Self-forces from generalized Killing fields

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Outline

- Review of the Newtonian problem
- Momentum in curved spacetime and generalized Killing fields
- Laws of motion (main result)
- New effects
- Generalized Detweiler-Whiting axiom

Introduction

- Self-force problems usually seek some kind of nearly universal “bulk” behaviors in a class of compact objects.

- Net forces typically look like

$$F_a \approx \int_\Sigma \rho \nabla_a \phi \, d^3x$$

- This simplifies if ϕ varies slowly enough that it can be adequately approximated inside Σ by a few terms in a Taylor expansion. Then,

$$F_a \approx q \nabla_a \phi(\gamma) + q^b \nabla_a \nabla_b \phi(\gamma) + \ldots$$

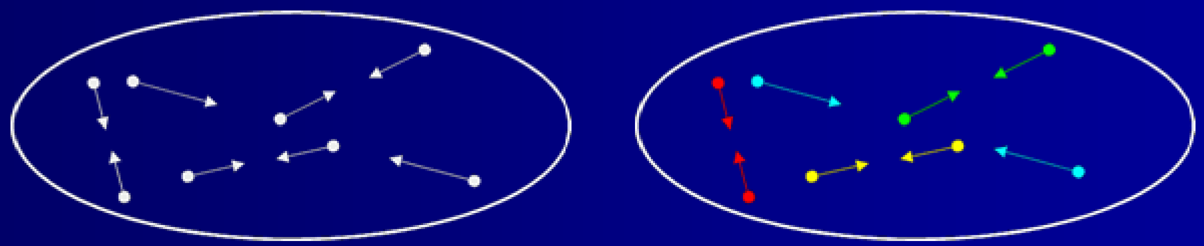
- This isn’t possible for the (retarded) self-field, yet something like it is still expected in some systems.

- Only a small part of the self-field affects a body's bulk motion. This usually does vary slowly on each cross-section.
- The "ignorable" portion of the field diverges in point particle approaches, although this is physically less relevant (and not sufficient to completely identify it).
- Such effects exist because of action-reaction cancellations in the sense of Newton's 3rd law.
- These ideas can be made precise in order to determine which part of the self-field affects the motion of an extended body.

Non-relativistic self-forces

- Self-forces and self-torques in Newtonian gravity or electrostatics (with uniform permittivity) always vanish due to Newton's 3rd law:

$$F^a_{self} = \sum_{i \neq j} f^a(i \rightarrow j) = \frac{1}{2} \sum_{i \neq j} \left[f^a(i \rightarrow j) + f^a(j \rightarrow i) \right] = 0$$



$$N^{ab}_{self} = 2 \sum_{i \neq j} (x_j - \gamma)^{[a} f^{b]}(i \rightarrow j) = \sum_{i \neq j} (x_j - x_i)^{[a} f^{b]}(i \rightarrow j) = 0$$

$$f_a(i \rightarrow j) = m_i m_j \frac{\partial}{\partial x^a_j} G(x_j; x_i)$$

Newton's 3rd law in geometric form

- More concisely,

$$F^{a}_{self}\xi_a + \frac{1}{2} N^{ab}_{self} \nabla_{[a}\xi_{b]} = \frac{1}{2}\sum_{i \neq j} m_i m_j \mathcal{L}_{\xi} G$$

  for all Killing fields ξ^a:

- Newton's 3rd law is therefore equivalent to

$$\mathcal{L}_{\xi} G(x; x') = \xi^a \nabla_a G + \xi^{a'} \nabla_{a'} G = 0$$

  Translational invariance of the Green function implies the weak form of the 3rd law. Adding rotational invariance implies the strong form.

- This shows non-perturbatively - and without necessarily knowing *G* - that the quickly-varying (or "singular") self-field doesn't explicitly affect the motion at all in this theory. The same ideas carry over for relativistic systems in arbitrary spacetimes.

Momentum in curved spacetime

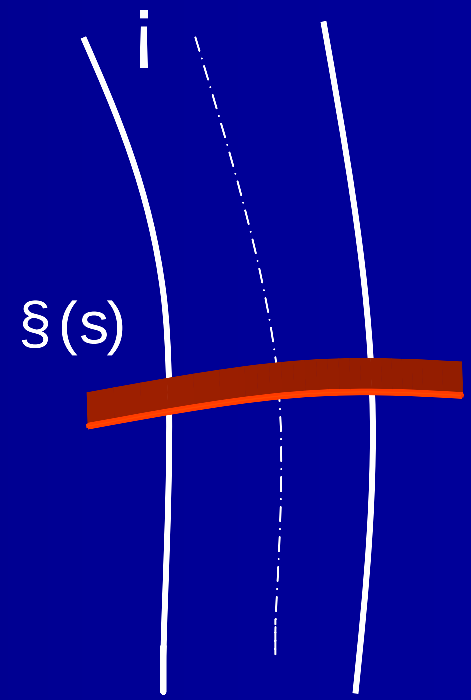
- Usually define some $p_a(s)$ and $S_{ab}(s)$ on a timelike worldline ¡ :

- Use a family of scalars instead

$$P_{\text{»}}(s) = \text{–}_{\text{§}(s)} T^{a}{}_{b}\,\text{»}^{b}\,dS_{b}$$

- These are defined with respect to a ten-parameter family of generalized Killing fields »^a: Dixon's momenta are related via

$$P_{\text{»}}(s) = \left(\text{»}_a p^a + \frac{1}{2} r_{[a}\text{»}_{b]} S^{ab}\right)_{\text{°}(s)}$$



Properties of the Generalized Killing fields (GKFs)

- Form a ten-dimensional group in every four-dimensional spacetime. (generalized Poincaré group). Momentum is a map

$$P_\xi : GP \times \Gamma \longrightarrow \mathbb{R}$$

- Any real Killing fields that may exist are also GKFs.
- