Investigating consequences of the self-force

Steve Detweiler
University of Florida
Orléans, June, 2008

- Status of self-force analyses: Past, future and present.

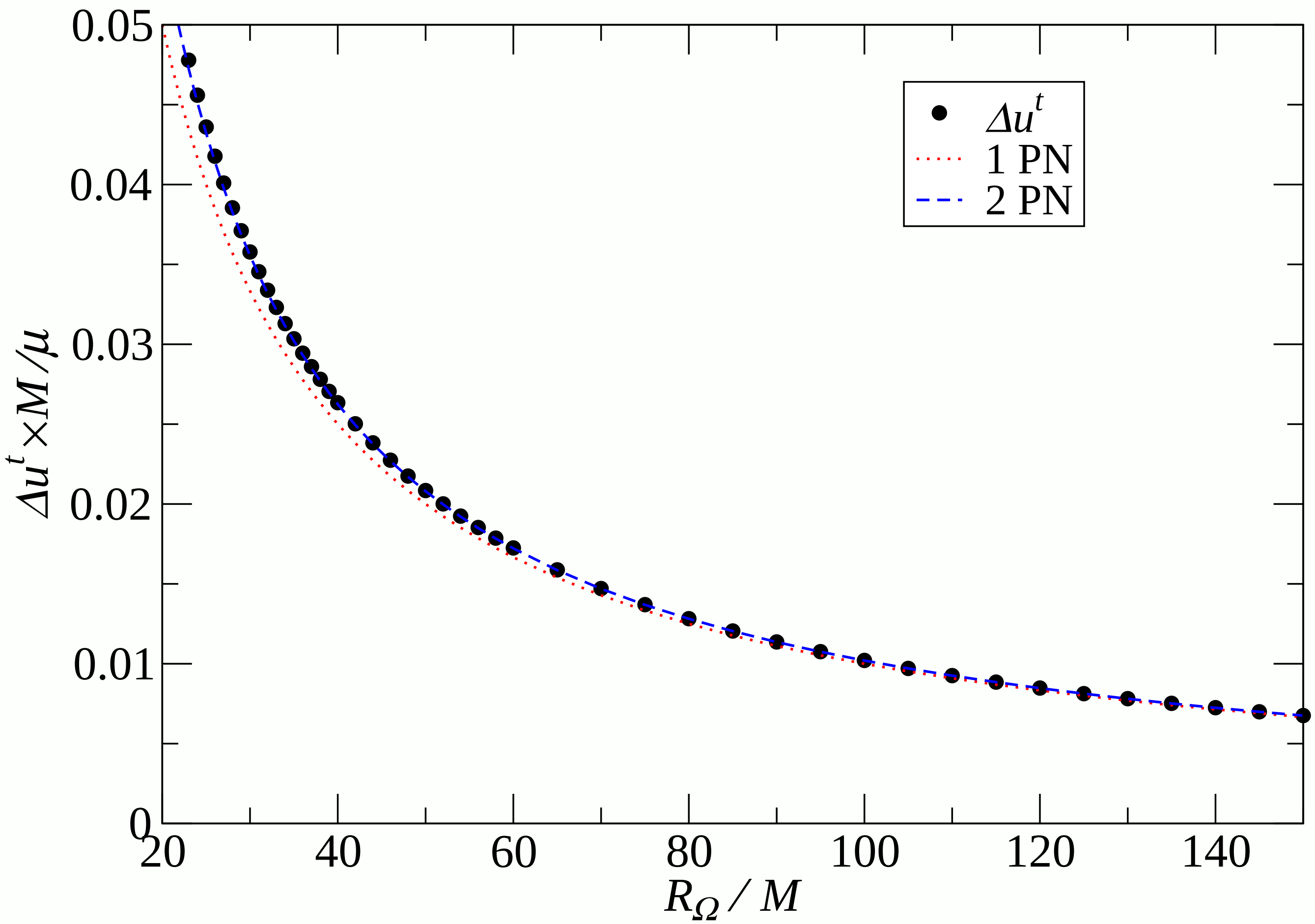
- Introduction to self-force calculations in general relativity

