Radiation reaction and energy-momentum conservation

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General setting

- In classical fields theories one usually deals with point particles interacting via massless fields. For linear theories in Minkowski space, such as classical electrodynamics (CED), the set of equations of motion (EOM) consists of the equations for particles moving in the collective field created by all of them (including that of the selected particle), and the linear field equations sourced by the particles.
- Generically, the particles motion results in radiation, which can then propagate freely and leave the region where particles are located. From the conservation equations it is expected that the momentum transferred to radiation is borrowed from the particles kinetic energy, and consequently, a radiation reaction force must exist.
- The overall conservation of the momentum implies that the action of the reaction force must be balance by the momentum of radiation. However, the problem is a bit more complicated since the field generated by the particle is partly bound to it, so we have to consider three ingredients: kinetic particles momenta, the field bound momentum and the radiation momentum.
- The bound momentum is described by the so-called Schott term which is generically a total higher derivative term. Special care is needed to understand the momentum balance in some situation, like Born paradox, self-accelerating solutions etc.
- For non-linear theories and/or theories in curved space and one can still maintain basic treatment of flat-space linear theories provided spacetime has symmetries ensuring the existence of conservation equations and asymptotically flat regions to identify radiation.

Schott term in Maxwell-Lorentz CED

The system of N point charges interacting via the Maxwell field is described by the coupled system of equations

$$\partial_\nu F^{\mu\nu} = 4\pi \int \sum_{a=1}^{N} \dot{z}^\mu \delta(x - z_a(\tau)) d\tau$$

$$m_a \ddot{z}_a^\mu = e_a F^\mu{}_\nu \dot{z}_a^\nu$$

It has $3N + \infty$ degrees of freedom, where ∞ stands for the Maxwell field. The corresponding energy-momentum conservation equation is

$$\partial_\nu (\overset{m}{T}{}^{\mu\nu} + \overset{F}{T}{}^{\mu\nu}) = 0$$

where

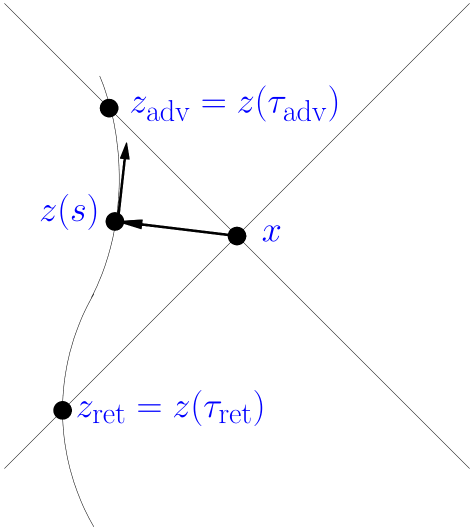
$$\overset{m}{T}{}^{\mu\nu} = \int \sum_{a=1}^{N} \dot{z}^\mu \dot{z}^\nu \delta(x - z_a(\tau)) d\tau$$

is particles stress-tensor and

$$\overset{F}{T}{}^{\mu\nu} = \frac{1}{4\pi} \left(F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right)$$

For each given charge e_a the field generated by the other $N-1$ charges is an external field $F^{\mu\nu}_{\text{ext}}$, which is regular at its location. But the whole field $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + F^{\mu\nu}_{\text{ret}}$ contains the singular contribution of the chosen charge $F^{\mu\nu}_{\text{ret}}$, which has to be dealt with somehow. Note that this decomposition holds only in the linear theory. In what follows we omit external filed for brevity. The energy-momentum conservation is ensured by the EOM-s in spite of infinities.

The retarded potential



$z_{\rm adv} = z(s_{\rm adv})$ and $z_{\rm ret} = z(s_{\rm ret})$ are intersection points of the future/past light cone of the point x with the world-line; $z(s)$ is intersection point of the plane orthogonal to the world-line and passing through x. The retarded potential at x depends on variables taken at $s_{\rm ret}(x)$ defined as the solution to the equation $R^\mu R_\mu = 0, \quad R^\mu = x^\mu - z^\mu(s_{\rm ret})$, satisfying $x^0 > z^0$. The advanced solution to the same equation with $z^0 > x^0$ refers to the advanced proper time $s_{\rm adv}(x)$. Introducing the invariant distance $\rho = v_\mu(s_{\rm ret})R^\mu, \quad v^\mu = \frac{dz^\mu}{ds}$, which is equal to the spatial distance $|\mathbf{R}| = |\mathbf{x} - \mathbf{z}(s_{\rm ret})|$ between the points of emission and observation in the momentarily co-moving Lorenz frame at the time moment $x^0 = z^0(s_{\rm ret})$, one can present the retarded potential as

$$A^\mu_{\rm ret}(x) = \frac{ev^\mu}{\rho}\Big|_{s_{\rm ret}(x)}.$$

Field split versus stress tensor split

- Introduce the null vector $c^\mu = R^\mu/\rho$, whose scalar product with v^μ is equal to unity, and the unit space-like vector $u^\mu = c^\mu - v^\mu$. Thus we have (signature + - - -)

$$v^2 = 1, \quad c^2 = 0, \quad vc = 1, \quad u^2 = -1.$$

One has
$c_\mu = \partial_\mu s_{\rm ret}(x), \quad \partial_\mu \rho = v_\mu + \lambda c_\mu, \quad \partial_\mu c^\nu = \frac{1}{\rho}\left(\delta_\mu^\nu - v_\mu c^\nu - c_\mu v^\nu - \lambda c_\mu c^\nu, \right)$ where $\lambda = \dot\rho = \rho(ac) - 1$. Then the field tensor

$$F_{\mu\nu} = \frac{e\left(\rho(ac) - 1\right)}{\rho^2} v_{[\mu} c_{\nu]} - \frac{e}{\rho} a_{[\mu} c_{\nu]}$$

- The retarded potential in Minkowski space admits a natural decomposition with respect to T-parity:

$$A^\mu_{\rm ret} = A^\mu_{\rm self} + A^\mu_{\rm rad}$$

with $A^\mu_{\rm rad} = \frac{1}{2}\left(A^\mu_{\rm ret} - A^\mu_{\rm adv}\right)$ satisfying an homogeneous equation, while $A^\mu_{\rm self} = \frac{1}{2}\left(A^\mu_{\rm ret} + A^\mu_{\rm adv}\right)$ is sourced at $x = z(s)$

- One could expect that T-symmetric $A^\mu_{\rm self}$ corresponds to bound field, but in flat space it is only a singular term, absorbed by the mass renormalization. In curved space-time $A^\mu_{\rm self}$ also contains a finite part (DeWitt-DeWitt-Smith-Will force)

