A Rigorous Derivation of Gravitational Self-force. II

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Perturbed Motion: two preliminary remarks

1. The definition of a world line for an extended body at finite » is an extremely hard problem. We will only define the world line perturbatively. To see what we mean by this, suppose we had succeeded in defining an actual world line finite ». We could then expand in »,

$$x^i(\lambda, t) = z^i(\lambda, t) = \lambda Z^i(t) + O(\lambda^2)$$

The perturbed motion is then most naturally described as the spatial components of a vector field Z^a defined on the background world line at x^i=0. This *deviation vector* gives the “infinitesimal displacement to the perturbed motion”.

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2. The deviation vector will depend on the choice of gauge for the metric perturbations. Changing by the metric smoothly at first-order,

$$x^\mu \rightarrow \hat{x}^\mu = x^\mu - \lambda A^\mu(x^\nu) + O(\lambda^2)$$

it is clear that the finite motion will also change to first order,

$$\hat{z}^i(t) = z^i(t) - A^i(t, x^j = 0) + O(\lambda^2)$$

so that the deviation vector changes by

$$Z^i(t) \rightarrow \hat{Z}^i(t) = Z^i(t) - A^i(t, x^j = 0)$$

Physical observables are constructed from knowledge of both Z and h.

Multipole Moments

As already noted, the scaled metric at »=0 (the “body metric”) is always stationary and asymptotically flat. Stationary, asymptotically flat metrics have well-defined sets of multipole moments. This gives a natural definition of the mass/spin/quadrupole/etc. of the body at time $t_{0,}$ as the mass/spin/quadrupole/etc. of the body metric computed at time t_0.

The mass dipole moment of a stationary, asymptotically flat spacetime is pure gauge and represents the extent to which the coordinates are “off center”. In particular, if the dipole vanishes one says that the coordinates are mass-centered. To define the perturbed motion we seek coordinates for which scaling yields mass-centered body metrics.

Definition of Pertubed Motion

Our assumptions imply that (independent of gauge choice) the time-time component of the body metric (scaled metric at »=0) takes the form,

$$\bar{g}_{\bar{t}\bar{t}}(\lambda = 0; t_0) = -(1 - \frac{2M}{\bar{r}}) + O(1/\bar{r}^2) \ .$$

M is the mass of the particle. Consider a (far-zone) smooth gauge transformation,

$$x^\mu \to \hat{x}^\mu = x^\mu - \lambda A^\mu(x^\nu) + O(\lambda^2) \ .$$

This affects the scaled metric by

$$\bar{x}^\mu \to \hat{\bar{x}}^\mu = \bar{x}^\mu - A^\mu(t_0, x^i = 0) + O(\lambda)$$

which, according to the simple formula,

$$\frac{1}{\bar{r}} = \frac{1}{|\hat{\bar{x}}^i + A^i(t_0, 0)|} = \frac{1}{\hat{\bar{r}}} - \frac{A_i x^i}{\hat{\bar{r}}^3} + O(1/\hat{\bar{r}}^3) \ ,$$

changes the mass dipole by $-MA^i$. Thus, it is always possible to remove the mass dipole for all time by a far-zone gauge transformation. This “displacement to mass-centered coordinates” defines the perturbed motion,

$$Z^i(t) = A^i(t, x^j = 0)$$

Calculation of Perturbed Motion

We now use our assumptions to calculate the perturbed motion via Einstein's equation. We will use Einstein's equation at 0^{th}, 1^{st}, and 2^{nd} orders. We will compute exclusively in the far-zone for 0^{th} and 1^{st} orders, and use a mixture of far-zone and near-zone at 2^{nd} order. This choice of which "zone" to use is purely for conveneince.