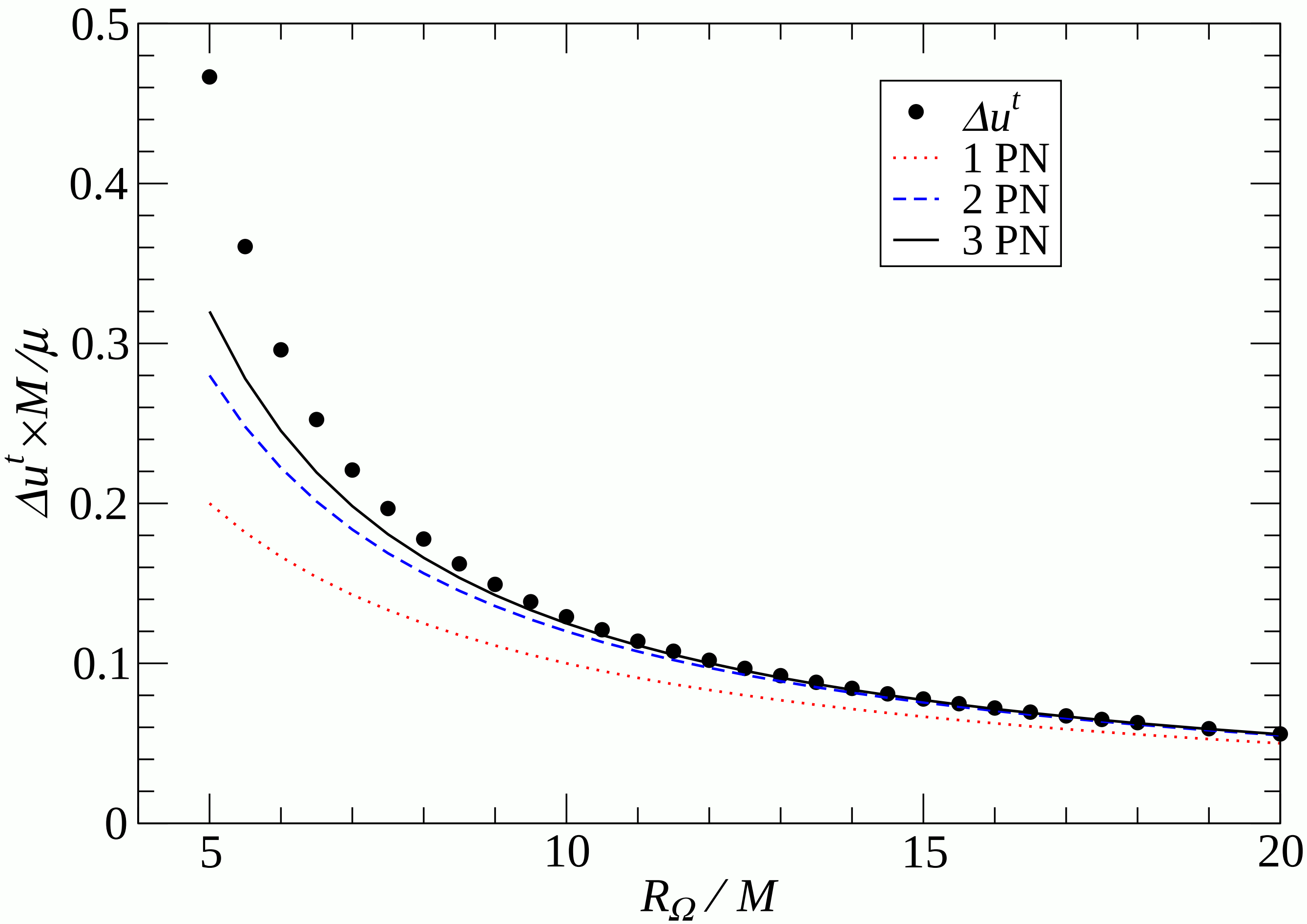
- Actual results for gravitational self-force effects on circular orbits in the Schwarzschild geometry.

- The most interesting part: A new approach to smoothing out δ-function sources in curved spacetime.

