# Templates of generic extreme mass ratio inspirals in the frequency domain

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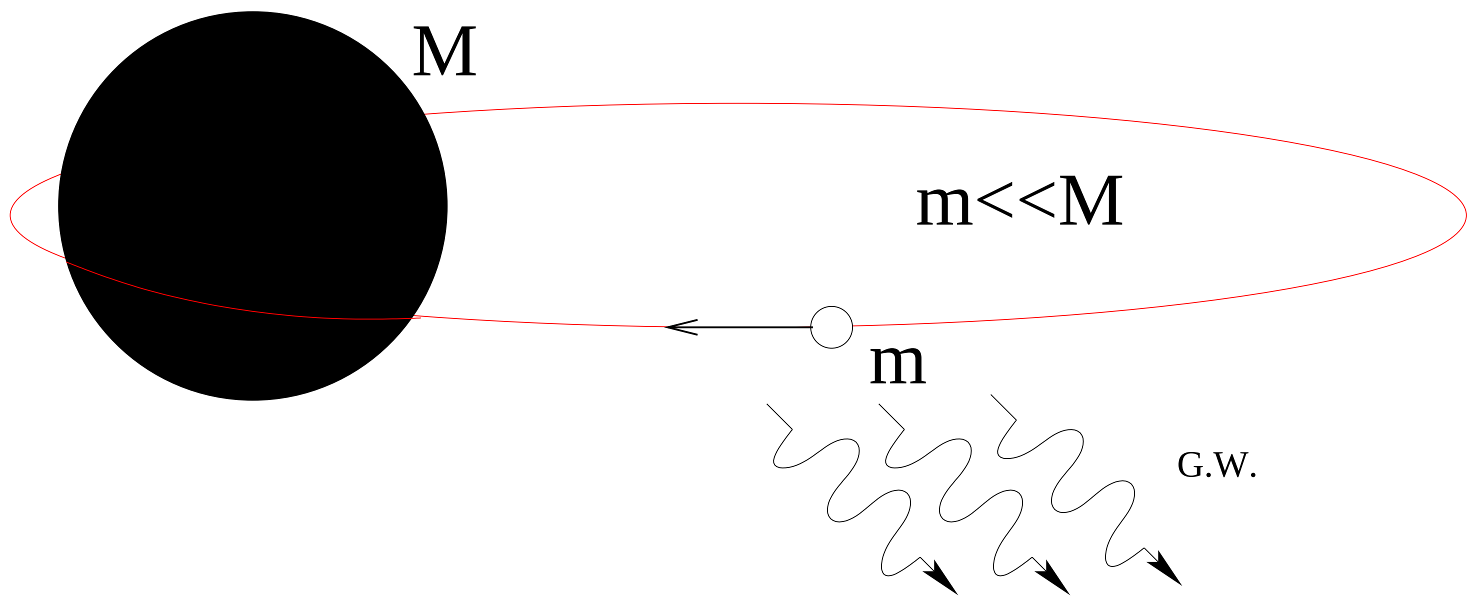
# Motivation

- Extreme Mass Ratio Inspirals (EMRI)
  - One of main targets of LISA $\Rightarrow$ 100 events/yrs




- Motivation
  Construct templates of generic extreme mass ratio inspirals in the frequency domain

# Black hole perturbation equation

- Teukolsky equation (Frequency domain) [Teukolsky(1973)]

  $$\Psi = \sum_{\ell m} \int d\omega e^{-i\omega t + im\varphi} \, {}_{-2}S^{a\omega}_{\ell m}(\theta) Z^{\infty}_{\ell m\omega}(r), \ \Psi : \ \text{Weyl scalar} \sim \ddot{h}$$

- Recent works for the eccentric and the inclined orbits
  - Numerical : Drasco and Hughes (2005)
  - Semi-analytical : Barack and Cutler (2004) $\Rightarrow$ MLDC

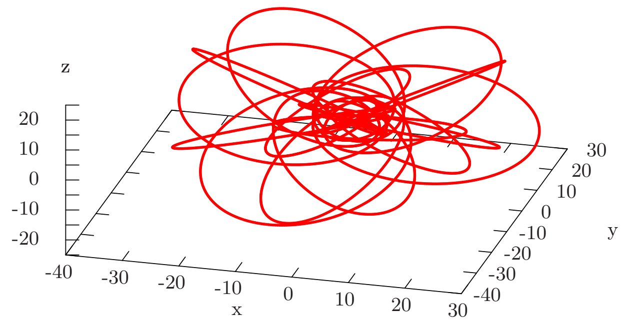
It is useful if we have templates for EMRI
in the frequency domain

