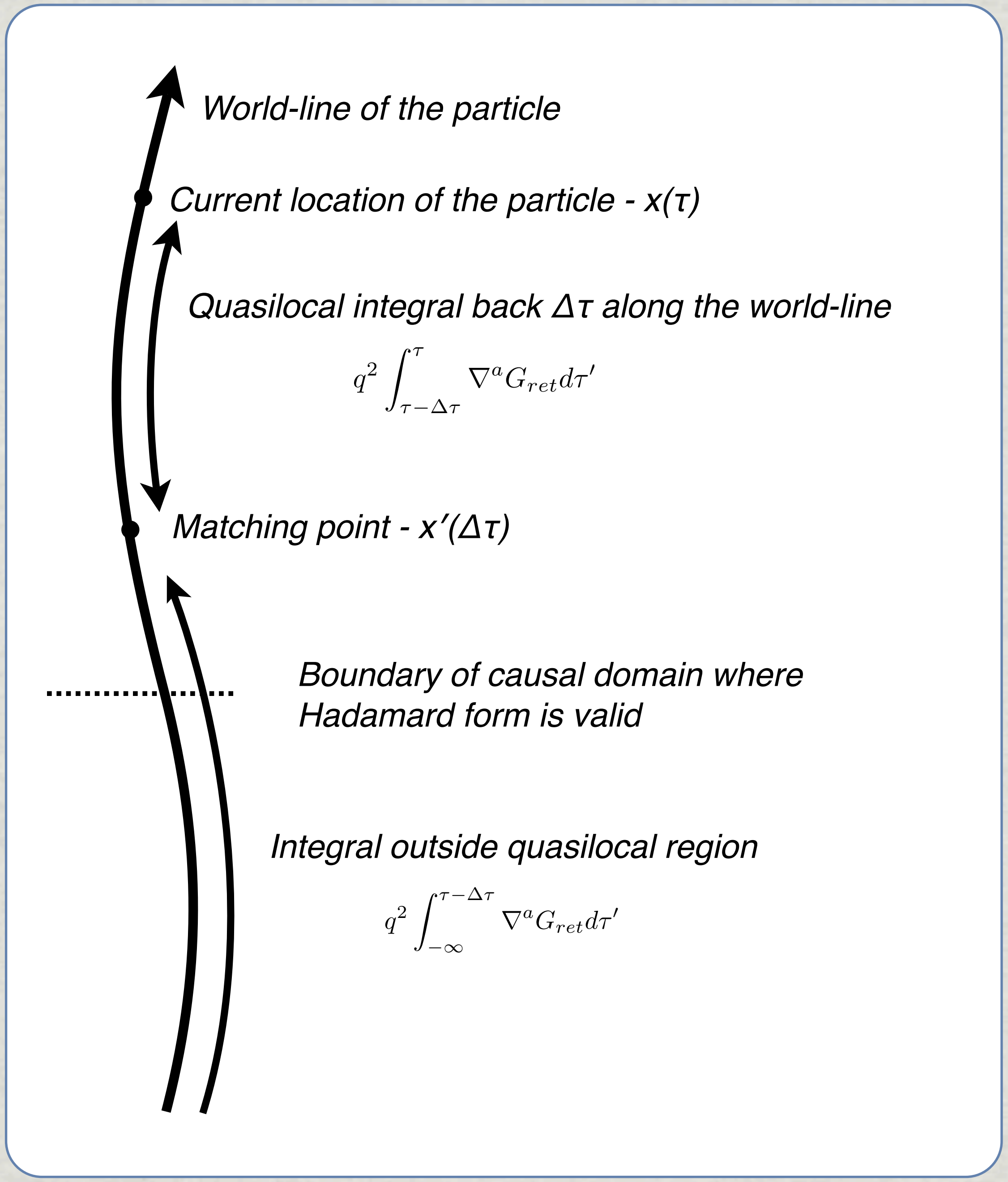
- So far looking at scalar SF for simplicity - method works for gravitational case with few changes.
- Start with Quinn expression for scalar SF

$$f^{\mu}(x) \quad = \quad q^2\left(\frac{1}{3}\left(\dot{a}^{\mu}-a^2u^{\mu}\right)+\frac{1}{6}\left(R^{\mu\beta}u_{\beta}+R_{\beta\gamma}u^{\beta}u^{\gamma}u^{\mu}\right)-\frac{1}{12}\left(1-6\xi\right)R\right)u^a+\lim_{\epsilon\to 0}q^2\int_{-\infty}^{\tau-\epsilon}\nabla^{\mu}G_{ret}(x,x')d\tau$$

- Mainly interested in calculating the tail integral of the derivative of the retarded Green's function over the past world-line of the particle