Δu^t is the $O(\mu/M)$ contribution to the difference between the proper time as measured near μ and the Schwarzschild time. This is also proportional to the redshift of a photon emitted near μ and received far away on the z-axis.

R_Ω via $\Omega^2 = m/R_\Omega^3$





$$ {}_1u^t = \frac{\mu}{m}\left[-\left(\frac{m}{R_\Omega}\right) - 2\left(\frac{m}{R_\Omega}\right)^2 - 5\left(\frac{m}{R_\Omega}\right)^3 - (27.61 \pm 0.3)\left(\frac{m}{R_\Omega}\right)^4 + \cdots\right], $$

which includes analytic coefficients up to terms of order v^6/c^6.

My personal opinion about the laws of classical physics

The laws of physics are determined by nature and are local. Partial differential equations govern the evolution of fields, and ordinary differential equations govern the motion of objects.

Initial conditions and boundary conditions are determined by physicists in experiments and in calculations.

• What does Jackson say about radiation reaction?
Abraham-Lorentz force:

$\mathbf{F}_{\text{rad}} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}$ calculates $\mathbf{F}_{\text{rad}}$ but does not "explain" it.

This expression for radiation reaction involves boundary conditions and should not be thought of as a "law of physics."

I prefer part of Dirac's analysis.

How we do a self-force calculation:

1. For a geometry g_{ab}, pick a geodesic for μ.
2. Use numerical methods to find "$\Box h^{\mathrm{ret}}_{ab}$" $= 8\pi T_{ab}$
3. h^{ret}_{ab} is singular at the particle — it must be regularized:

   a. Calculate analytically the **S**ingular, **S**ource field $h^{\mathrm{S}}_{ab} \approx$ "μ/r" in the neighborhood of μ, with locally-inertial, Cartesian coordinates.

   b. Subtract $h^{\mathrm{ret}}_{ab} - h^{\mathrm{S}}_{ab} \equiv h^{\mathrm{R}}_{ab}$, to obtain the **R**egular **R**emainder.

   c. By construction, h^{R}_{ab} is guaranteed to be a solution of the vacuum, perturbed Einstein equations in a neighborhood of μ.
4. The Bianchi identity implies that μ moves along a geodesic of $g_{ab} + h^{\mathrm{R}}_{ab}$. This appears as "acceleration" of $O(\mu)$ in the background geometry:

$$u^a \underset{(g)}{\nabla}_a u^b = u^a u^c (g^{bd} + u^b u^d)(\underset{(g)}{\nabla}_a h^{\mathrm{R}}_{cd} - \frac{1}{2} \underset{(g)}{\nabla}_d h^{\mathrm{R}}_{ac}).$$

and the RHS is the "gravitational self force/μ."

- What does the Gravitational self force mean?

- How do we measure the acceleration from the gravitational self-force?

Construction of h^{S}_{ab}

- We use a matched asymptotic expansion: In the matching region the geometry looks like (a) the external geometry being perturbed by the black hole and *simultaneously* (b) the black hole geometry being tidally distorted by the external geometry

- Use special coordinates (THZ) which are locally-inertial, Cartesian, harmonic and defined in a neighborhood about a geodesic in a vacuum spacetime. A Taylor expansion of the background metric gives

$$\begin{aligned} g^{\mathrm{o}}_{ab} &= \eta_{ab} + {}_2H_{ab} + \ldots \quad r \to 0, \\ {}_2H_{ab}dx^a dx^b &= -\mathscr{E}_{ij}x^i x^j (dt^2 + \delta_{kl}dx^k dx^l) + \frac{4}{3}\varepsilon_{kpq}\mathscr{B}^q{}_i x^p x^i dt\, dx^k, \end{aligned}$$

$\mathscr{E}$ and $\mathscr{B}$ are quadrupole tidal fields and are spatial, symmetric, tracefree and related to the Riemann tensor and its derivatives evaluated on the geodesic; and, $\mathscr{E}_{ij} = R_{titj}$ and $\mathscr{B}_{ij} = \varepsilon_i{}^{pq}R_{pqjt}/2$.