# This talk

- Motivation
- Template in the frequency domain
  - Geodesic orbit in Kerr space time
  - Wave form in the time domain
    - The lowest order quadrupolar wave form
    - The effect of the inclination angle
  - Template in the frequency domain
    - Orbital evolution with PN results
    - LISA antenna beam pattern
- (PRELIMINARY) Results : SNR and parameter estimation
- Summary and future work

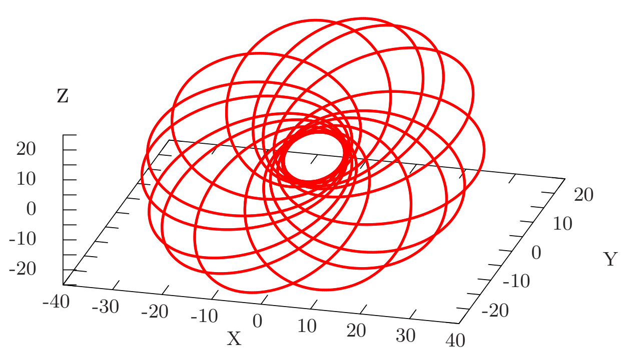
# Geodesic orbit in Kerr space time

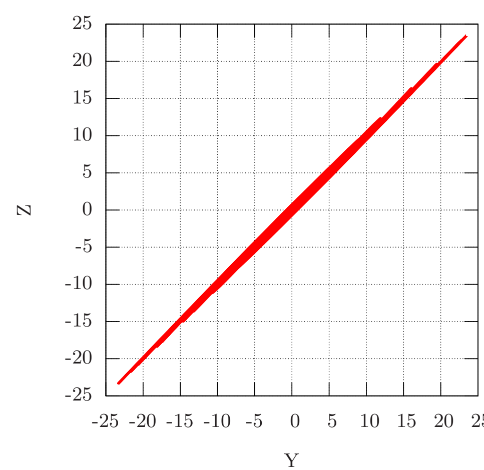
- Geodesic orbit $(q = 0.9, p/M = 10, e = 0.7, \iota = 45^\circ)$




- Geodesic orbit expressed by co-rotating frame

$$X + i\,Y = (x + i\,y)\,e^{-i\,\Omega_{\rm LT}\,t},\ \Omega_{\rm LT} = \Omega_\phi - \Omega_\theta.$$



# Gravitational waveforms in time domain

The lowest order quadrupolar wave form

The effect of the inclination angle

# Gravitational waveforms in time domain

- Using three fundamental frequencies

$$A^{+}(t) - iA^{\times}(t) = -\frac{2}{r}\sum_{\ell mkn} \tilde{Z}_{\ell mkn}(p,e,Y) S_{\ell mkn}(\hat{J}\cdot\hat{N}) \times \exp[-i(m\Omega_\phi + k\Omega_\theta + n\Omega_{\rm r})t + im\phi]\,,$$

$$\text{where}\quad Y = \cos\iota,$$

$$\hat{N}:\ \text{source location},$$

$$\vec{J} := \vec{L} + \vec{S}:\ \text{total angular, momentum},$$

$$\hat{L}:\ \text{orbital angular momentum of compact object},$$

$$\hat{S}:\ \text{massive black hole's spin}.$$

# Gravitational waveforms in time domain

- Using the phase functions

$$A^+(t) - iA^\times(t) = -\frac{2}{r}\sum_{\ell mkn} \tilde{Z}_{\ell mkn}(p,e,Y) S_{\ell mkn}(\hat{J}\cdot\hat{N}) \times \exp\left[-i\left((m+k+n)\chi_\phi(t) - n\chi_{\mathsf{PP}}(t) - k\chi_{\mathsf{LT}}(t)\right)\right] ,$$

$$\text{where } \frac{d\chi_\phi(t)}{dt} = \Omega_\phi,\ \frac{d\chi_\theta(t)}{dt} = \Omega_\theta,\ \frac{d\chi_r(t)}{dt} = \Omega_r\ ,$$

$$\frac{d\chi_{\mathsf{PP}}(t)}{dt} = \Omega_\phi - \Omega_r \equiv \Omega_{\mathsf{PP}},$$

$$\frac{d\chi_{\mathsf{LT}}(t)}{dt} = \Omega_\phi - \Omega_\theta \equiv \Omega_{\mathsf{LT}}.$$