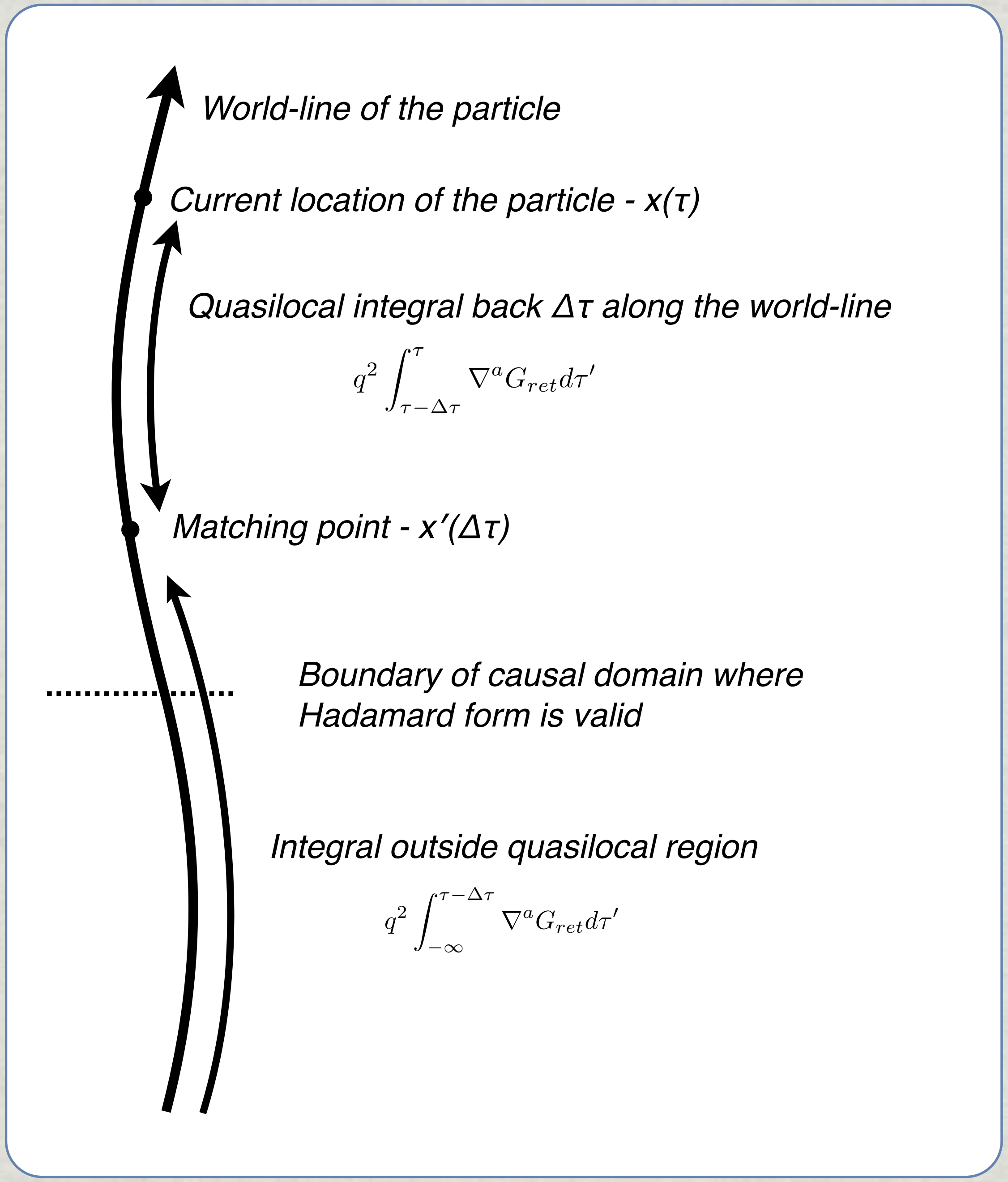
Matched Expansion

- Anderson & Wiseman (CQG 22 (2005))
- Select point Δτ along the world-line
- Separate tail integral into two regimes:
- 1. Quasilocal region from the recent past
- 2. Contribution from "distant" past



Matched Expansion

- Calculate Green's function in each region separately
- Analytically in quasilocal region
- Numerically in "distant" past region
- Match them up at the point Δτ



Quasilocal Contribution

Hadamard form

- Provided *x* and *x′* are sufficiently “close” together, the Hadamard Form of the Green’s function can be used

$$G_{ret}(x,x') = \theta_-(x,x')\left\{U(x,x')\delta\left(\sigma(x,x')\right) - V(x,x')\theta\left(-\sigma(x,x')\right)\right\}$$

- Only part with *V(x,x′)* contributes to the self-force

$$f^a_{\mathrm{QL}} = -q^2 \int_{\tau-\Delta\tau}^{\tau} \nabla^a V(x,x')d\tau'$$

- The problem is now to calculate *V(x,x′)*

Calculating *V(x,x′)*

- Since *x* and *x′* close, write *V(x,x′)* as a series in σ:

$$V(x,x') = \sum_{n=0}^{\infty} V_n(x,x')\sigma^n(x,x')$$

- The coefficients $V_n(x,x')$ are related by a set of recursion relations

$$(n+1)(2n+4)V_{n+1} + 2(n+1)V_{n+1;\mu}\sigma^{;\mu} - 2(n+1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} + \Box_x V_n = 0$$

- Along with the boundary condition

$$2V_0 + 2V_{0;\mu}\sigma^{;\mu} - 2V_0\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} = -\Box_x\Delta^{1/2}$$

Calculating *V(x,x′)*

- Write $V_n(x,x')$ as a (covariant) series expansion about the particle's position:

$$V_n(x,x') = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} v_{n\alpha_1\ldots\alpha_p}(x)\sigma^{;\alpha_1}(x,x')\ldots\sigma^{;\alpha_p}(x,x')$$

- Also need to compute series expansions of the other terms that appear in the recursion relations

$$\Delta^{1/2},\ \Box\Delta^{1/2},\ \Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu},\ \sigma^{;\mu\nu},\ g_{\alpha\beta';\gamma}$$

- Match up powers of $\sigma^{;\alpha}$ to get the expression for the coefficients in the expansion of $V_n(x,x')$.

Calculating $V(x,x')$

- Coefficients are geometric quantities at x (i.e.. polynomials in R_{abcd} , R_{ab} , R and their derivatives).
- Traditionally, recursive methods of DeWitt are used to calculate these expansions - painful after the first couple of orders (Christensen, MathTensor)

Calculating $V(x,x')$

- There is a much better way to compute these expansions
- Based on the non-recursive algorithm of Avramidi
- Implemented (by hand) for a scalar field by Décanini and Folacci (Phys. Rev. D 73, 044027) to calculate $V(x,x')$ to 4th order for scalar case.
- Can be extended one order (relatively) easily by symmetry of Green's function

Calculating *V(x,x′)*

- We've modified the Avramidi approach to a recursive form and implemented it in Mathematica
- Also able to expand the approach to calculate expansions of other fundamental bitensors such as

$$g_{\alpha\beta';\gamma}$$

- Works well - easily calculates up to (and beyond) 20th order in $\sigma^{;\alpha}$ without too much difficulty
- Useful beyond self-force calculations