- There is no term in the expansion which is linear in x^i. If there were, then the Christoffel symbols would not vanish on Γ, and Γ would not be a geodesic.

- With THZ coordinates,

$$\sqrt{-g}\,\Box(q/r) = 4\pi q \delta^3(x^i) + O(x^i/\mathscr{R}^4).$$

And they make life in Florida rather easy.

Very low frequency perturbations of a Schwarzschild black hole

Put a small black hole, $\mu \ll \mathscr{R}$, on the geodesic.

The metric is perturbed by ${}_2h_{ab}$, where "$\nabla^2_{\mathsf{Schw}}({}_2h) = 0$ with ${}_2h_{ab} \to {}_2H_{ab}$ for $\mu \ll r \ll \mathscr{R}$. In the time independent limit, Zerilli (1970) shows us that

$$
\begin{aligned}
{}_2h_{ab}dx^a dx^b \;\; = \;\; & -\mathscr{E}_{ij}x^i x^j \big[(1-2\mu/r)^2 dt^2 + dr^2 \\
& \qquad + (r^2 - 2\mu^2)(d\theta^2 + \sin^2\theta d\phi^2)\big] \\
& + \frac{4}{3}\varepsilon_{kpq}\mathscr{B}^q{}_i x^p x^i (1-2\mu/r)dt\,dx^k
\end{aligned}
\tag{1}
$$

• The black hole is "at rest" on the geodesic, from its own perspective.

• The black hole is acted upon by no external force to move it off the geodesic.

A Simple expansion for h^{S}_{ab}

- An expansion of h^{S}_{ab} about the worldline of the black hole includes all terms which are linear in μ. These terms are singular or nondifferentiable, but exert no "force" on the black hole itself—the black hole is moving along a geodesic of the background geometry.

- Thus, in the special coordinates

$$
\begin{aligned}
h^{\mathsf{S}}_{ab}dx^a dx^b &= \frac{2\mu}{r}(dt^2+dr^2) \\
&\quad + \frac{4\mu}{r}\mathcal{E}_{ij}x^i x^j dt^2 - \frac{8\mu}{3r}\varepsilon_{kpq}\mathcal{B}^q{}_i x^p x^i dt\, dx^k
\end{aligned}
$$

- The $\mu x^i x^j \mathcal{E}_{ij}/r$ term and the similar $\mathcal{B}^q{}_i$ terms in h^{S}_{ab} result from the tidal distortion of the hole's monopole field by the background geometry, and are not differentiable on the worldline; these must be included in h^{S}_{ab} in order that the resulting h^{R}_{ab} is differentiable.

- For calculating the self force effect on the worldline, it is sufficient to use just these terms to approximate h^{S}_{ab}.

- Higher order terms in the expansion all have vanishing first derivative on the world line.

- But, including higher order terms in the analysis significantly improves convergence of numerical results.

Radiation reaction and gravitational waves

A small mass μ orbits a large black hole and slowly emits gravitational waves.
A local observer measures the gravitational field and distinguishes two parts:

1. The μ/r field distorted by tidal forces, h^{S}_{ab}.
2. An external field consisting of the background combined with a small perturbing field, $g_{ab} + h^{\mathsf{R}}_{ab}$.

- The local observer cannot distinguish h^{R}_{ab} from g_{ab}. Together they satisfy the Einstein equations through first order in μ.
- The mass μ moves along a geodesic of this external gravitational field.

With local measurements, the observer sees:

- No radiation,
- No local source for the "external field,"
- No effect that he would be compelled to describe as radiation reaction.

The observer sees only a small mass coasting along a geodesic of the "external" gravitational field with acceleration of order μ^2, or smaller.

How we do a self-force calculation (again):

1. For a geometry g_{ab}, pick a geodesic for μ.
2. Use numerical methods to find "$\Box h^{\mathrm{ret}}_{ab}$" $= 8\pi T_{ab}$
3. h^{ret}_{ab} is singular at the particle — it must be regularized:

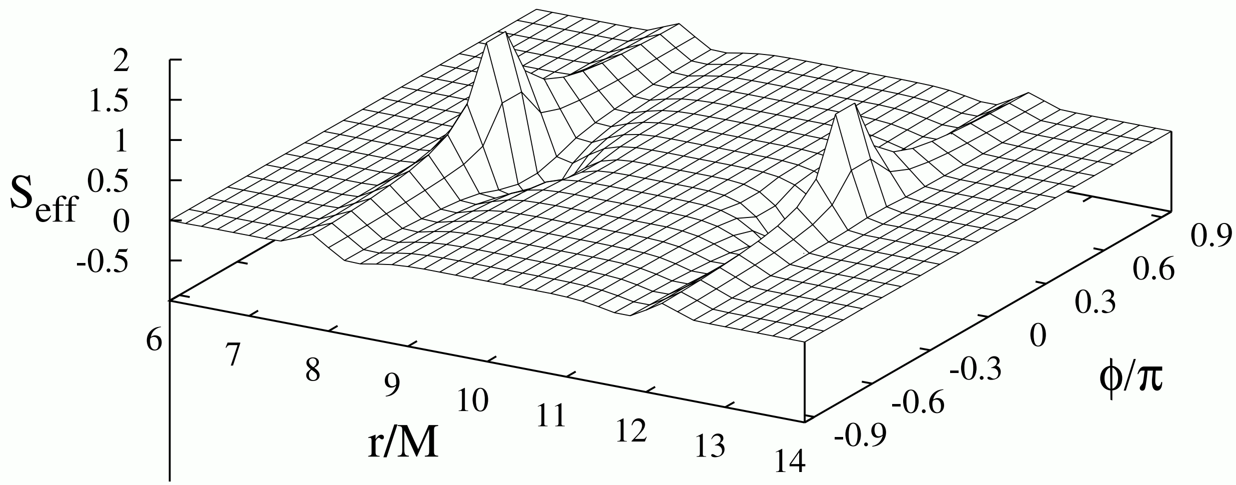
   a. Calculate analytically the **S**ingular, **S**ource field $h^{\mathrm{S}}_{ab} \approx$ "μ/r" in the neighborhood of μ, with locally-inertial, Cartesian coordinates.

   b. Subtract $h^{\mathrm{ret}}_{ab} - h^{\mathrm{S}}_{ab} \equiv h^{\mathrm{R}}_{ab}$, to obtain the **R**egular **R**emainder.

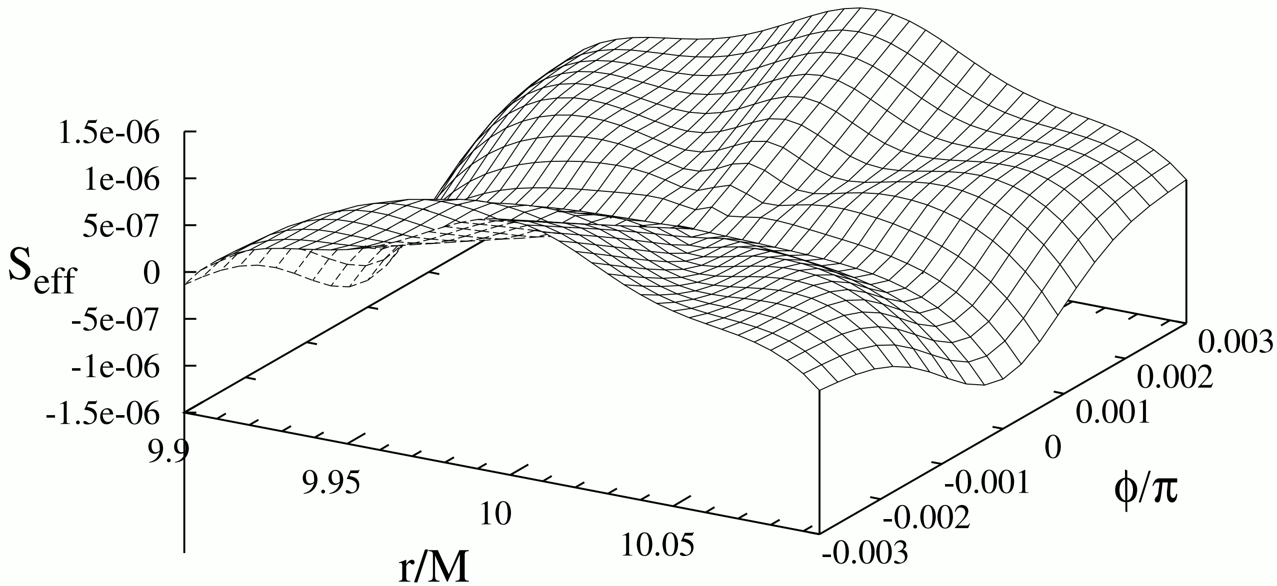
   c. By construction, h^{R}_{ab} is guaranteed to be a solution of the vacuum, perturbed Einstein equations in a neighborhood of μ.
4. The Bianchi identity implies that μ moves along a geodesic of $g_{ab} + h^{\mathrm{R}}_{ab}$.
5. Alternatively, from "$\Box h^{\mathrm{ret}}_{ab}$" $=$ "$\Box(h^{\mathrm{R}}_{ab} + h^{\mathrm{S}}_{ab})$" $= 8\pi T_{ab}$ solve