# The lowest order quadrupolar wave form

- On the equatorial plane [e.g. Peters and Mathews (1963)]

$$\tilde{Z}_{2\pm 20n}\Big|_{Y=1} = (\omega_{20n})^2 \frac{1}{\sqrt{20}} \mu \frac{p^2}{(1-e^2)^2} \frac{2}{n+2} \left\{ g_1(n+2,e) \mp i g_2(n+2,e) \right\}$$

where
$$\begin{aligned} g_1(n,e) &= J_{n-2}(ne) - 2eJ_{n-1}(ne) + \frac{2}{n} J_n(ne) \\ &\quad + 2eJ_{n+1}(ne) - J_{n+2}(ne)\,, \\ g_2(n,e) &= (1-e^2)^{1/2} \left[ J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne) \right]\,. \end{aligned}$$

# The effect of the inclination angle

- The effect of the inclination angle

$$\tilde{Z}_{\ell(m-k)kn} = \frac{\left(\omega_{(m-k)kn}\right)^2}{\left(\omega_{m0n}\right)^2} \mathcal{D}^{\ell}_{(m-k)m}(\arccos Y)\ \tilde{Z}_{\ell m0n}\Big|_{Y=1} .$$

$$\text{where}\ \ \mathcal{D}^{\ell}_{m'm}(\theta)\ :\ \text{Wigner D} - \text{function},$$

$$\mathcal{D}^{\ell}_{m'm}(\theta) = \sum_{q=\max(0,m'-m)}^{\min(\ell-m,\ell+m')} (-1)^q \left(\cos\frac{\theta}{2}\right)^{2\ell+m'-m-2q} \left(-\sin\frac{\theta}{2}\right)^{m-m'+2q}$$

$$\times \frac{\sqrt{(\ell+m')!(\ell-m')!(\ell+m)!(\ell-m)!}}{q!(\ell-m-q)!(\ell+m'-q)!(m-m'+q)!} .$$

# The lowest order quadrupolar wave form

- Including these effects...

$$A^+(t) - iA^\times(t) = -\frac{2}{r}\mu\left(\pi M\nu\right)^{2/3}\sum_{k,n}[f_{kn}(\nu)]^2\frac{2}{n+2}\left(A^+_{kn}(t) - iA^\times_{kn}(t)\right),$$

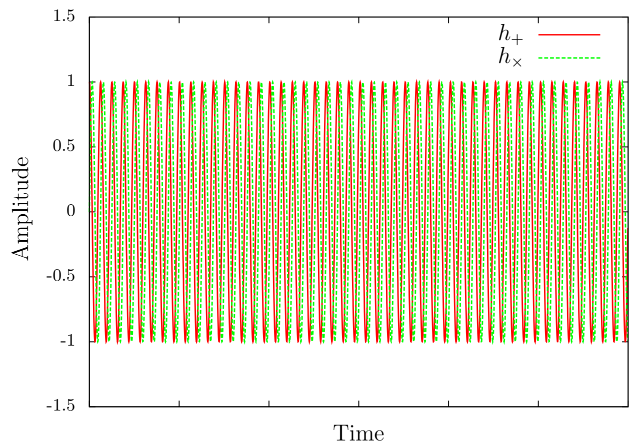
$$\text{where}\quad f_{kn}(\nu) = \frac{\omega_{(2-k)kn}}{2\pi\nu} = \left(1+\frac{n}{2}\right) - \frac{n}{2}\frac{\Omega_{\mathsf{PP}}}{\Omega_\phi} - \frac{k}{2}\frac{\Omega_{\mathsf{LT}}}{\Omega_\phi},$$

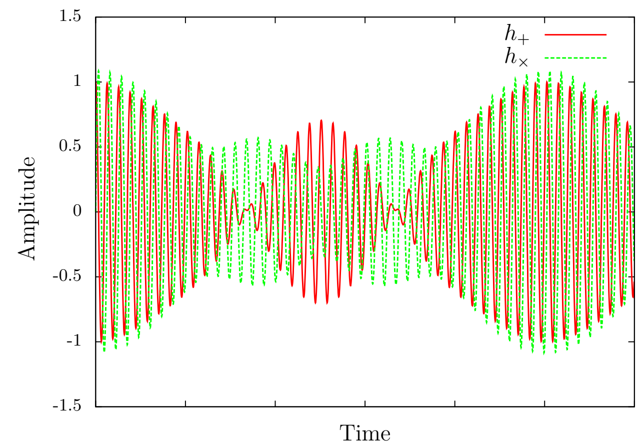
$$A^+_{kn}(t) - iA^\times_{kn}(t) = g(n,e)\left[\alpha_k\cos\Phi_{kn}(t) - i\beta_k\sin\Phi_{kn}(t)\right],$$

$$\Phi_{kn}(t) = \left(1+\frac{n}{2}\right)\phi(t) - n\chi_{\mathsf{PP}}(t) - k\chi_{\mathsf{LT}}(t) + \gamma_n(e).$$