Bound momentum

To define bound part of the field we have to split the stress-tensor as function of the retarded (physical) field, rather than the field itself (Rohrlich)

$$\overset{F}{T}{}^{\mu\nu} = T^{\mu\nu}_{\rm emit} + T^{\mu\nu}_{\rm bound},$$

where the first term is proportional to ρ^{-2}:

$$\frac{4\pi}{e^2} T^{\mu\nu}_{\rm emit} = -\frac{(ac)^2 + a^2}{\rho^2} c^\mu c^\nu,$$

while the second contains higher powers of ρ^{-1}:

$$\frac{4\pi}{e^2} T^{\mu\nu}_{\rm bound} = \frac{a^{(\mu} c^{\nu)} + 2(ac)c^\mu c^\nu - (ac)v^{(\mu} c^{\nu)}}{\rho^3} + \frac{v^{(\mu} c^{\nu)} - c^\mu c^\nu - \eta^{\mu\nu}/2}{\rho^4}.$$

"Emitted" part $T^{\mu\nu}_{\rm emit}$ is distinguished by the following properties:

- Its geometric structure is the tensor product of two null vectors c^μ,
- It is traceless: $T^{\mu}_{\nu\,\rm emit} = 0$,
- It falls down as $|\mathbf{x}|^{-2}$ as $|\mathbf{x}| \to \infty$,
- It is independently conserved $\partial_\nu T^{\mu\nu}_{\rm emit} = 0$.

All these features indicate that $T^{\mu\nu}_{\rm emit}$ describes an outgoing radiation. An independent conservation of this quantity means that the bound part satisfies

$$\partial_\nu (T^{\mu\nu}_{\rm bound} + \overset{m}{T}{}^{\mu\nu}) = 0.$$

Momentum balance and split of the reaction force

Conservation of the total four-momentum implies that the sum of the mechanical momentum and the momentum of the electromagnetic field is constant $\frac{dp^\mu_{\rm mech}}{ds} + \frac{dp^\mu_{\rm em}}{ds} = 0$. Here $p^\mu_{\rm mech} = m_0 v^\mu$, while the field part is given by $p^\mu_{\rm em} = \int T^{\mu\nu} d\Sigma_\nu$, where the retarded field must be used. According to the split of the stress-tensor

$$\frac{dp^\mu_{\rm mech}}{ds} = -\frac{dp^\mu_{\rm em}}{ds} = f^\mu_{\rm emit} + f^\mu_{\rm bound},$$

where

$$f^\mu_{\rm emit} = -\frac{d}{ds}\int T^{\mu\nu}_{\rm emit} d\Sigma_\nu, \quad f^\mu_{\rm bound} = -\frac{d}{ds}\int T^{\mu\nu}_{\rm bound} d\Sigma_\nu.$$

On the other hand, the derivative of the bare mechanical momentum can be expressed using the equation of motion of the charge in which the electromagnetic field is decomposed into the self part and the radiation parts

$$\frac{dp^\mu_{\rm mech}}{ds} = eF^{\mu\nu}_{\rm ret} v_\nu = \left(F^{\mu\nu}_{\rm self} + F^{\mu\nu}_{\rm rad}\right) v_\nu = f^\mu_{\rm self} + f^\mu_{\rm rad}.$$

Clearly, the following energy-momentum conservation identity should hold:

$$f^\mu_{\rm self} + f^\mu_{\rm rad} = f^\mu_{\rm bound} + f^\mu_{\rm emit}$$

Now, somewhat unexpectedly, $f^\mu_{\rm rad} \neq f^\mu_{\rm emit}$ and $f^\mu_{\rm self} \neq f^\mu_{\rm bound}$, differing by the Schott term:

$$f^\mu_{\rm rad} = f^\mu_{\rm emit} + f^\mu_{\rm Schott}, \quad f^\mu_{\rm self} = f^\mu_{\rm bound} - f^\mu_{\rm Schott}.$$

The identity is satisfied as expected.

World-line calculation of the reaction force

The retarded and advanced potential taken on the world-line $x^\mu = z^\mu(s)$ of a charge can conveniently be written in terms of Green's functions

$$G_{\rm self}(Z) = \delta(Z^2), \quad G_{\rm rad}(Z) = \frac{Z^0}{|Z^0|}\delta(Z^2),$$

where $Z^\mu = Z^\mu(s,s') = z^\mu(s) - z^\mu(s')$. Substitution of the electromagnetic field of the charge on its world-line leads to the following integrals

$$f^\mu(s) = 2e^2 \int Z^{[\mu}(s,s')v^{\nu]}(s')v_\nu(s)\frac{d}{dZ^2}G(Z)ds',$$

Due to the presence of delta-functions in $G_{\rm self}$ and $G_{\rm rad}$, one may expand the integrands in $\sigma = s - s'$. Taking into account that $Z^2 = \sigma^2 + O(\sigma^4)$, one can write

$$G_{\rm self}(Z) = \delta(\sigma^2) + O(\sigma^4), \quad G_{\rm rad}(Z) = \frac{\sigma}{|\sigma|}\left(\delta(\sigma^2) + O(\sigma^4)\right).$$

Expanding the rest of the integrands in σ, one encounters the following integrals:

$$A_l = \int_{-\infty}^{\infty} \sigma^l \frac{d}{d\sigma^2}\delta(\sigma^2)\, d\sigma, \quad B_l = \int_{-\infty}^{\infty} \sigma^l \frac{d}{d\sigma^2}\left(\frac{\sigma}{|\sigma|}\delta(\sigma^2)\right) d\sigma$$

with $l \geq 2$. Regularizing by 'point-splitting' $\delta(\sigma^2) \to \delta(\sigma^2 - \epsilon^2)$ we obtain $A_2 = -\frac{1}{2\epsilon}, \quad B_3 = -1$, so one finds $f^\mu_{\rm self} = -\frac{e^2}{2\epsilon}a^\mu, \quad f^\mu_{\rm rad} = \frac{2e^2}{3}(v^\mu a^2 + \dot{a}^\mu)$. After mass renormalization, $m_0 - A_2 = m$, we get the Lorentz-Dirac equation

$$ma^\mu = \frac{2e^2}{3}(v^\mu a^2 + \dot{a}^\mu)$$

Self-force vs radiation reaction force

- The Abraham reaction force contains the radiation recoil term

$$f^{\mu}_{\rm emit} = \frac{2}{3} e^2 a^2 v^{\mu},$$

 and the Schott term

$$f^{\mu}_{\rm Schott} = \frac{2}{3} e^2 \dot{a}^{\mu}$$

 This latter is a total derivative, so it does not correspond to an irreversible loss of momentum by the particle, but plays an important role in the momentum balance between the radiation and particle momentum loss. If $f^{\mu}_{\rm rad} = 0$, this does not necessarily mean that there is no radiation (recall the of a uniformly accelerated charge), but if there is no radiation, the Abraham force is zero. Indeed, if one has $a_{\mu} a^{\mu} = 0$ at any time, then it is easy to show that the three-acceleration is zero, $\mathbf{a} = 0$, and therefore $\dot{a}^{\mu} = 0$. Thus, no radiation reaction force is possible in absence of radiation. It seems natural to have this property maintained in curved space too in physically reasonable terminology.