$$\text{“}\Box h^{\mathrm{R}}_{ab}\text{”} = -\text{“}\Box h^{\mathrm{S}}_{ab}\text{”} + 8\pi T_{ab}$$

where the right hand side is known analytically and guaranteed to be continuous in a neighborhood of the particle



The *distributed source* on the righthand side is $-“\Box h^{\mathsf{S}}_{ab}” + 8\pi T_{ab}$ is continuous in a neighborhood of the particle and has structure with a length scale of order of the $\mathscr{R}$.



1.5e-06
1e-06
5e-07
S_{eff}
0
-5e-07
-1e-06
-1.5e-06
9.9
9.95
10
10.05
r/M
-0.003
-0.002
-0.001
0
0.001
0.002
0.003
ϕ/π

Second order perturbation theory with a point mass

Define the parts of the Einstein tensor of various orders in h by
$G(g+h) = G(g) + G^{(1)}(g,h) + G^{(2)}(g,h) + G^{(3)}(g,h) + \dots$

- $G^{(1)}(g,h)$ looks like a wave operator on h
- $G^{(2)}(g,h)$ looks like "$\nabla h \nabla h$" or "$h \nabla\nabla h$".

At second order solve $G(g) + G^{(1)}(g,h) + G^{(2)}(g,h) = 8\pi T$ or

$$G(g+h^{\mathsf{R}}) + G^{(1)}(g+h^{\mathsf{R}}, h^{\mathsf{S}}) + G^{(2)}(g+h^{\mathsf{R}}, h^{\mathsf{S}}) = 8\pi T$$

$$G(g) + G^{(1)}(g,h^{\mathsf{R}}) + G^{(2)}(g,h^{\mathsf{R}}) + G^{(1)}(g+h^{\mathsf{R}}, h^{\mathsf{S}}) + G^{(2)}(g+h^{\mathsf{R}}, h^{\mathsf{S}}) = 8\pi T$$

This results in
$$G^{(1)}(g,h^{\mathsf{R}}) = -G^{(2)}(g,h^{\mathsf{R}}) - G^{(1)}(g+h^{\mathsf{R}}, h^{\mathsf{S}}) - G^{(2)}(g+h^{\mathsf{R}}, h^{\mathsf{S}}) + 8\pi T$$

Schematically this is

$$_g\Box h^{\mathsf{R}} = \nabla h^{\mathsf{R}}\,\nabla h^{\mathsf{R}} + [_{(g+h^{\mathsf{R}})}\nabla^2 h^{\mathsf{S}} - {}_g\nabla h^{\mathsf{S}}{}_g\nabla h^{\mathsf{S}} - 16\pi T] + O(h^3)$$

The righthand side is C^0 at $O(\mu)$ and finite but discontinuous at $O(\mu^2)$.

Numerical relativists could solve this nonlinear field equation, while simultaneously having μ move along a geodesic of $g + h_R$.

Note that there is no mention of any gauge choice.