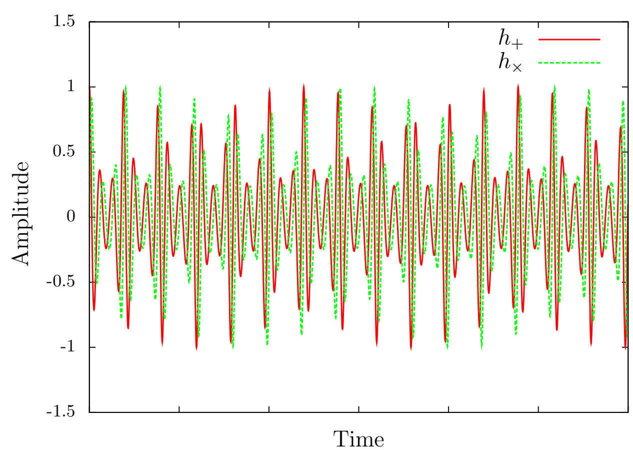
# Gravitational waveforms : $A^{+}(t)$ & $A^{\times}(t)$

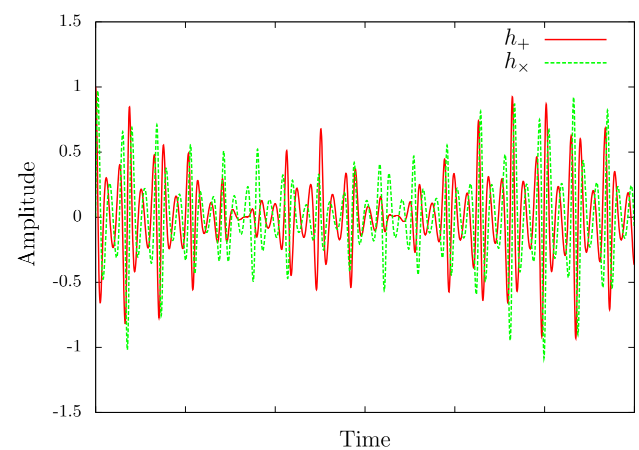
- $e = 0$ and $\iota = 60^{\circ}$: $q = 0$ (left) and $q = 0.9$ (right)




- $e = 0.2$ and $\iota = 60^{\circ}$: $q = 0$ (left) and $q = 0.9$ (right)



# Template in the frequency domain

Orbital evolution with PN results

LISA antenna beam pattern

# Orbital evolution

- We include orbital evolution as

$$(p,\, e,\, Y) \;\rightarrow\; (p(t),\, e(t),\, Y(t))\,,$$
$$\chi_\alpha(t) = \int^t \Omega_\alpha\left(p(t'), e(t'), Y(t')\right)\, dt'\,.$$

- Orbital evolution in the frequency domain [Ganz et al.,(2007)]

$$e^2(\nu) = \tilde{e}_I^2 x^{-19/3}\left(1 + \frac{3215}{1008}x^2 + \left\{-\frac{377}{72}\pi + q\left(\frac{38}{9} - \frac{287}{54}Y_I\right)\right\}x^3 + \left\{\frac{35705555}{1524096} + q^2\left(\frac{731}{576} - \frac{19}{6}Y_I + \frac{1675}{576}Y_I^2\right)\right\}x^4\right)\,,$$
$$\phi(\nu) = \cdots$$

where $x \equiv (\pi M\nu)^{1/3}(1-e^2)^{-1/2}, e_I$ : the value of $e$ at $\nu = 0$.

# LISA effects

- Strain amplitude in LISA

$$h_\alpha(t) = \frac{\sqrt{3}}{2}\left[A^+(t)F_\alpha^+(t) + A^\times(t)F_\alpha^\times(t)\right],$$

$$\text{where } F_\alpha^{+,\times} \;:\; \text{LISA antenna beam pattern}$$

$$\alpha = I, II \;:\; \text{two independent Michelson outputs}$$

# LISA effects

- Amplitude-and-phase representation

$$h_\alpha(t) = \sum_{k=0}^{4} \sum_{n=-1} h_{\alpha,kn} \, ,$$

$$h_{\alpha,kn} = -\frac{\sqrt{3}}{r} \mu (\pi M \nu)^{2/3} [f_{kn}(\nu)]^2 \frac{2}{n+2} g(n, e(t)) A_{\alpha,k}(t)$$
$$\times \cos\left[\Phi_{kn}(t) + \varphi^{p}_{\alpha,k}(t) + \varphi^{D}_{kn}(t)\right] \, ,$$

- Templates in the frequency domain

We perform Fourier transformation
using stationary phase approximation

# Templates in the frequency domain

- Templates in the frequency domain

$$\tilde{h}(F) = \sum_{\alpha=I,II} \sum_{k=0}^{4} \sum_{n=0} \tilde{h}_{\alpha,kn}(f_{kn}(\nu)\,\nu)\,, \quad (\nu_{\text{min}} \leq \nu \leq \nu_{\text{max}}),$$

with

$$\tilde{h}_{\alpha,kn}\left(f_{kn}(\nu)\,\nu\right) = \frac{[f_{kn}(\nu)]^{5/2}\nu^{-1/6}\tilde{\mathcal{N}}_n(e(\nu))}{\sqrt{\nu^2 f_{kn}(\nu)[\nu f_{kn}(\nu)]'}} \times A_{\alpha,k}[t(\nu)]\exp\left\{i\left(\Psi_{kn}(\nu) - \varphi^{p}_{\alpha,k}[t(\nu)] - \varphi^{D}_{kn}[t(\nu)]\right)\right\}\,,$$

# Results

# Algorithm

- 14 parameters of binary system

$$\lambda^i \equiv \{t^{(\mathrm{c})}, M, \mu, q, \mu_J, \phi_J, e^{(\mathrm{c})}, Y^{(\mathrm{c})}, \phi^{(\mathrm{c})}, \chi^{(\mathrm{c})}_{\mathrm{PP}}, \chi^{(\mathrm{c})}_{LT}, \mu_N, \phi_N, D_L\},$$

- Set up fiducial frequencies

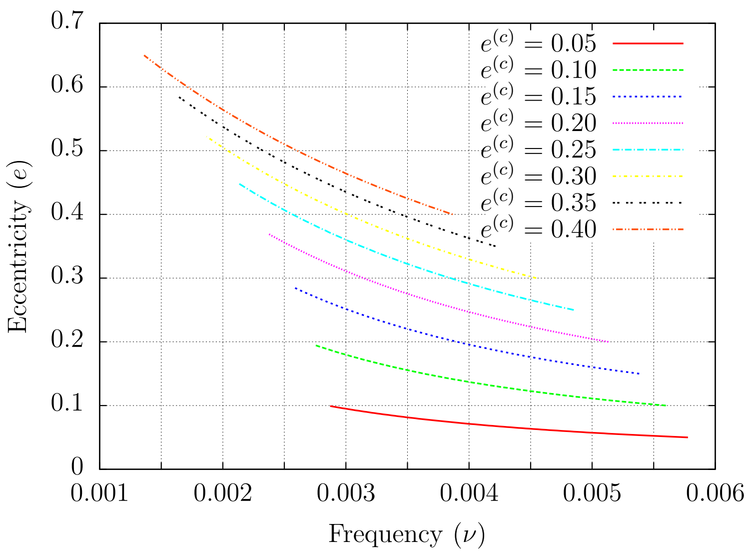
$$\nu_{\max} = \nu_{\mathrm{ISCO}}, \;\; \nu_{\min} = \min[\nu_{1\mathrm{yr}}, 10^{-5}].$$