- The self-force, given by the T-even Green function, in flat is infinite and proportional to acceleration, thus it can be absorbed by mass renormalization. This is no more so in the theories with (flat) extra dimensions. Neither it is so in 4D curved space: apart from the mass renormalization one obtains a finite part, which is known as DeWitt-DeWitt-Smith-Will force.

Physical interpretation of the Schott term

- Correct interpretation and a constructive derivation of the Schott term was given by Dirac. Still in the literature there were doubts because of use of the advanced potential. Havas and Rohrlich clarified that actually only the physical retarded field was involved in the calculation of the bound electromagnetic momentum, and the whole expression for the Abraham vector can be obtained using only the retarded field. This became especially transparent after a later investigation of the nature of the Schott term by Teitelboim, who emphasized that this term originates from the bound electromagnetic momentum, though didn't provided a detailed calculation (later done by Galtsov and Spirin 04').
- Note, that the Schott term does not show up if integration of the bound momentum is performed using the retarded coordinates of Newman and Unti (Poisson) (since the sequence of space-like hypersurfaces are not well defined in this case).
- Schott term is sometimes included into the modified momentum of the "dressed" particle. Such an interpretation immediately gives rise to the problem of self-accelerating solutions, Born paradox, and even a conclusion about inconsistency of CED (Landau and Lifshitz)
- Rather we deal with a point particle with a variable coat whose momentum has still to be regarded as the field momentum. Then the momentum conservation equation of the Maxwell-Lorentz theory holds. Changing acceleration entails the momentum exchange between the charge and the coat. This explains the Born paradox.

Remarks on the "orthogonalization"

- Confusion about the Schott term is partially related to the 'phenomenological' derivation of the Lorentz-Dirac equation (Landau and Lifshitz) adding the Schott term by hand from the requirement of orthogonality of the reaction force to the particle four-velocity. The first term of the Abraham force is obtained by computing the rate of radiation as a flux in the wave zone, $f^{\mu}_{\rm emit} = \frac{2}{3}e^2a^2v^{\mu}$, and it is not orthogonal to the 4-velocity
$$f^{\mu}_{\rm emit}v_{\mu} = \frac{2}{3}e^2a^2 \neq 0.$$
Adding δf^{μ} of the corresponding mass dimension so that $(f^{\mu}_{\rm emit} + \delta f^{\mu})v_{\nu} = 0$ one finds $\delta f^{\mu} = f^{\mu}_{\rm Schott} = \frac{2}{3}e^2\dot{a}^{\mu}$
- Formally, this procedure leads to the correct equation (though does not explain origin of the Schott term), so *per se* it does not contradict to the correct interpretation it should be viewed as finite part of the bound electromagnetic momentum. But if one does not relate the Schott term to the bound momentum, the energy-momentum balance equations become contradictory.
- The difference between this 'phenomenological' derivation of the Schott term and its consistent treatment as the derivative of the bound electromagnetic momentum is very clear in higher dimensions: generically the number of possible momentum 'counterterms' in higher dimensions is larger than the number of equations arising from the requirement of the orthogonality. So the Schott term(s) can not be obtained by orthogonalization in even dimensions higher than six.

Integration of the electromagnetic momentum

To calculate the four-momentum carried by the electromagnetic field of the charge for a given moment of the proper time s on the particle world-line $z^\mu(s)$ one has to choose a space-like hypersurface $\Sigma(s)$ intersecting the world-line at the point $z^\mu(s)$ and to integrate the electromagnetic energy-momentum flux as follows

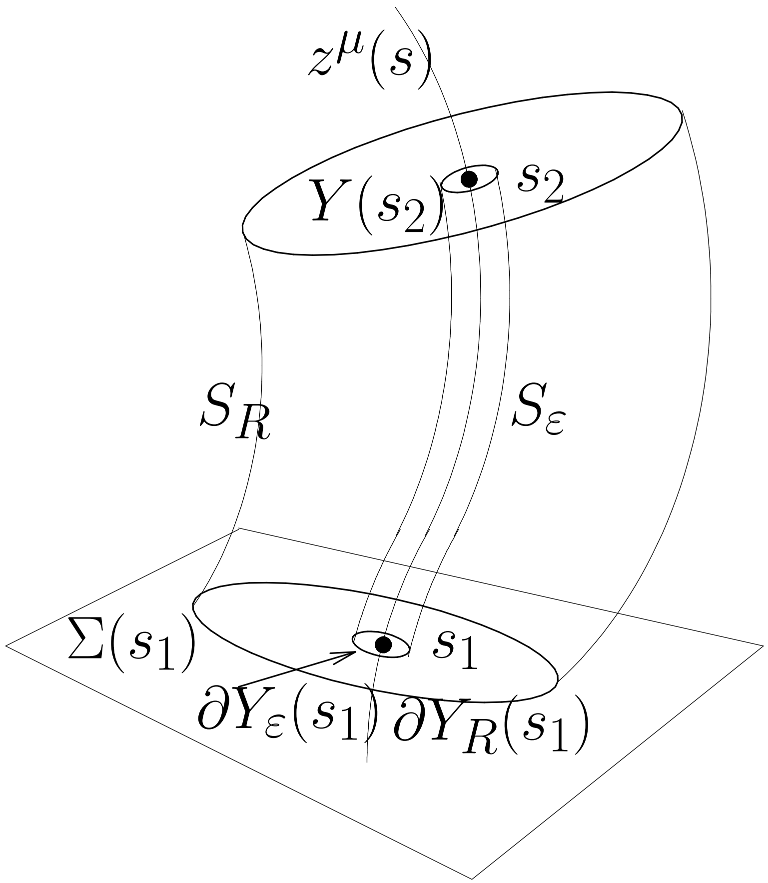
$$p^\mu_{\rm em}(s) = \int\limits_{\Sigma(s)} T^{\mu\nu} d\Sigma_\nu .$$