Calculating *V(x,x′)*

- *V(x,x′)* now looks like:

$$V(x,x') = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} v_{\alpha_1 \ldots \alpha_p}(x) \sigma^{;\alpha_1}(x,x') \ldots \sigma^{;\alpha_p}(x,x')$$

- For time-like geodesics,

$$\sigma^{;a} = -(\tau - \tau')\, u^a$$

- Coefficients of expansion are all local quantities at *x*

Quasilocal Self-Force

- Coefficients of expansion are all local quantities at *x*, integral is at *x*′

$$f^a_{\mathrm{QL}} = -q^2 \int_{\tau-\Delta\tau}^{\tau} \nabla^a V(x,x')d\tau'$$

- So the integration is trivial and we get the result

$$f^a_{\mathrm{QL}} = -q^2 \left(A^a \Delta\tau^1 + \frac{1}{2} B^a \Delta\tau^2 + \frac{1}{3} C^a \Delta\tau^3 + \frac{1}{4} D^a \Delta\tau^4 + \frac{1}{5} E^a \Delta\tau^5 + O\left(\Delta\tau^6\right) \right)$$

Quasilocal Self-Force

- As an example (see PRD 77 104002 (2008)) for more detail) of the expressions we get, in vacuum spacetime (up to third order is identically 0),

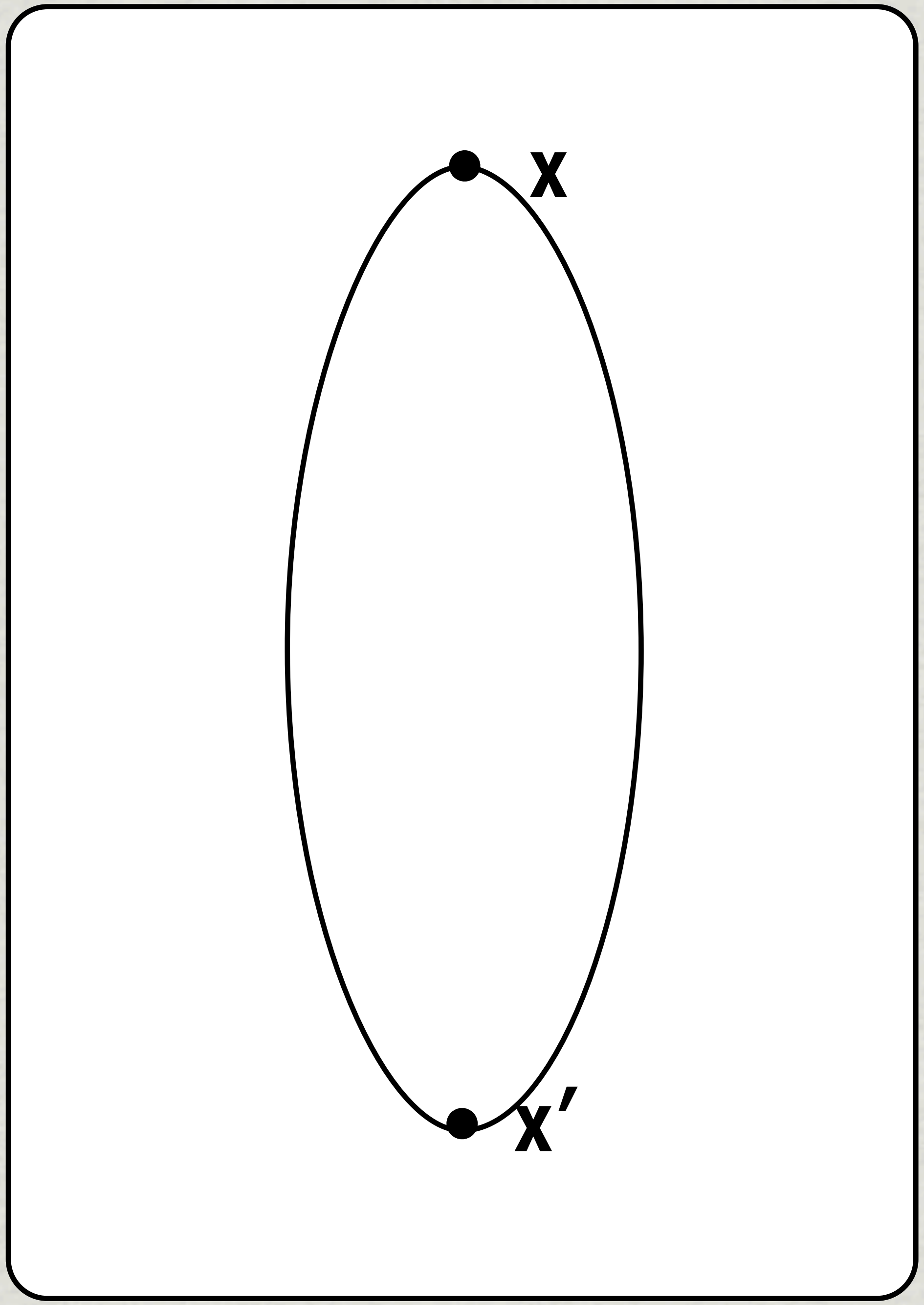
$$
\begin{aligned}
D^{\mu} \;=\;& \Big(-\frac{2}{525} C^{\rho}{}_{(a|\sigma|b} \Box C^{\sigma}{}_{c|\rho|d)} - \frac{2}{105} C^{\rho\sigma\tau}{}_{(a} C_{|\rho\sigma\tau|b;cd)} - \frac{1}{280} C^{\rho}{}_{(a|\sigma|b}{}^{;\tau} C^{\sigma}{}_{c|\rho|d);\tau} \\
& - \frac{1}{56} C^{\rho\sigma\tau}{}_{(a;b} C_{|\rho\sigma\tau|c;d)} - \frac{2}{1575} C^{\rho\sigma\tau\kappa} C_{\rho(a|\tau|b} C_{|\sigma|c|\kappa|d)} - \frac{2}{525} C^{\rho\kappa\tau}{}_{(a} C_{|\rho\tau|}{}^{\sigma}{}_{b} C_{|\sigma|c|\kappa|d)} \\
& - \frac{8}{1575} C^{\rho\kappa\tau}{}_{(a} C_{|\rho|}{}^{\sigma}{}_{|\tau|b} C_{|\sigma|c|\kappa|d)} - \frac{4}{1575} C^{\rho\tau\kappa}{}_{(a} C_{|\rho\tau|}{}^{\sigma}{}_{b} C_{|\sigma|c|\kappa|d)}\Big) g^{\mu a} u^b u^c u^d
\end{aligned}
$$