For the background and first-order perturbations, make the convenient coordinate choices of Fermi normal coordinates (about the background geodesic γ) and the Lorenz gauge condition. Following DB,MST,QW, one finds,

$$
\begin{aligned}
g_{\alpha\beta}(\lambda; t, x^i) &= \eta_{\alpha\beta} + B_{\alpha i\beta j}(t) x^i x^j + O(r^3) \\
&\quad + \lambda \left(\frac{2M}{r} \delta_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}(t) + h^{\text{tail}}_{\alpha\beta,i}(t) x^i + M \mathcal{R}_{\alpha\beta}(t, x^i) + O(r^2) \right) + O(\lambda^2)
\end{aligned}
$$

where B and R are background curvature quantities and h^{tail} is the Lorenz gauge "tail term" <u>of the background geodesic</u>. Now supplement this expression by using the Einstein equation at second-order, but just for the leading behavior in 1/r. One finds…

$$
\begin{aligned}
g_{\alpha\beta}(\lambda;t,x^i) &= \eta_{\alpha\beta} + B_{\alpha i\beta j}(t)x^i x^j + O(r^3) \\
&\quad + \lambda\left(\frac{2M}{r}\delta_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}(t) + h^{\text{tail}}_{\alpha\beta,i}(t)x^i + M\mathcal{R}_{\alpha\beta}(t) + O(r^2)\right) \\
&\quad + \lambda^2\left(\frac{M^2}{r^2}(-2t_\alpha t_\beta + 3n_\alpha n_\beta) + \frac{2}{r^2}P_i(t)n^i\delta_{\alpha\beta} + \frac{1}{r^2}t_{(\alpha}S_{\beta)j}(t)n^j \right. \\
&\qquad \left. + \frac{1}{r}K_{\alpha\beta}(t,\theta,\phi) + H_{\alpha\beta}(t,\theta,\phi) + O(r)\right) + O(\lambda^3)
\end{aligned}
$$

H and K are arbitrary—we have not used the Einstein equation for these orders. The parameters P and S will turn out to be the mass and current dipoles of the body metric(s). Now introduce the smooth gauge transformation,

$$
\boxed{x^\mu \to \hat{x}^\mu = x^\mu - \lambda A^\mu(x^\nu) + O(\lambda^2)\ .}
$$

and compute

$$
\begin{aligned}
g_{\hat\alpha\hat\beta} &= \eta_{\alpha\beta} + B_{\alpha i\beta j}(\hat t)\hat x^i \hat x^j + O(r^3) \\
&\quad + \lambda\left(\frac{2M}{\hat r}\delta_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}(\hat t) + h^{\text{tail}}_{\alpha\beta,i}(\hat t)\hat x^i + M\mathcal{R}_{\alpha\beta}(\hat t,\hat x^i) + 2A_{\alpha,\beta}(\hat t,x^i) + 2B_{\alpha i\beta j}(\hat t)\hat x^i A^j(\hat t,\hat x^i) + O(r^2)\right) \\
&\quad + \lambda^2\left(\frac{M^2}{r^2}(-2t_\alpha t_\beta + 3n_\alpha n_\beta) + \frac{2}{r^2}\left[P_i(\hat t) - MA_i(\hat t,0)\right]n^i\delta_{\alpha\beta} + \frac{1}{r^2}t_{(\alpha}S_{\beta)j}(\hat t)n^j \right. \\
&\qquad \left. + \frac{1}{r}K_{\alpha\beta}(\hat t,\theta,\phi) + H_{\alpha\beta}(\hat t,\theta,\phi) + O(r)\right) + O(\lambda^3),
\end{aligned}
$$