The most practical choice for $\Sigma(s)$ is the hypersurface orthogonal to the world-line

$$v_\mu(s)\left(x^\mu - z^\mu(s)\right) = 0.$$

To control divergence near the world line introduce the small tube with the radius ϵ, of the 2-sphere $\partial Y_\epsilon(s)$ (Fig. 1), defined by the intersection of the hyperplane with the hyperboloid $(x - z(s))^2 = -\varepsilon^2$. We also introduce the sphere $\partial Y_R(s)$ of a large radius R defined by the intersection of $\Sigma(s)$ with the hyperboloid $(x - z(s))^2 = -R^2$. The electromagnetic momentum can then be obtained by taking the limit $\epsilon \to 0,\ R \to \infty$ of the integral over the domain $Y(s) \subset \Sigma(s)$ between the boundaries $\partial Y_\epsilon(s)$ and $\partial Y_R(s)$.
Let us evaluate the variation of this quantity between the moments s_1 and s_2 of the proper time on the world-line

$$\Delta p^\mu_{\rm em} = \int\limits_{Y(s_2)} T^{\mu\nu} d\Sigma_\nu - \int\limits_{Y(s_1)} T^{\mu\nu} d\Sigma_\nu .$$



Integration of the bound electromagnetic momentum. Here $\Sigma(s_1)$ is the space-like hyperplane transverse to the world-line $z^\mu(s)$ intersecting it at the proper time s_1 (similarly $\Sigma(s_2)$). The hypersurfaces S_ε and S_R are small and large tubes around the world-line formed by sequences of the 2-spheres $\partial Y_\varepsilon(s)$ and $\partial Y_R(s)$ for $s \in [s_1,\, s_2]$. The domain $Y(s_2) \subset \Sigma(s_2)$ (similarly $Y(s_1)$) is the 3-annulus between $\partial Y_R(s_2)$ and $\partial Y_\varepsilon(s_2)$.

Bound momentum

For the bound momentum it is convenient to consider the tubes S_ϵ and S_R formed as sequences of the spheres $\partial Y_\epsilon(s)$ and $\partial Y_R(s)$ on the interval $s \in [s_1,\, s_2]$ and to transform this quantity to

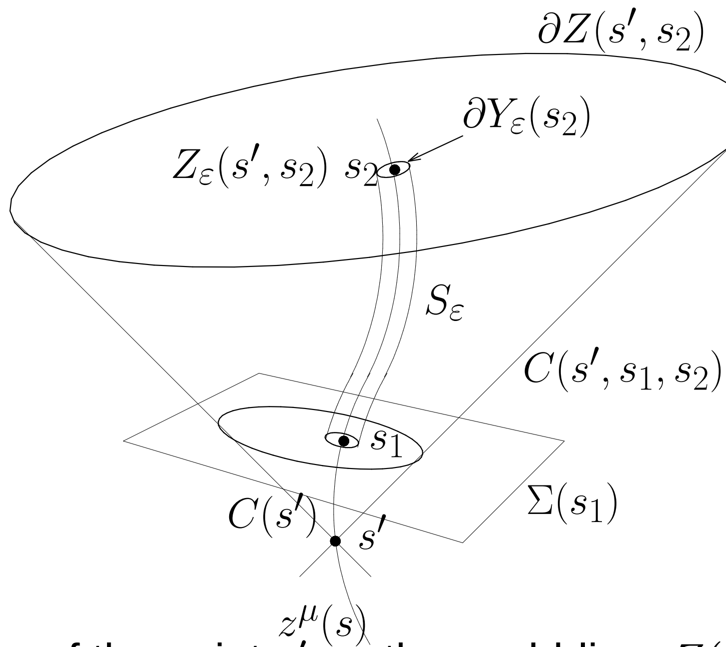
$$\Delta p^\mu_{\text{bound}} = \int\limits_{S_R} T^{\mu\nu}_{\text{bound}} dS_\nu - \int\limits_{S_\epsilon} T^{\mu\nu}_{\text{bound}} dS_\nu \tag{1}$$

in view of the conservation equation for $T^{\mu\nu}_{\text{bound}}$. Here normal vectors in dS_ν are directed outwards with respect to the world-line. The contribution from the infinitely distant surface S_R vanishes if one assumes that the charge acceleration is zero in the limit $s \to -\infty$. This assertion is somewhat non-trivial, since, in spite of the fact that the stress tensor (**??**) decays as R^{-3}, the corresponding flux does not vanish *a priori*, because the surface element contains a term (proportional to the acceleration) which asymptotically grows as R^3. As a consequence, the surviving term will be proportional to the acceleration taken at the moment s_{ret} of the proper time, where $s_{\text{ret}} \to -\infty$ in the limit $R \to \infty$. Finally we are left with the integral over the inner boundary only

$$\Delta p^\mu_{\text{bound}} = -\int\limits_{S_\epsilon} T^{\mu\nu}_{\text{bound}} dS_\nu .$$

Radiation

For integration of the emitted momentum it is convenient to take the light cone boundary $C(s')$ instead of S_R



$C(s')$ is the future light cone of the point s' on the world-line, $Z(s', s_2)$ (similarly $Z(s', s_1)$) is the annulus between the intersections of the light cone with the outer boundary $\Sigma(s_2)$ and the inner boundary $\partial Y_\varepsilon(s_2)$. The change of the emitted momentum in the whole three-space corresponds to the limits $s' \to -\infty, \epsilon \to 0$. Since the normal to the light cone lies on it, the flux of the energy-momentum through the null boundary $C(s', s_1, s_2)$ is zero for any s', therefore

$$\Delta p^\mu_{\rm emit} = -\int_{S_\epsilon} T^{\mu\nu}_{\rm emit} dS_\nu$$

with $\epsilon \to 0$, $s' \to -\infty$. An integration gives

$$f^\mu_{\rm emit} = \frac{2}{3} e^2 a^2 v^\mu$$

To compute bound momentum all quantities depending on retarded time must be expanded in ϵ-series

$$R^\mu = x^\mu - z^\mu(s_{\rm ret}) = \varepsilon n^\mu + v^\mu \sigma - \frac{1}{2} a^\mu \sigma^2 + \frac{1}{6} \dot{a}^\mu \sigma^3 + \mathcal{O}(\sigma^4)$$

where $\sigma = s - s_{\rm ret} > 0$ and all the vectors in the last line are taken at s. This is an expansion in powers of σ, but we need an expansion in powers of ε. The relation between the two can be found from the condition $R^2 = 0$:

$$R^\mu = (n^\mu + v^\mu)\varepsilon + ((an)v^\mu - a^\mu)\,\frac{\varepsilon^2}{2} + \left[\left(9(an)^2 + a^2 - 4\dot{a}n\right) v^\mu - 12(an)a^\mu + 4\dot{a}^\mu\right] \frac{\epsilon^3}{24}.$$

Similar expansions can be obtained for the velocity and the acceleration at $s_{\rm ret}$ Using them we obtain

$$\Delta p^\mu_{\rm bound} = \frac{e^2}{4\pi} \int_{s_1}^{s_2} ds \{ \frac{-n^\mu}{2\varepsilon^2} + \frac{a^\mu}{2\varepsilon} + [\left((an)^2 + a^2/3\right) v^\mu + \left((an)^2 + a^2/2\right) n^\mu - 2\dot{a}^\mu/3 + 3(an)a^\mu/4]$$