- For e.g. circular orbit in Schwarzschild

$$
\begin{aligned}
f^{r}_{\mathrm{QL}} &= -\frac{3q^2M^2\,(r-2M)\left(53M^3+54rM^2-81Mr^2+20r^3\right)}{11200\,(r-3M)^2\,r^{11}}\Delta\tau^5 \\
f^{\phi}_{\mathrm{QL}} &= \frac{9q^2M^2\,(r-2M)\,(3r-5M)}{2240r^{10}\,(r-3M)}\sqrt{\frac{M}{r-3M}}\Delta\tau^4 \\
f^{t}_{\mathrm{QL}} &= -\frac{3q^2M^2\,(r-2M)\,(5r-M)}{2240r^9\,(r-3M)}\sqrt{\frac{r}{r-3}}\Delta\tau^4
\end{aligned}
$$

Where can we put $\Delta\tau$?

- Hadamard form of Green's function only valid within a convex normal neighborhood, i.e x and x' must be in a domain where they are separated by a unique geodesic within the domain
- This puts an upper limit on how big $\Delta\tau$ can be



Convex Normal Neighborhood

- Van Vleck determinant might give a good idea of where the region of validity of the Hadamard form ends
- It blows up when when neighboring geodesics from a point converge back to a point

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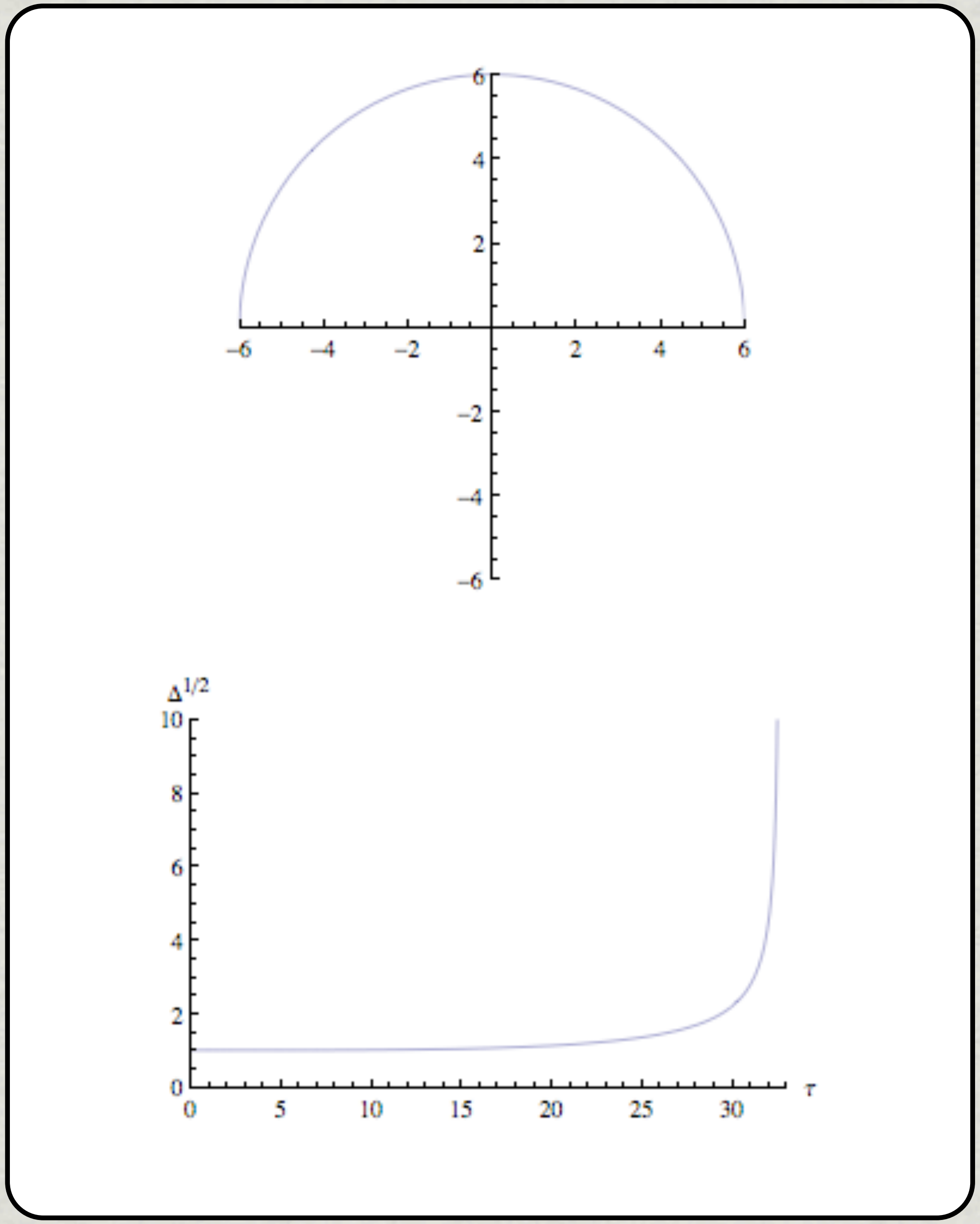
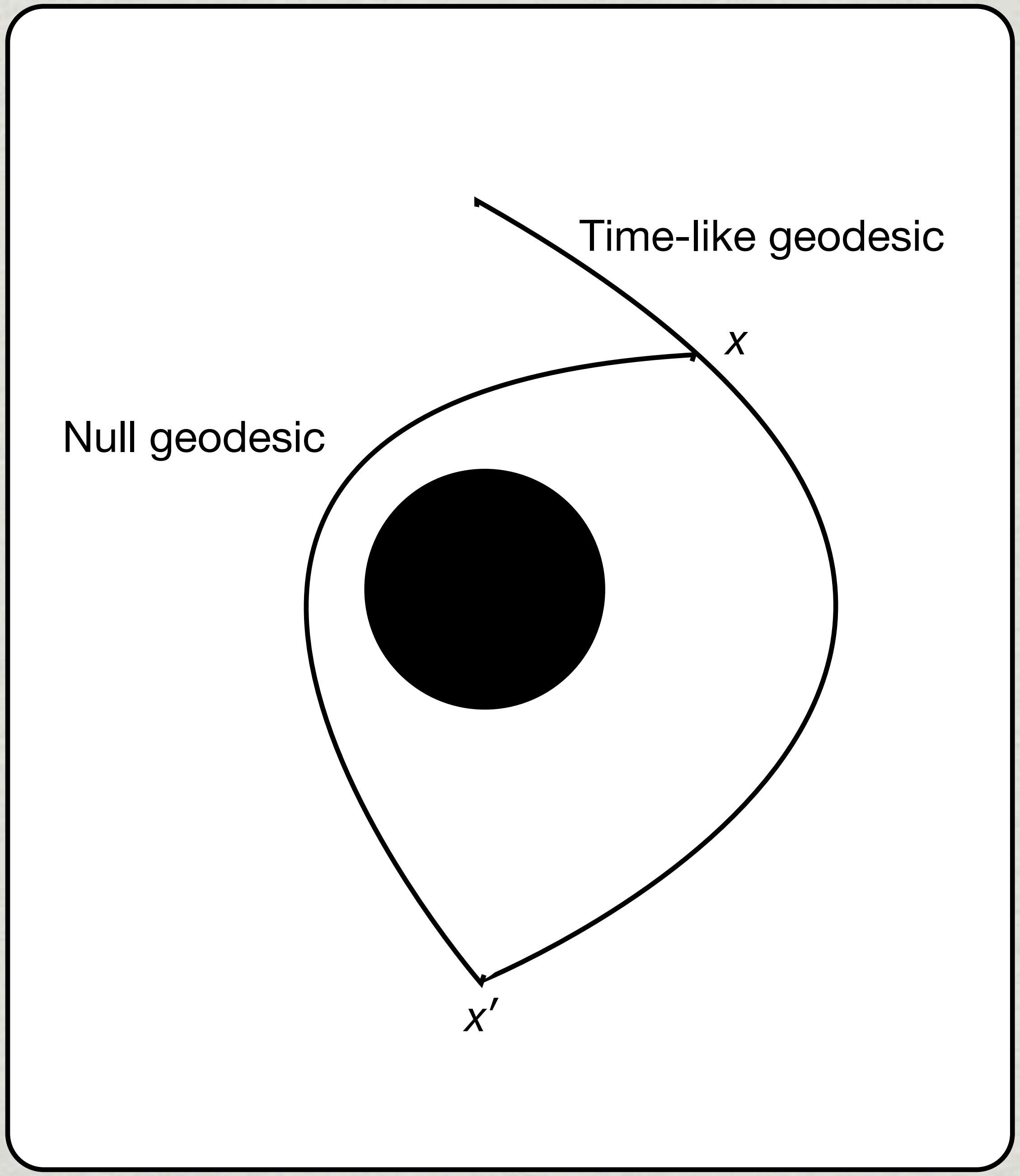


Convex Normal Neighborhood

- Suggestion by Anderson & Wiseman
- Consider a time-like circular geodesic intersecting a null geodesic coming from x'
- Calculate the proper time when they re-intersect at x