The effects of A have been “absorbed” in to H,K

Compute the scaled metric, applying the “no mass dipole” condition to get,

$$\bar{g}_{\bar{\alpha}\bar{\beta}}(\hat{t}_0) = \eta_{\alpha\beta} + \frac{2M}{\bar{r}}\delta_{\alpha\beta} + \frac{M^2}{\bar{r}^2}\left(-2t_\alpha t_\beta + 3n_\alpha n_\beta\right) + \frac{1}{\bar{r}^2}t_{(\alpha}S_{\beta)j}n^j + O\left(\frac{1}{\bar{r}^3}\right)$$
$$+ \lambda\left[h^{\text{tail}}_{\alpha\beta} + 2A_{(\alpha,\beta)} + \frac{1}{r}K_{\alpha\beta} + \frac{\bar{t}}{\bar{r}^2}t_{(\alpha}\dot{S}_{\beta)j}n^j + O\left(\frac{1}{\bar{r}^2}\right) + \bar{t}\,O\left(\frac{1}{\bar{r}^3}\right)\right]$$
$$+ \lambda^2\Bigg[B_{\alpha i\beta j}\bar{x}^i\bar{x}^j + h^{\text{tail}}_{\alpha\beta,\gamma}\bar{x}^\gamma + M\mathcal{R}_{\alpha\beta}(\bar{x}^i) + 2B_{\alpha i\beta j}A^i\bar{x}^j + 2A_{(\alpha,\beta)\gamma}\bar{x}^\gamma$$
$$+ H_{\alpha\beta} + \frac{\bar{t}}{\bar{r}}\dot{K}_{\alpha\beta} + \frac{\bar{t}^2}{\bar{r}^2}t_{(\alpha}\ddot{S}_{\beta)j}n^j + O\left(\frac{1}{\bar{r}}\right) + \bar{t}\,O\left(\frac{1}{\bar{r}^3}\right) + \bar{t}^2\,O\left(\frac{1}{\bar{r}^3}\right)\Bigg] + O(\lambda^3)$$

Now “plug in” this form to the near-zone Einstein equation. Because the far-zone Einstein equation was already used extensively, the near-zone Einstein equation is satisfied trivially for most terms. However, the *l*=1, electric parity, even under time reversal, 1/r^2 and t/r^3 part of the second-order Einstein equation gives…

(BIG MATRIX)

$$
\begin{pmatrix} R_i^A \\ R_i^B \\ R_i^C \\ R_i^D \\ R_i^E \\ R_i^F \end{pmatrix}
= -\frac{1}{2}
\begin{pmatrix}
\frac{-16}{5} & -3M & 0 & -M & -M & -2 & 0 & 0 & 0 & -2 & 2 \\
0 & 0 & -6M & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 \\
0 & 0 & 2M & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\frac{-6}{5} & -6M & 0 & 0 & -6M & -3 & 3 & 0 & 3 & -6 & 0 \\
\frac{-16}{5} & 0 & 4M & 0 & 4M & 2 & -2 & 0 & -2 & 2 & -2 \\
2 & 0 & 3M & -3M & 9M & 1 & -3 & 0 & -3 & 2 & 0
\end{pmatrix}
\begin{pmatrix} F_i \\ D_i^A \\ D_i^C \\ D_i^E \\ D_i^F \\ H_i^A \\ H_i^D \\ H_i^E \\ H_i^F \\ \dot{K}_i^B \\ \dot{K}_i^C \end{pmatrix}
$$

linearized Ricci tensor

terms appearing in the perturbations

$$D_{\alpha\beta 0} \equiv \gamma^{\text{tail}}_{\alpha\beta,0} + 2A_{(\alpha,\beta)0}$$

$$D_{\alpha\beta i} \equiv \gamma^{\text{tail}}_{\alpha\beta,i} + 2A_{(\alpha,\beta)i} + 2B_{\alpha i\beta j}A^j$$

$$F_i \equiv S^{kl}R_{kl0i}$$

The labels A-F correspond to decomposition in to *l*=1 tensor spherical harmonics:

$$Q_{00} = Q_i^A n^i$$

$$Q_{i0} = Q_j^B n^j n_i + Q_i^C + Q_k^M \epsilon_{ij}{}^k n^j$$

$$Q_{ij} = Q_k^D n^k n_i n_j + Q^E_{(i} n_{j)} + Q_k^F \delta_{ij} n^k + Q_k^N \epsilon^k{}_{l(i} n_{j)} n^l,$$