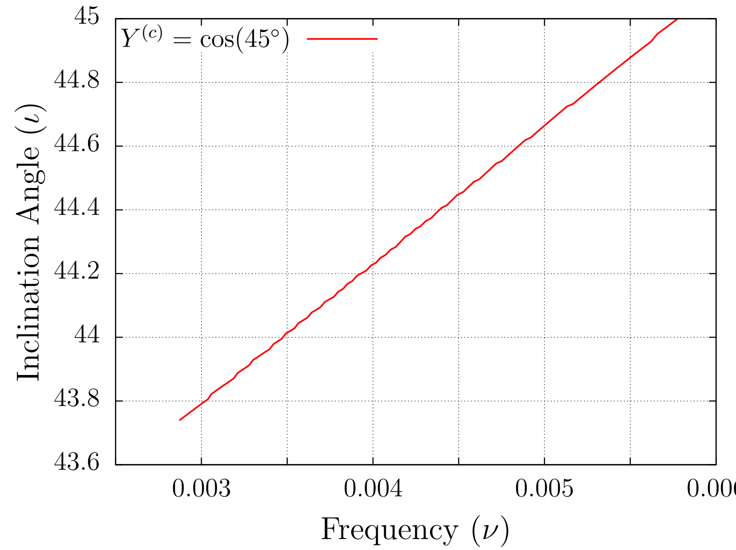
- Evoluve the orbits
- Compute $\tilde{h}_\alpha(\nu)$
- Compute SNR and parameter estimation errors using LISA detector noise

$$\rho^2 = 4\sum_{\alpha,k,n}\int_0^\infty dF\frac{|\tilde{h}_{\alpha,kn}(F)|^2}{S_n(F)}, \Gamma_{ij} \equiv \sum_{\alpha,k,n}\left(\partial_i\tilde{h}_{\alpha,kn}\middle|\,\partial_j\tilde{h}_{\alpha,kn}\right).$$

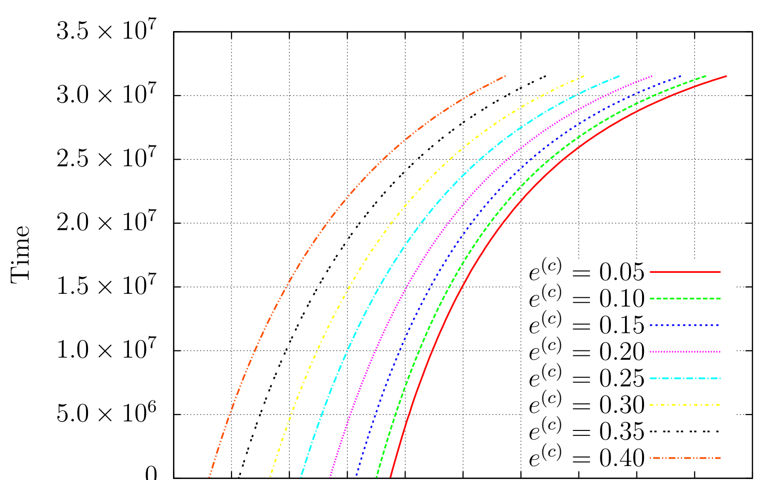
# Evolution of the time and phase functions

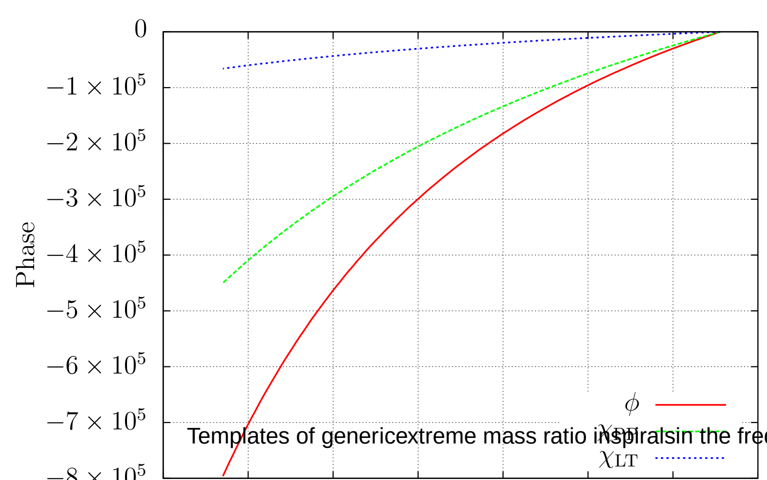
- $M = 10^6 M_\odot$, $\mu = 10 M_\odot$, $\iota^{(c)} = 45°$, $q = 0.9$