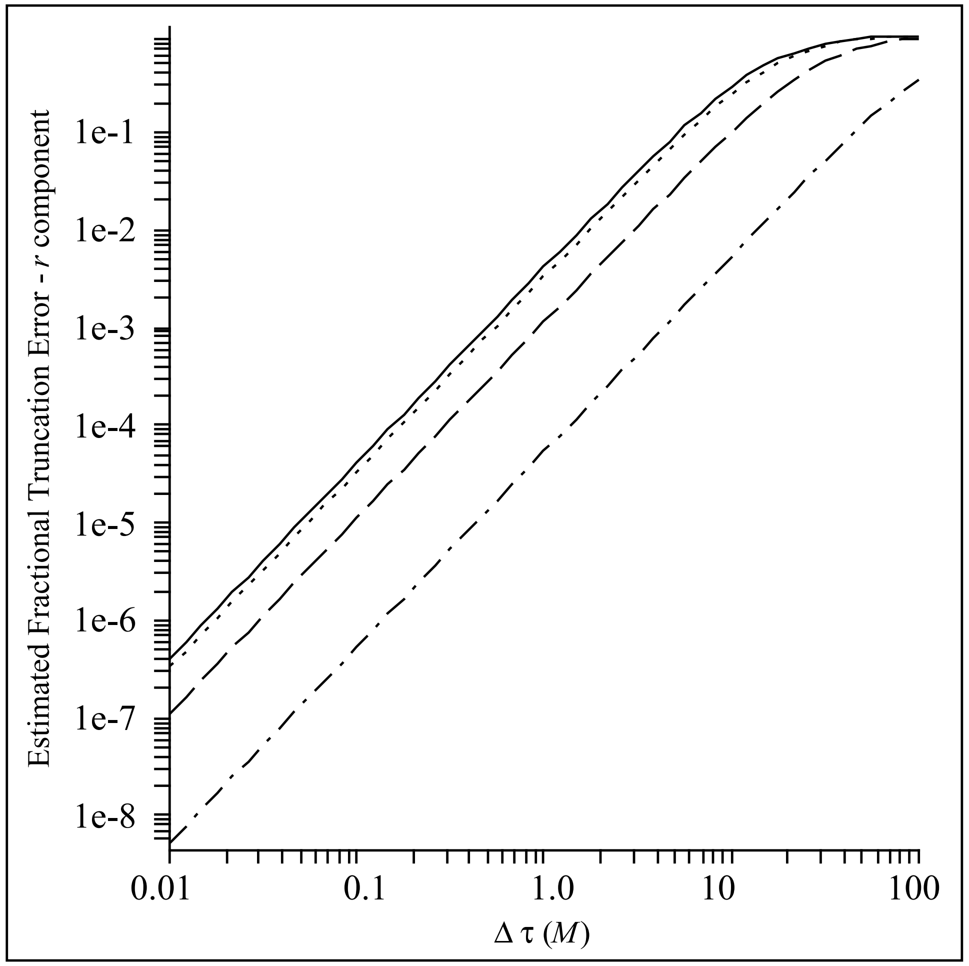


Truncation Error

- We are truncating our series, so would like to estimate the truncation error as it affects how far we can push Δτ

- Best we can do is local truncation error

$$\epsilon \equiv \frac{f^a_{\mathrm{QL}}\left[n\right]}{\sum_{i=0}^{n} f^a_{\mathrm{QL}}\left[i\right]}$$



Local truncation error at O(Δτ^7) for circular geodesic motion in Schwarzschild at r=6M, 10M, 20M,100M.

Conclusions, Future

- Quasilocal contribution to scalar self-force to 5th order in expansion in $\Delta\tau$ (PRD 77 104002 (2008))
- Recently been able to take many of the necessary expansions to extremely high order (20th order in $\sigma^{;\alpha}$ without too much difficulty) - hopefully this will give good accuracy a long way out towards the boundary of the normal neighborhood
- Gravitational case shouldn't pose many problems
- What's a good measure of the domain of validity of Hadamard form? Van Vleck? Intersecting null and time-like geodesics?

Contribution From “Distant” Past

“Distant” contribution

- How can we calculate the distant part of the retarded Green’s function?
- What is the (singularity-) structure of the tail part?

“Distant” contribution

- Teukolsky(1973): Separation of variables of spin-field perturbations in Kerr

$$\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right)+(a\omega)^2(x^2-1)-\frac{(m+hx)^2}{1-x^2}+\lambda+2a\omega(m-hx)+h\right]{}_hS_{lm\omega}=0$$

$$\left[\Delta^{-h}\frac{d}{dr}\left(\Delta^{h+1}\frac{d}{dr}\right)+\frac{K^2-2ih(r-M)K}{\Delta}+4ih\omega r-\lambda+a\omega(2m-a\omega)\right]{}_hR_{lm\omega}=0$$

where $x\equiv\cos\theta$, $\Delta\equiv r^2-2Mr+a^2$, $K\equiv(r^2+a^2)\omega-am$ and λ:eigenvalue.
h:helicity

“Distant” contribution

- In the scalar (h=0) case:

$$G^{\rm ret}(x,x') = \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} \underbrace{i\frac{\omega}{|\omega|}\sum_{lm} e^{+im(\phi-\phi')}{}_hS_{lm\omega}(\theta){}_hS_{lm\omega}(\theta'){}_hR^{\rm up}_{lm\omega}(r_>){}_hR^{\rm in}(r_<)_{lm\omega}}_{G_\omega(\vec{x},\vec{x}')}$$

- where b. c. :

$${}_hR^{\rm up}_{lm\omega}(r) \sim e^{i\omega r_*} \text{ at } r \sim +\infty, \qquad {}_hR^{\rm in}_{lm\omega}(r) \sim e^{-i\omega r_*} \text{ at } r \sim r_+.$$

$$\frac{dr_*}{dr} = \frac{(r^2+a^2)}{\Delta}$$

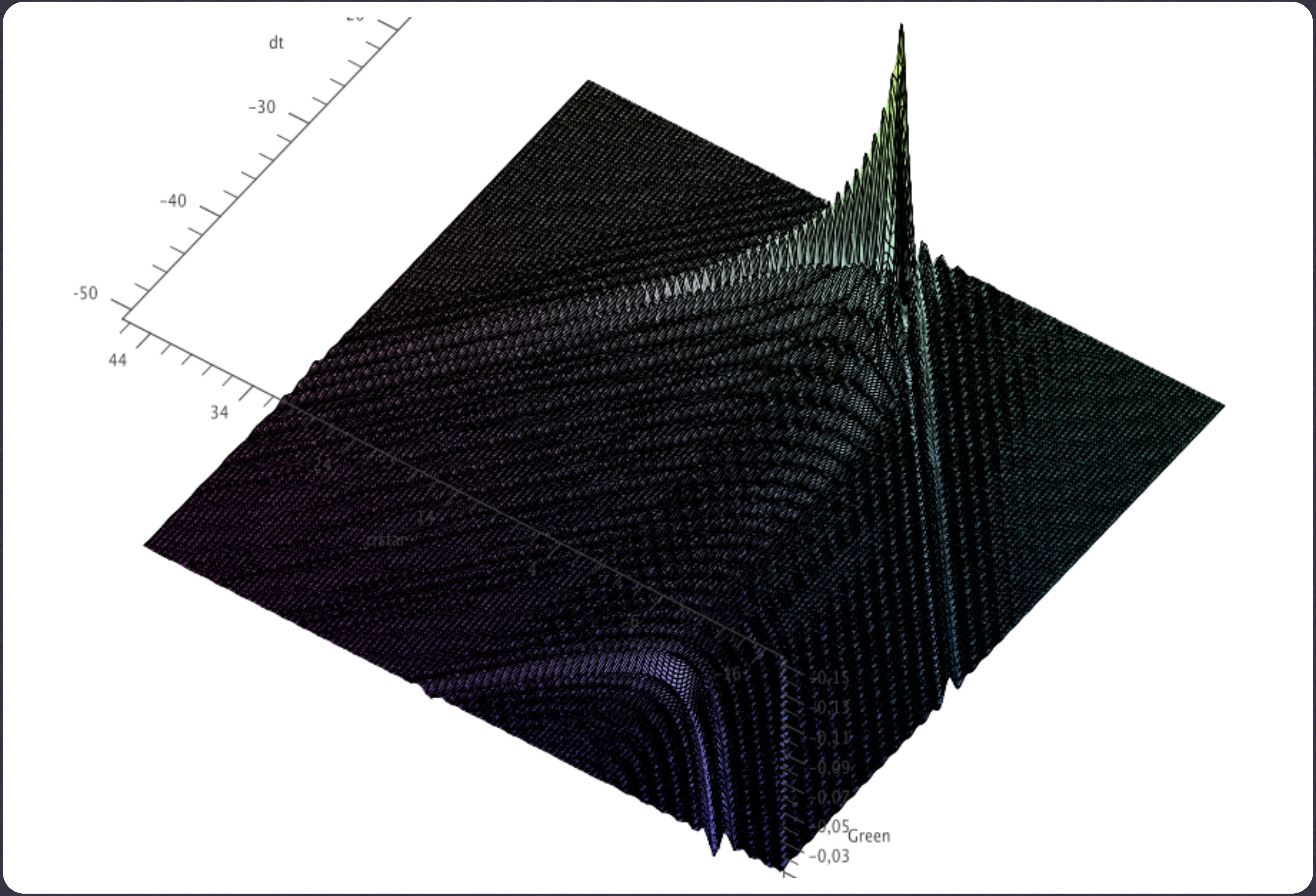
- Use of open Simpson’s rule for ω-integration

Results

Scalar case, $a=0$, $\Phi=\Phi'$, $\theta=\theta'=\pi/2$, $r'\sim 10.3M$

Movie 1

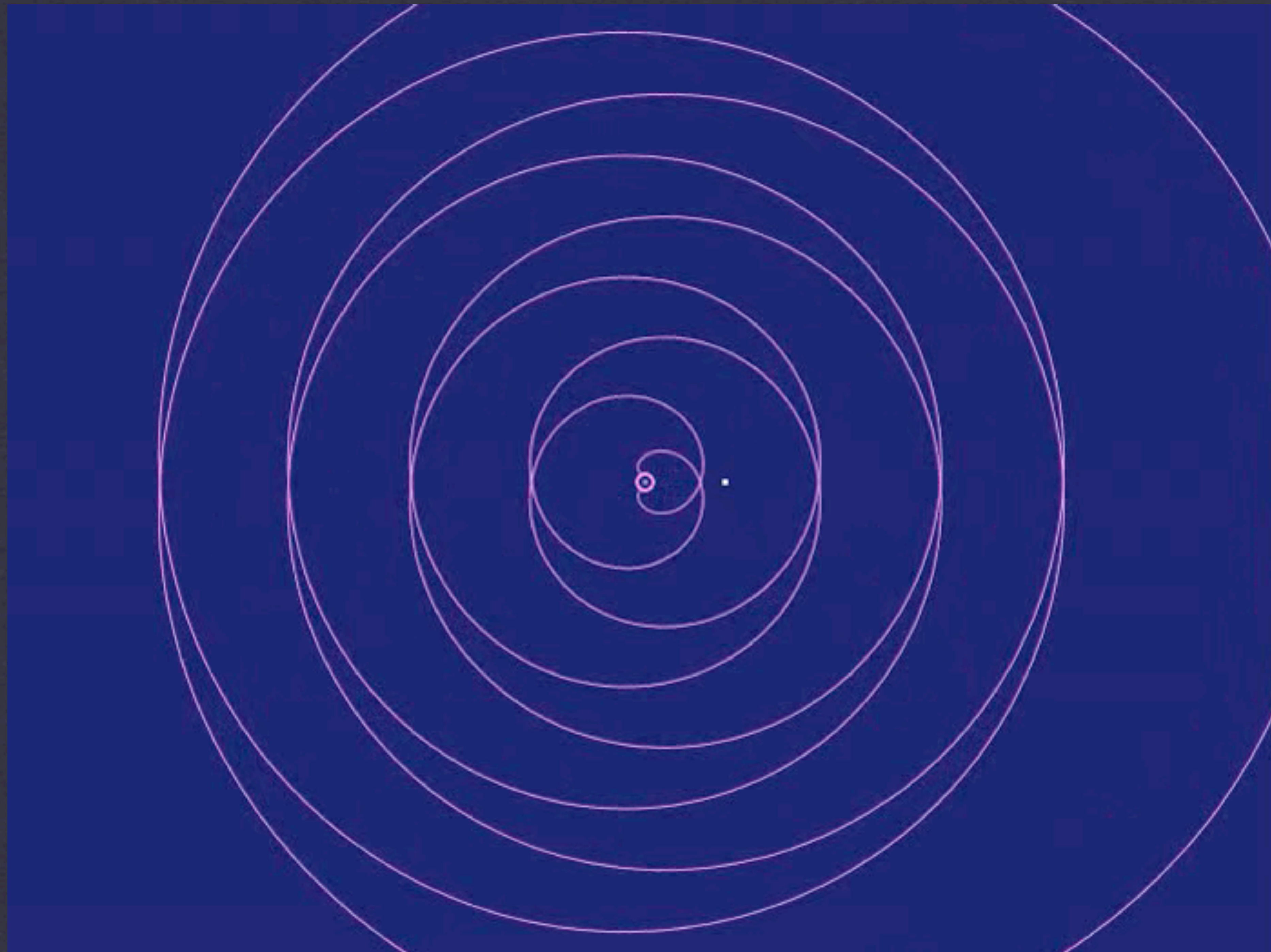
Director: M.C.
Actors $G^{ret}(x,x')$ at $r=r'\sim 10.3M$ as function of $t-t'>0$
Title: You think it's dead and then...



Green’s function as a function of t and r*

Movie 2

Director: Kirill Ignatiev
Actors:
Title:



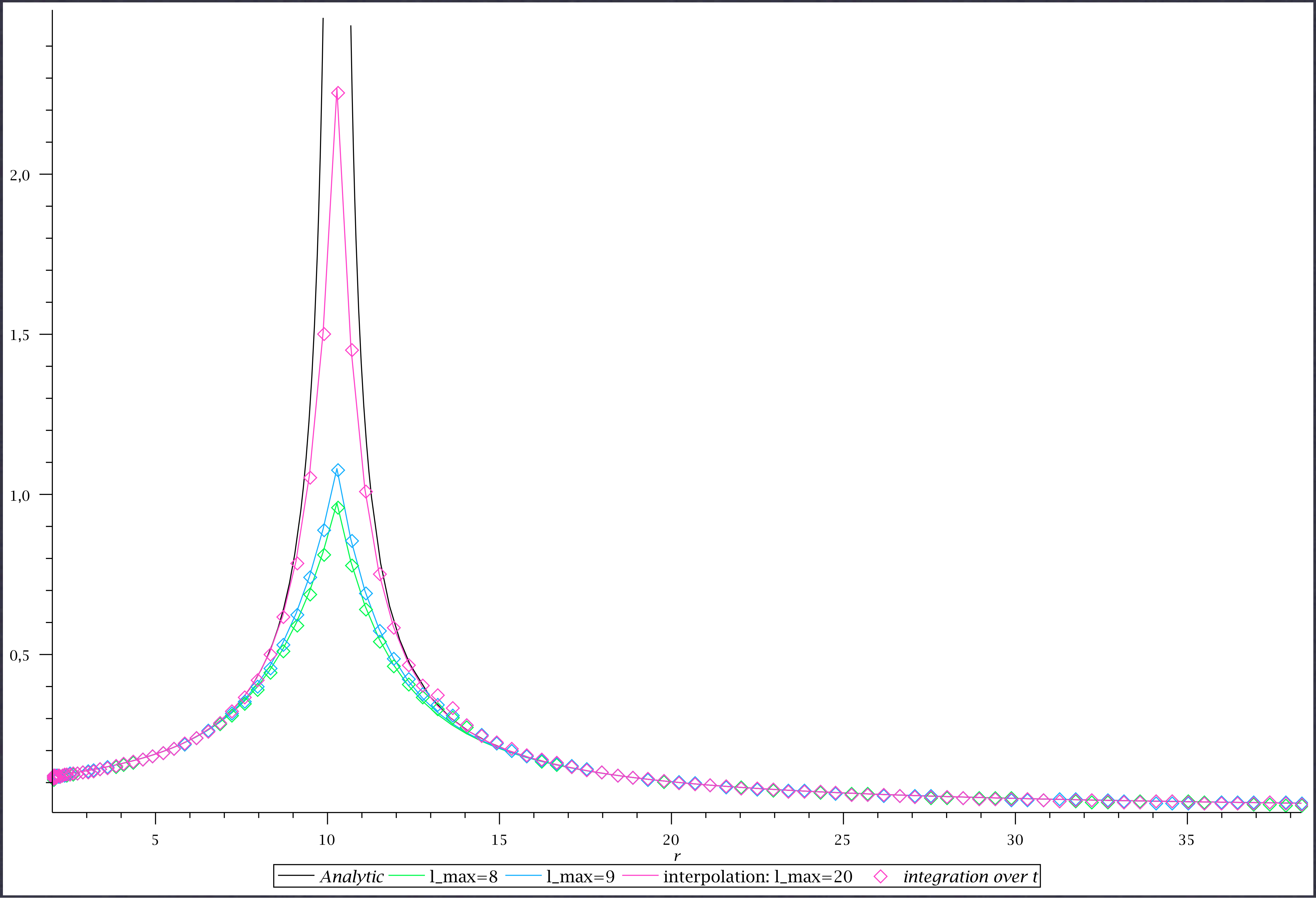
Check

- Analytic - Wiseman PRD61(2000):

$$G_{\omega=0}(x,x') = \left(1 - \frac{M^2}{(r' - M + \sqrt{r'^2 - 2r'M})^2}\right)$$
$$\left((r-M)^2 - 2(r-M)(r'-M)\cos\gamma + (r'-M)^2 - M^2\sin^2\gamma\right)^{-1/2}$$

- Numerical: two integrations using open Simpsons rule:

$$G_{\omega=0}(x,x') = \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')} G_\omega(\vec{x},\vec{x}')$$



2,0
1,5
1,0
0,5
5
10
15
20
25
30
35
r
Analytic
l_max=8
l_max=9
interpolation: l_max=20
integration over t

Questions & Improvements

- Q1: Determine the convergence in l and in ω

- I1: Use WKB (and/or Green-Liouville method) for finding high-ω, high-l asymptotics in order to improve convergence.....it kind of defeats the purpose though!

- Q2: Green function has a branch point at $\omega=0$, value which we have interpolated - what integration contour in the complex ω-plane are we choosing?

- I2: Alternatively, use of Leaver method of integration over the complex ω-plane.

- Three distinct contributions:

- (1) QNMs, (2) along branch cut, (3) large complex ω.

Questions & Improvements

- Q3: What determines the scale of the oscillations of G(x,x') in-between the wavefronts?
- I3: Look for an analytic approximation to the value of G(x,x') at the appearance of the 2nd (and following) wavefront(s), i.e., value of U(x,x').