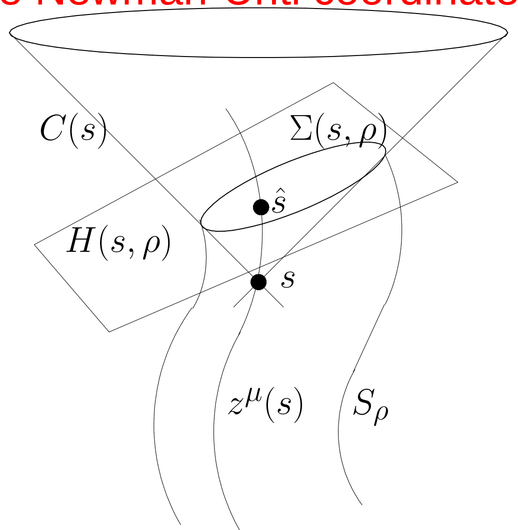
Integrating over the angles, one can see that the leading divergent term proportional to $1/\epsilon^2$ vanishes and the result reads

$$\Delta p^\mu_{\rm bound} = e^2 \int_{s_1}^{s_2} ds \left(\frac{1}{2\varepsilon} a^\mu - \frac{2}{3} \dot{a}^\mu \right).$$

Therefore the bound part of the self-force is

$$-\frac{dp^\mu_{\rm bound}}{ds} = -\frac{e^2 a^\mu}{2\epsilon} + \frac{2e^2}{3} \dot{a}^\mu = f^\mu_{\rm div} + f^\mu_{\rm Schott}.$$

Integration of the bound flux in the Newman-Unti coordinates



$C(s)$ is a future light cone with an apex at $z(s)$, $H(s,\rho)$ - hyperplane transverse to $v(s)$ intersecting the world-line at $\hat{s}$, ρ as an affine length parameter on a null geodesic. The two-dimensional surface $\Sigma(s,\rho)$ is an intersection $\Sigma(s,\rho) = C(s) \bigcap H(s,\rho)$, and S_ρ is a lateral hypersurface of constant ρ, formed by all $\Sigma(s',\rho)$. The orbit of the constant $\rho(s,x)$ on the cone forms a two-dimensional manifold $\Sigma(s,\rho)$. The open tube is defined as the sequence $\Sigma(u,\rho)$ of hypersurfaces $\rho = \mathrm{const}$ for $u \in (-\infty, s]$. Integration of the flux over the tube for fixed ρ with the subsequent limit $\rho \to 0$ has a problem when applied to the bound momentum because of the singular nature of the integrand at $\rho = 0$. This means to consider the sequence of tubes of variable radius ρ, which do not give the field momentum associated with any given moment of the proper time. With changing ρ, the spheres $\Sigma(s,\rho)$ move across the light cone and do not lie on a definite space-like hypersurface. Such a (wrong) calculation gives for the bound momentum only the leading divergent term $\frac{dp^\mu_{\mathrm{bound}}}{ds} = \frac{e^2 a^\mu}{2\epsilon}$, but fails to produce the second finite Schott term. For the emitted momentum the procedure works well, since the answer in this case is given entirely by the leading term.

Energy balance for accelerated charge

- Consider a charged moving under action of an external force. The total momentum consists of the particle kinetic momentum $p^{\rm mech} = mv^\mu$, the bound momentum of the field $p^\mu_{\rm bound}$ and the momentum stored in the radiated field $p^\mu_{\rm emit}$. All the three can be associated with the given moment of the proper time s on the particle world-line
- If acceleration is not constant (in the laboratory frame) the charge kinetic momentum can be transfered to the bound momentum and back. The derivative of the bound momentum is given by $\frac{dp^\mu_{\rm bound}}{ds} = -f^\mu_{\rm Schott} = -\frac{2}{3}e^2\dot{a}^\mu$
- When the acceleration increases from zero to some value $\dot{a}^\mu > 0$, this means that the bound field momentum is decreased according to

$$\frac{dp^\mu_{\rm bound}}{ds} = -\frac{2}{3}e^2\dot{a}^\mu.$$

At the same time,the momentum stored in the radiation changes according to

$$\frac{dp^\mu_{\rm emit}}{ds} = -\frac{2}{3}e^2a^2v^\mu$$

- If acceleration becomes constant in the charge proper frame, one has $\frac{dp^\mu_{\rm bound}}{ds} = -\frac{dp^\mu_{\rm emit}}{ds}$, so the kinetic particle momentum remains constant. If consequently acceleration decreases to zero, the kinetic momentum is transferred to bound momentum, so finally the kinetic momentum is lost on the amount equal to the radiated momentum

Field split vs momentum split

Two type of splits are different in the following:

- The split on the half-sum and the half-difference of the retarded and advanced potentials is *decomposition of the field*. It can be proved generally (in flat space) that the flux of the momentum of radiation in the wave zone coincides with the work produced by the *radiative* component of the self-force on a charge:

$$\int f^{\mu}_{\mathrm{rad}} dt = \int_{S^2_\infty} \overset{F}{T}{}^{\mu\nu} d\Sigma_\nu dt$$

  (where $T^{\mu\nu}$ may contain or not the bound part: it does not contribute)

- In curved space-time with asymptotically flat regions (and/or black hole horizons) one can prove similar relation in presence of the Killing vectors K^i_μ, replacing $f^{\mu}_{\mathrm{rad}} \to f^{\mu}_{\mathrm{rad}} K^i_\mu$, and $\overset{F}{T}{}^{\mu\nu} \to \overset{F}{T}{}^{\mu\nu} K^i_\mu$ and the integration must be taken over the two-surfaces which radiation eventually crosses.

- The split of the stress-tensor on the emitted and bound parts is the split of stresses of the *retarded* field. It is decomposition of the momentum of essentially the same field. Evolution of the retarded field causes evolution of its bound and emitted momentum in a certain proportion, that is why the "transmission" of the kinetic energy of the the particle to its electromagnetic coat (bound momentum) is accompanied by transmission of momentum to radiation. Both are described by the unique retarded field, but the relative momentum stored in these two components can change.

Non-relativistic limit

- In the rest frame of a charge the recoil force has no spatial component. Hence the total radiative self-force is presented by the Schott term, namely

$$\mathbf{f}_{\text{Schott}} = \frac{2}{3} e^2 \dot{\mathbf{a}}$$

The work done by this force

$$\int \mathbf{f}_{\text{Schott}} \cdot \mathbf{v} dt = \int \frac{2}{3} e^2 \dot{\mathbf{a}} \cdot \mathbf{v} dt = - \int \frac{2}{3} e^2 \mathbf{a}^2 dt + \text{ boundary} \quad \text{terms}$$

correctly reproduces the radiative losses (Boundary terms should vanish by appropriate asymptotic switching on/off conditions, or their absence be ensured by periodicity condition in which case the normalization to a period must be done).