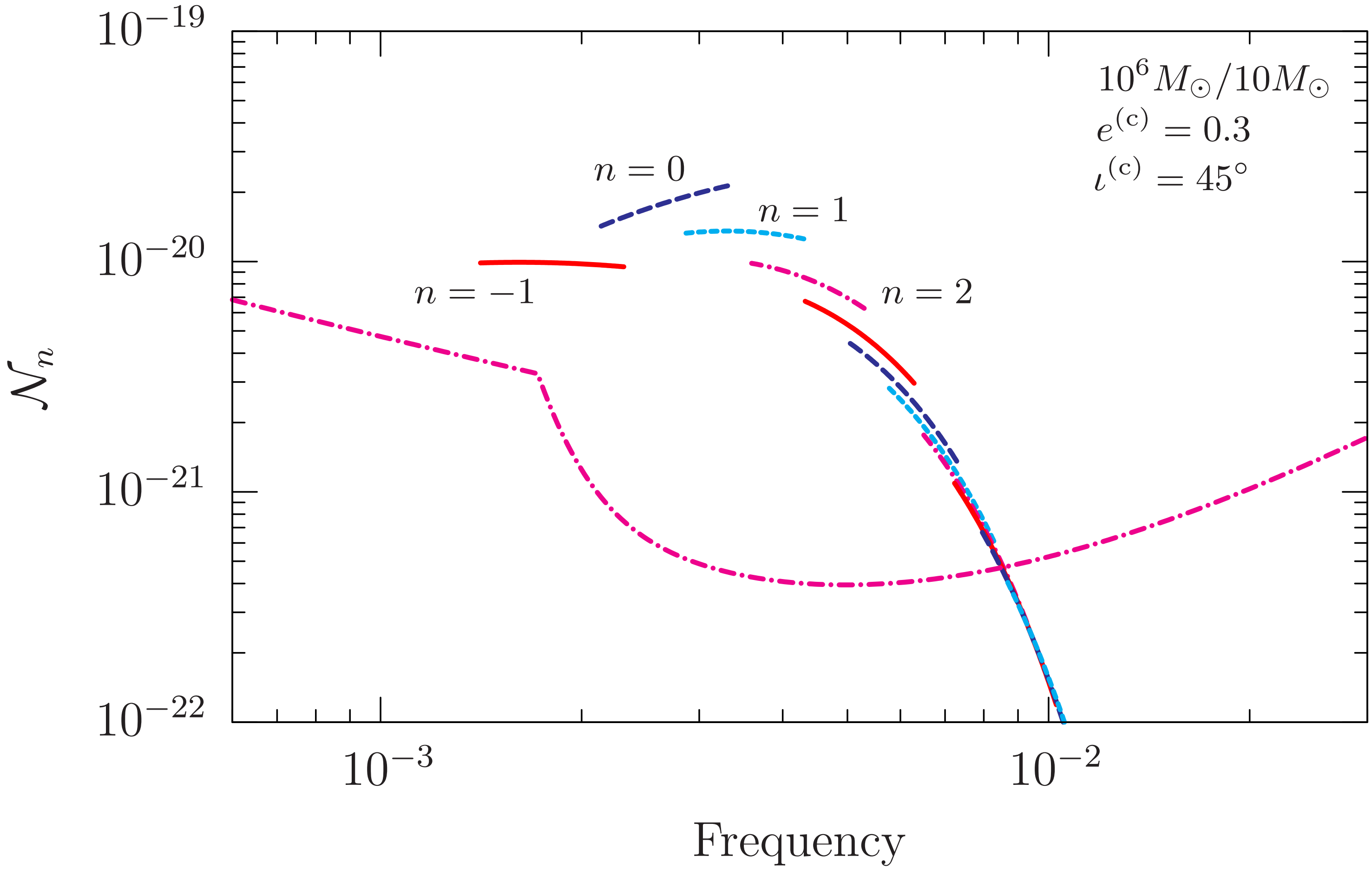

- $M = 10^6 M_\odot$, $\mu = 10 M_\odot$, $\iota^{(c)} = 45°$, $q = 0.9$



# Dependence of the $n$ modes

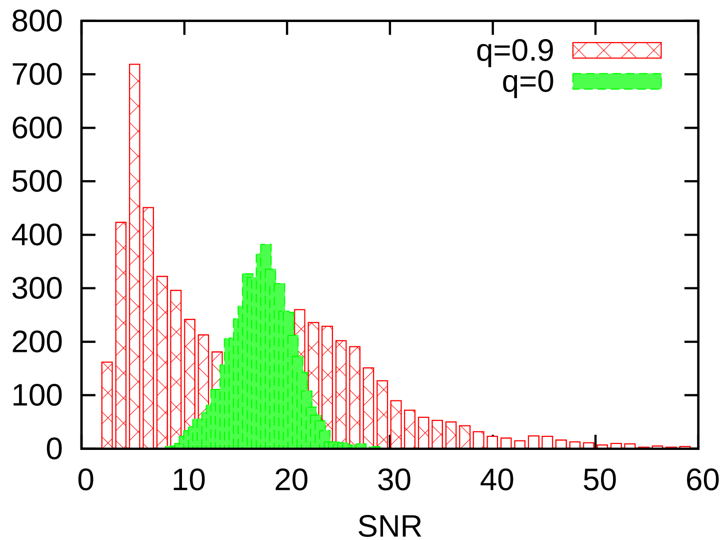
- $M = 10^6 M_\odot$, $\mu = 10 M_\odot$, $e^{(c)} = 0.3$, $\iota^{(c)} = 45°$