There are two relations independent of H and K. They are

$$-4F_i - 3MD_i^A + 2MD_i^C - 2MD_i^E + 4MD_i^F = 0$$

$$-F_i - MD_i^A + 2MD_i^C = 0$$

The first tells things unrelated to the motion. The second gives (plugging in),

$$-S^{kl}R_{kl0i} - M(h^{\mathrm{tail}}_{00,i} + 2R_{0j0i}A^j + 2A_{0,0i}) + 2M(h^{\mathrm{tail}}_{i0,0} + A_{i,00} + A_{0,i0}) = 0$$

Using equality of mixed partials, we then have

$$A_{i,00} = \frac{1}{2M}S^{kl}R_{kl0i} - R_{0j0i}A^j - \left(h^{\mathrm{tail}}_{i0,0} - \frac{1}{2}h^{\mathrm{tail}}_{00,i}\right)$$

Our notation has A with no argument evaluated on the world-line. Therefore we have found a second-order differential equation for the deviation vector,

$$\boxed{\frac{d^2Z^i}{dt^2} = \frac{1}{2M}S^{kl}R_{kl0}{}^i - R_{0j0}{}^iZ^j - \left({h^{\mathrm{tail}}}^i{}_{0,0} - \frac{1}{2}h^{\mathrm{tail},i}_{00}\right)}$$

Interpretation of Results

$$u^c\nabla_c(u^b\nabla_b Z^a) = \frac{1}{2M}R_{bcd}{}^a S^{bc}u^d - R_{bcd}{}^a u^b Z^c u^d - (g^{ab}+u^a u^b)(\nabla_d h^{\rm tail}_{bc} - \frac{1}{2}\nabla_b h^{\rm tail}_{cd})u^c u^d$$

spin force — geodesic deviation — self-force

The tail integral in this perturbative result is taken over the background geodesic.

The spin and self-forces appear at the same order because our scaling assumptions force the body multipole moments to scale according to their physical dimension.

Why the geodesic deviation equation? Suppose our family consists of a body whose “initial position” smoothly varies with ». For very small mass and spin, the world lines of the body are just (neighboring) geodesics, and the perturbative description is of course just the geodesic deviation equation.

This equation gives the description of motion in the Lorenz gauge. The change in description under change of gauge may be expressed in terms of the (possibly non-regular) vector relating the gauges—we derive the transformation law (not shown in this talk).

Self-consistent Equation

The perturbative result is mathematically guaranteed to approximate the motion in the » $ 0 limit. However, this guarantee is only useful—i.e., the approximation is only accurate—if the true motion is “close” to the background motion at small but finite ». It is clear that this can only happen for a short time, since particles lose energy and “spiral in”. That is, we expect the convergence of our perturbation series to be highly non-uniform in time.

Of course, when a particle has deviated from a particular geodesic and the solution off of that geodesic is no longer accurate, it should then be close to a *new* geodesic, perturbing off of which should give a better approximation to the motion for that period of time. One could then attempt to “patch together” the two solutions. In the limit of many patches with small times between them, one would expect the motion to be described by a single “self-consistently perturbed” equation. We will argue that the original MiSaTaQuWa equation is (a good candidate for) such an equation.

$$\nabla^c \nabla_c \tilde{h}_{ab} - 2R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}},$$

← gauge-relaxed equations

$$u^b \nabla_b u^a = -(g^{ab} + u^a u^b)(\nabla_d h^{\rm tail}_{bc} - \frac{1}{2} \nabla_b h^{\rm tail}_{cd}) u^c u^d \ ,$$

← geodesic deviation equation gone

$$h^{\rm tail}_{ab}(\tau) = M \int_{-\infty}^{\tau^-} \left(G^+_{aba'b'} - \frac{1}{2} g_{ab} G^{+}{}_{c}{}^{c}{}_{a'b'} \right) (z(\tau), z(\tau'))\, u^{a'} u^{b'} d\tau' \ ,$$

← tail taken over self-consistent motion

A more common self-consistent equation

A simple, familiar example helps to illustrate these ideas. Consider the cooling of a "black body"—specifically, a hot lump of coal enclosed in perfectly reflecting walls, but with a hole of area A cut out.