- This does not contradict to our previous discussion about transmission of the energy to the coat which is described by the Schott term: it automatically ensures transmission of energy to radiation according to an overall energy balance.
- Vanishing of the recoil force in the rest frame $\mathbf{v} = \mathbf{0}$ of the particle simply means that radiation in two opposite directions is the same so that the spatial momentum is not lost by radiation, though the energy is.

Gravitational Schott term

- It can be argued that the Schott term (bound momentum) must also exist in the system of gravitating bodies. We have to distinguish several cases
  1. non-gravitational radiation under geodesic motion
  2. gravitational radiation under geodesic motion
  3. gravitational radiation under non-geodesic motion

  In the relativistic regime these processes are qualitatively different. The first case is described by DeWitt-Brehme pure tail equation. Splitting the retarded field into *self* and *rad* parts allows to separate radiative effects and the tidal effects such as the DeWitt-DeWitt-Smith-Will force.

- Two latter cases are qualitatively different in the relativistic case, but should reduce to the quadrupole approximation is the slow-motion weak-field limit. It is expected, that gravitational radiation in the quadrupole approximation must result from the *gravitational Schott term* (Landau-Lifshitz, Field theory)

$$f^i_{\mathrm{Gshott}} = -\frac{G\mu}{15}\frac{d^5 D^{ij}}{dt^5}x_j$$

so that

$$\int f^i_{\mathrm{Gshott}}\dot{x}_i dt = -\int \frac{G}{45}\dddot{D}^{ij}\dddot{D}_{ij}dt + \text{ boundary } \quad \text{terms}$$

I will show that the gravitational tail term gives the quadrupole $\mathbf{f}_{\mathrm{Gshott}}$ indeed.

Dimensions other than four

Before to address the gravitational case it is instructive to consider flat space of dimensions other than four, which already introduce some features typical for four-dimensional curved space.

- Going to other dimensions is motivated by
  1. cosmological models with extra dimensions (compact and non-compact)
  2. string theory ten (eleven) dimensional settings
  3. search for better understanding of the four-dimensional theory (in particular, dimensional regularization)
- It turns out that radiation picture is substantially different in even and odd dimensions because of the different structure of the Green's functions for massless fields in the coordinate representation (they still look similarly in all dimensions in the momentum representation)
- In even dimensions Huygens principle holds so the situation is similar to 4D case, while in odd dimensions Huygens principle does not hold which results, in particular, in non-local radiation reaction equationa already in flat space
- Generically, quantum field theories are non-renormalizable in higher dimensions, while classical theories require introduction of (higher-derivative) counterterms to eliminate divergences

Green's functions with tails

The retarded Green's functions for massless fields in even-dimensional Minkowski space-times are localized on the future light cone. In odd-dimensional Minkowski space-times and in curved space of arbitrary dimensions the retarded Green's functions have support on the future light cone and inside the future light cone. This may be thought of as violation of the Huygens principle. As a result, the retarded fields substituted to the equations of motion generate the integral tail terms over the all the past history of the particle motion. In 3D, for instance, the scalar Green's function reads

$$G^{3D}_{\rm ret}(X) = \vartheta(X^0)\vartheta(X^2)(X^2)^{(-1/2)}, \quad X^\mu = x^\mu - x'^\mu,$$

It does not-contain the "direct" part singular on the light cone. Green's functions in higher odd dimensions $D = 2n+1$ can be obtained by the recurrent relation (see e.g. Gal'tsov 01')

$$G^{2n+1}_{\rm ret}(X) \sim \frac{dG^{2n-1}_{\rm ret}}{dX^2}$$

In particular, in 5D

$$G^{5D}_{\rm ret}(X) \sim \vartheta(X^0)\left(\frac{\delta(X^2)}{(X^2)^{1/2}} - \frac{1}{2}\frac{\vartheta(X^2)}{(X^2)^{3/2}}\right)$$

both the direct and the tail parts are present. It turns out that the direct part regularizes the tail contribution to the field stress (the derivative of G) which otherwise would be singular outside the world line! To eliminate divergences on the world-line one needs (local) counterterms the number of which increases with dimension.

More counterterms in higher dimensions

In 6D one has two divergent terms (which in terms of the field split correspond to $f_{\rm self}$:

$$f^\mu_{\rm div} = -\frac{1}{6\varepsilon^3}a_\mu + \frac{1}{2\varepsilon}\left(\frac{3}{4}v_\mu(\dot{a}a) + \frac{3}{8}a^2 a_\mu + \frac{1}{4}\ddot{a}_\mu\right)$$

the leading being eliminated by the mass renormalization and the subleading requires the counterterm (Kosyakov 99')

$$S_1 = -\kappa_0^{(1)}\int(\ddot{z})^2 ds$$

which leads to the Frenet-Serre dynamics unless the renormalized value of $\kappa^{(1)} = 0$. Each two space-time dimensions add one new higher-derivative counterterm needed to absorb divergencies. Thus, the CED in higher dimensions is classically renormalizable at the expence of introducing new counterterms in the action.

The radiation recoil force is

$$f^\mu_{\rm emit} = \frac{4}{45}e^2\left(\dot{a}^2 v^\mu + \frac{2}{21}(a\dot{a})a^\mu - \frac{2}{9}a^4 v^\mu - \frac{2}{105}a^2\dot{a}^\mu\right)$$

and the Schott terms is

$$f^\mu_{\rm Schott} = -\frac{4e^2}{45}\left(\dddot{a}^\mu + \frac{16}{7}a^2\dot{a}^\mu + \frac{60}{7}(a\dot{a})a_\mu + 4\dot{a}^2 v^\mu + 4(a\ddot{a})v^\mu\right),$$

the sum of two being orthogonal to the 6-velocity. Note that in 8D and higher one can not obtain the Schott term by renormalization.

Lessons from other dimensions: even

In even dimensional space-times (no tails) two alternative splits are possible:

- Split of the retarded field into T-even (self) part and T-odd (rad) part. Then one performs a local calculation substituting fields directly into the equations of motion. One always obtain
$$f^\mu_{\rm self} = f^\mu_{\rm div}, \quad f^\mu_{\rm rad} = {\rm finite}.$$
Divergent terms are Lagrangian type and can be absorbed by introducing suitable counterterms.

- Split the field stress-tensor built with retarded field only into the sum of the emitted and bound terms. Then by a world-tube calculation one gets a different representation of the total self-force:
$$f^\mu_{\rm self} + f^\mu_{\rm rad} = f^\mu_{\rm bound} + f^\mu_{\rm emit},$$
where $f^\mu_{\rm emit}$ is the radiation recoil force, and $f^\mu_{\rm bound}$ is the non-radiated bound field contribution due to exchange of the momentum between the particle and its coat. One always has
$$f^\mu_{\rm rad} = f^\mu_{\rm emit} + f^\mu_{\rm Schott}, \quad f^\mu_{\rm Schott} = f^\mu_{\rm bound} - f^\mu_{\rm self},$$
where the Schott term has to be identified via integration of the bound momentum.

- Acting self-force in entirely given by T-odd “rad” Green’s function within the local calculation

Lessons from higher dimensions: odd

- In odd dimensions one always have tail terms. Split of the retarded field into self and rad parts is always possible, and the substitution into the equations of motion leads to divergent and finite terms. Divergent ones can again be absorbed introducing (still local!) counterterms.
- However to split the stress tensor built on the retarded field into radiation (the emitted part) and the bound part is difficult already in 3D (though with some additional assumptions, it is still possible to define the emitted part Gal'tsov and Spirin 08').
- Therefore, the local calculation appealing to the fields on the world-line only is preferable, though the problem of physical interpretation of the tail term and identification of the emitted part has no general solution so far.
- No general formula for radiation like in odd dimensions is available in terms of the particle world-line local parameters (derivative of the velocity). For particular types of motion this is still possible.
- Tail term in odd dimensions has different origin from v-term is the Hadamard expansion in curved space: it is expanded in odd powers of σ contrary to even powers for even D and does not contain a logarithmic v-term (though there is still a w-term)

Radiation in the curved space

Passing to general relativity we face to various new problems basically related to

1. non-linearity
2. absence of the conservation laws
3. absence of “good” asymptotics

- Non-linearity makes problems in extracting radiation and self-field from the background and even in non-contradictory definition of the background itself suitable for further linearization. In some cases the radiation reaction problem can not be solved as a one-body problem at all
- Assuming stationarity (or even larger isometries ensuring the possibility of mode-sum representation for Green’s functions) and existence of flat or black hole horizon asymptotics one can avoid some of the problems, and to prove balance identities for conserved quantities like in the Kerr case (Galtsov 82’)
- One novel feature is that the T-symmetric (half sum of the retarded and advanced potentials) part of the reaction force is no more pure $\mathbf{f}_{\rm div}$, but contains some finite contribution originating from tidal deformation of the self-field by the background
- Another one is presence of tail terms in the EOM accounting for the reaction force. Their radiative part contains emitted and Schott terms, generically non-local, but extractable in the quadrupole approximation and in the mode-separable cases
- It is worth to consider separately 1) non-gravitational radiation under geodesic motion, 2) gravitational radiation under geodesic motion 3) gravitational radiation under non-geodesic motion

Local derivation of the self-force

- DeWitt-Brehme have initiated calculations of the reaction force based on integration of the stress-tensor of the field created by the particle(“world-tube” calculation). This is a curved space generalization of the computation of the bound and emitted momenta shown previously. The difference, however, is that in the curved space it is impossible to split the stress-tensor into the emitted and bound parts. So it does not make much sense to perform (quite tedious) calculations using the world-tube technique to get the result in which bound and emitted part can not be separated. In addition, this approach has problems which are partly technical, partly conceptual (cups, mass-renormalization etc) (see e.g. Poisson, 05’).
- Meanwhile, the calculation based on the substitution of the retarded field directly into the equations of motion given above for the flat-space CED perfectly works in the curved space too (Gal’tsov, Spirin and Staub 06’) for all spins 0,1,2. Technically it is much simpler (the complete derivation of the scalar force is a one-page calculation). Other advantages are:
  1. In the gravitational case no mass renormalization is involved: mass does not enter the geodesic equation. What is renormalized is the affine parameter along the world line (one-bein)
  2. split into the half-sum and the half-difference of the retarded and advanced potentials is possible, separating the WWSW type force from the radiative force
  3. split of the Detweiler-Whiting type is also possible opening a way to its further physical analysis

Geodesic motion, non-gravitational radiation: DeWitt-Brehme tail

Consider DeWitt-Brehme (WB) equation for a charge moving in a vacuum space-time (signature - + + +)

$$\mu \ddot{z}^\alpha = \frac{2}{3} e^2 (\dddot{z}^\alpha - \ddot{z}^2 \dot{z}^\alpha) + e^2 \, \dot{z}^\beta \int_{-\infty}^{\tau} f^\alpha_{\mathrm{ret}\,\beta\gamma} \dot{z}^\gamma (\tau') d\tau'.$$

In the case of geodetic motion local radiation reaction force (including the Schott term) vanishes, so radiative effects has to be contained in the tail term. Let us split the retarded potential

$$f^\alpha_{\mathrm{ret}\,\beta\gamma} \dot{z}^\gamma = f^\alpha_{\mathrm{self}\,\beta\gamma} \dot{z}^\gamma + f^\alpha_{\mathrm{rad}\,\beta\gamma} \dot{z}^\gamma$$

according T-parity. Then repeating calculations of DeWitt-DeWitt (WW) in the weak-field slow-motion approximation one finds for the corresponding parts of the self-force in terms of the flat space theory (spatial part)

$$\mathbf{f}_{\mathrm{self}} = \mathbf{f}_{\mathrm{div}} + \mathbf{f}_{\mathrm{WWSM}}, \qquad \mathbf{f}_{\mathrm{rad}} = \mathbf{f}_{\mathrm{Schott}}$$

where the divergent term is the same as in the flat space, and two finite terms

$$\mathbf{f}_{\mathrm{WWSM}} = \frac{GMe^2}{r^4} \mathbf{r}, \qquad \mathbf{f}_{\mathrm{Schott}} = \frac{2}{3} e^2 \dot{\mathbf{a}}$$

are the DeWitt-DeWitt-Smith-Will (WWSW) force and the Schott force. Therefore, the WWSW force is a finite part of the T-even (*self*) contribution to the WB tail, while the gravitational Schott term originates in the T-odd (*rad*) part and has the same form as in the flat space CED.Similar result holds for the scalar radiation.

Radiation reaction in Kerr

- Kerr is the unique stationary asymptotically flat solution of the vacuum Einstein equations possessing a regular event horizon, so it is likely to be the most general tractable case to check conservation equations for radiation reaction. This was done in *Gal'tsov, JPA 82'* using modified Chrzanowski Green's functions. The result was that the *radiative* part of the self-force (for the spins $s = 0, 1, 2$) exactly balances the sum of the fluxes of the energy and the angular momentum

$$\int f^{\mu}_{\rm rad} K^i_{\mu} dt = \mathcal{P}^i_{\infty} + \mathcal{P}^i_{\rm Hor}, \quad \mathcal{P}^i = \int T^{\mu\nu} K^i_{\mu}$$

going to infinity and absorbed at the horizon, $K^i_{\mu},\ i = t, \varphi$ being the Killing vectors.