We truncate $n$ mode upto $20$

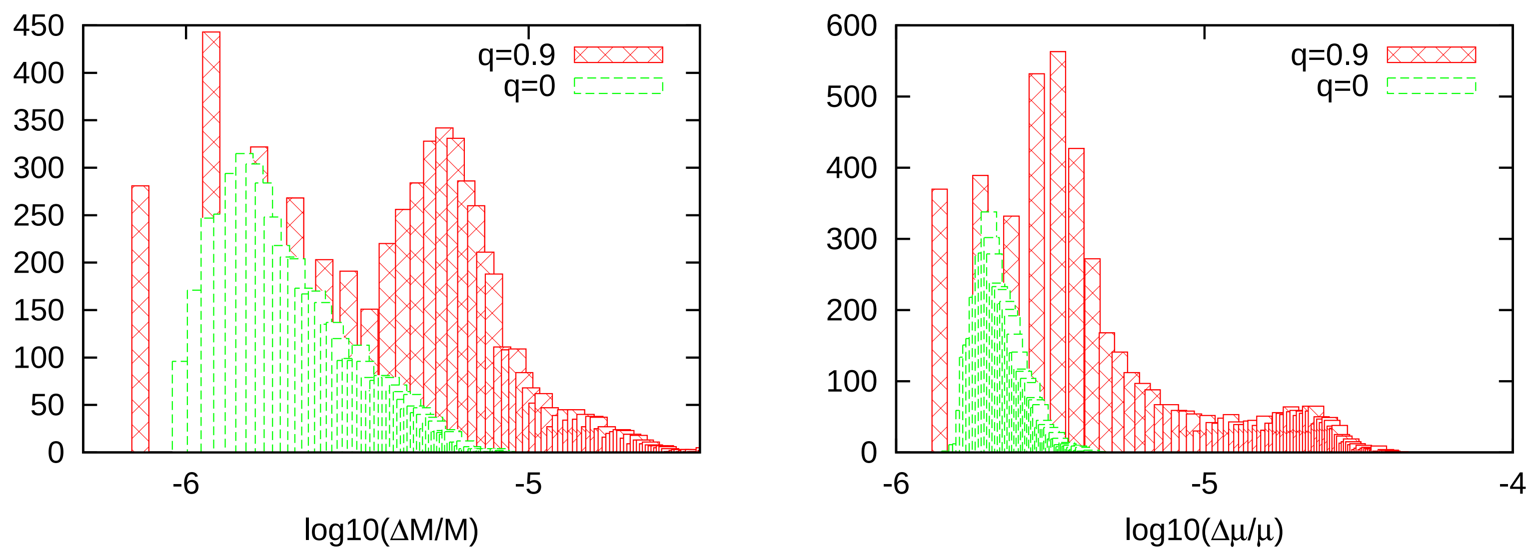
# Signal-to-noise (PRELIMINARY)

- $M = 10^6 M_\odot$, $\mu = 10 M_\odot$, $\iota^{(c)} = 45^\circ$



# Errors of masses (PRELIMINARY)

- $M = 10^6 M_\odot$, $\mu = 10 M_\odot$, $\iota^{(c)} = 45°$




$$\Delta M/M \sim 10^{-5\sim-6}, \Delta\mu/\mu \sim 10^{-5\sim-6}$$

The black hole's spin does not improve the accuracy?

# Summary and future work

Templates of EMRI in the frequency domain

- Quadrupolar wave amplitude
- All variables depend only on the gravitational frequency
  - Orbital evolution with PN and eccentricity expansion
  - LISA antenna beam pattern

Future work

- Debug
- The angular resolution of the source
- Higer PN and eccentricity?
- Application for (MOCK) LISA data analysis?

# Polarization amplitude $A_k$ for $e = 0.1$

- $\iota = 0°$, $\iota = 60°$, $\iota = 120°$ and $\iota = 180°$