At finite A, this is a very difficult problem!

However, let us consider a family of cavities A(»), where A(») $ 0 as » $ 0. In the limit, no energy escapes and the body remains at a constant temperature $T^{(0)}$ for all time. As a first perturbative correction, one should find

$$\frac{dE^{(1)}}{dt} = -\sigma A^{(1)} T_0^4$$

Only $T^{(0)}$ may appear on the RHS because $A^{(1)}$ is already first order. Thus perturbation theory in fact gives *linear decrease*,

$$E(\lambda, t) = E_0 - \lambda \sigma A^{(1)} T_0^4 t \ + O(\lambda^2)$$

This perturbative result is clearly only accurate for a short time. One might think one should go to second-order, but better is to pass to the self-consistent equation,

$$\boxed{\frac{dE}{dt} = -\sigma A T^4(t)}$$

← This (correct) equation includes some higher-order terms and not others!

About self-consistent equations

Self-consistent equations are *ubiquitous*—any dissipative process (I can think of) will be described by such an equation. Examples from relativity are black hole evaporation and energy balance ("70's style" or higher PN) calculations.

Self-consistent equations do *not* come (directly) from perturbation theory. Rather, they come from attempting to incorporate perturbative effects off of *different* backgrounds in to a single equation. Viewed off of a single background, these effects are effectively higher-order than linear. **Self-consistent equations include some higher-order terms but not others.**

This is not a bug but a feature—these are precisely the higher-order terms expected to accumulate secularly and dominate. For example, blackbody radiation energy loss will dominate over non-equilibrium effects.

Is the MiSaTaQuWa equation really the right self-consistent equation?

We don't know. Our perturbative result is rigorous. Beyond perturbation theory into the domain of self-consistent equations, we only have some initial feelings on the matter. As a rough outline of the criteria a self-consistent equation ought to satisfy,

1. It should have a well-posed initial value formulation

2. It should have the same number of degrees of freedom as the first-order perturbative system, so that a correspondence can be made between initial data for the self-consistent perturbative equation and the first-order perturbative system.

3. For corresponding initial data, the solutions to the self-consistent perturbative equation should be close to the corresponding solutions of the first order perturbative system over the time interval for which the first order perturbative system should be accurate.

It appears plausible that the MiSaTaQuWa equation satisfies 1,2,3.

Electromagnetic Case

A perturbatively valid derivation of electromagnetic self-force should give,

$$m(a^{(0)})^\alpha = F^\alpha_{\rm ext} \qquad (a^{(1)})^\alpha = \frac{2}{3}\frac{q^2}{m}\left[\delta^\alpha{}_\beta + (u^{(0)})^\alpha (u^{(0)})_\beta\right](\dot{a}^{(0)})^\beta$$

The perturbed motion is *perfectly well-behaved*. However, as usual it is only accurate for short times. So, pass to self-consistent equation,

$$ma^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}q^2(\delta^\alpha{}_\beta + u^\alpha u_\beta)\dot{a}^\beta$$

$$ma^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}q^2(\delta^\alpha{}_\beta + u^\alpha u_\beta)\frac{\dot{F}^\beta_{\rm ext}}{m}$$

The Abraham-Lorentz equation is not an allowed choice—according to criteria 1,2,3, it is <u>wrong</u>. The "reduced order" version, on the other hand, appears to satisfy 1,2,3.

Conclusion (of such a derivation): runaway solutions are an artifact of the improper use of perturbation theory.

“Practical” implications

The MiSaTaQuWa equation will never be derived from perturbation theory, at any order.

There is no more reason to distrust the waveforms from the MiSaTaQuWa equations than there is to distrust the energy flux predicted by the blackbody law.

It can be dangerous to attempt “skip” the rigorous perturbative result and proceed straight to the self-consistent equation—see the 100 year history of the Abraham-Lorentz force.

My truly practical conclusion: For waveforms incorporating gravitational self-force, there is no need to “break our backs” with second-order perturbation theory. This is good news!