- It is crucial that only the *radiative* part of the self-force is relevant in full analogy with the flat space linear theories (including other dimensions than four). Meanwhile, during more that ten years (1995-2006) in the literature there was a consensus that my paper have been wrong because no WWSW force was present! This criticism was incorrect: the WWSW force tidal force has nothing to do with radiation being the finite part of the T-even *half-sum* of the retarded and advanced contributions.
- Apparently this misunderstanding was overcome since 2006 when Mino and Tanaka et al. started using my proposal to calculate the evolution of the Carter's constant. Thus, because of the "errors conservation equation"

$$(-) \cdot (-) = +,$$

we now have a couple of dozens of incorrect papers!.

Gravitational Schott term

The above calculation can be used to extract the quadrupole radiation reaction force in the weak-field slow-motion approximation. Using the expression for $f^{\varphi}_{\rm rad}$ and substituting the radial functions in the approximation $M\omega \ll 1$ (hypergeometric functions) one can perform a series computation in terms of M/r. Then using the non-relativistic EOM-s (in flat space) one can show that the result is equivalent (up to total derivatives) to the quadrupole Schott term

$$f^i_{\rm Gshott} = -\frac{G\mu}{15}\frac{d^5 D^{ij}}{dt^5}x_j$$

projected on azimuthal direction. In the Hadamard expansion notation deal here with pure $v-$tail contribution. Therefore the tail term is shown to contain indeed the quadrupole radiation reaction force. Note again, that this force originates from the half difference of the retarded and advanced potentials! There is no trace of presence of the WWSW force in the balance equation for gravitational radiation.

Tidal friction

- Let us also mention the existence of another “static” force of the type different from WWSW which results from the radiative potentials (half-difference of the retarded and advanced fields). It turns out that in the static limit the azimuthal component of the reaction force remains non-zero (Gal’tsov 82’). In this case both the energy and angular momenta fluxes to infinity are zero (no radiation), the energy flux through the horizon is also zero, but the angular momentum flux through the horizon is finite and it is balances by the azimuthal component of the reaction force

$$f^{\varphi}_{\rm rad} = \frac{8}{5}\frac{G\mu M}{r^2}\frac{a}{M}\left(\frac{M}{r}\right)^5\left(1+3\frac{a^2}{M^2}\right)$$

  This is not surprising since in the physical frame rotating on the horizon one still have radiation falling to the black hole. This force is a counterpart to Hawking’s tidal friction acting on the hole.

- Contrary to the WWSW force, this force is non-lagrangian type (it is a dissipative force)

Non-geodesic motion, gravitational radiation

In this case one has no one-body equation to describe a reaction force. Naively, for a non-geodesic motion one obtains a putative “11/3” antidamping term as a local part of the *rad* contribution to the self-force

$$f^{\mu}_{\rm rad} = -\frac{11}{3} 8\pi G (g^{\mu\nu} + \dot{z}^{\mu}\dot{z}^{\nu})\dddot{z}_{\nu} \quad + \quad \text{tail.}$$

But the reason is simply that the source of gravitational radiation is incomplete: if the force is non-gravitational, one has to take into account the contribution of stresses of the field causing the body to accelerate. For instance, to describe gravitational radiation of an electron in the atom, it is insufficient to consider the motion of the electron only, but one has to construct as the source term in the linearized equation for gravitational field the sum of the contributions of masses and the Maxwell field stresses (spatial components, non-relativistic motion)

$$\Box \psi^{ij} = 16\pi G T^{ij}, \qquad T^{ij} = T^{ij}_{\rm mass} + T^{ij}_{\rm stress}$$

where

$$T^{ij}_{\rm mass} = \sum_{a=1,2} \mu_a \dot{z}^i_a \dot{z}^j_a \delta^3(X_a), \quad X^i_a = x^i - z^i_a(t), \quad T^{ij}_{\rm stress} = -\frac{e_1 e_2}{4\pi} \frac{X^i_1 X^j_2}{(X^2_1 X^2_2)^{3/2}}$$

Using this source one can calculate the gravitational force (Galtsov 84’) and find again the gravitational Schott term

$$f^i_{\rm Gshott} = -\frac{G\mu}{15} \frac{d^5 D^{ij}}{dt^5} x_j, \quad \mu = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}$$

but this time, the derivation does not result form the tail term, the two-body treatment being necessary.

Discussion and conclusions

- In the balance equations for radiation reaction force actually three and not just two ingredients are involved:
  1. Mechanical momentum of a particle
  2. Momentum carried by radiation
  3. Bound moment

  This is explicit in CED and other linear theories in flat space, and implicit in the curved space. Still, the gravitational Schott term can be extracted in some limiting cases.
- For this reason, the concept of "dressed particle" in CED as a composite object incorporating the "mechanical" point particle and the coat (bound momentum) obscures the conservation equations. The coat (described by the Schott term) has variable momentum and it is intrinsically related both to particle and radiation. To maintain unambiguous conservation equations one has still to think of the usual kinetic momentum and the electromagnetic momenta split into the bound and radiated parts. Similar argument can be applied in General Relativity.
- Radiation reaction force is given solely by the retarded minus advanced potentials. This can be expected from the T-parity argument, and also confirmed by the balance equations relating radiative loss through asymptotic regions and local friction. In the curved space such balance equations are only possible in the case of isometries (at least stationarity)

- Derivation of the equations with the self-force in curved space can be performed substituting the self field directly into EOM-s. This not only greatly simplifies the derivation, but clarify the nature of renormalization and opens the way to avoid problems associated with the DeWitt type world-tube method. Moreover., in this approach one can use field decomposition into self/rad parts or Detweiler-Whiting R/S parts providing additional insights into the physical nature of different contributions to the self-force.
- The WWSW tidal force is entirely given by the retarded plus advanced potential (after elimination of divergency). Contrary to the dissipative radiation reaction force, it can be incorporated into some Lagrangian
- Gravitational Schott term is hidden in the radiative part of the tail term and it can be extracted explicitly in some limiting cases, in particular, in the weak field slow motion approximation. In the non-relativistic limit the Schott term is the only contribution to the spatial part of the radiation reaction force
- For gravitational radiation under non-geodesic motion the non-relativistic gravitational Schott term can not be extracted from any one-body equation. Thus, the gravitational self force generally does not follow from a one-body equation being essentially a collective phenomenon

- Necessity for time integration or some averaging procedure is related to physical nature of the Schott term describing the effect of momentum exchange between the bound and radiated momenta.
- In general curved space-time no conserved quantities exist already at the level of primary (non-iterated) equations, clearly the situation is not better for equations with the self-force. On the other hand, once Killing symmetries are present, the corresponding balance equations for the radiation reaction force also hold.

Thanks!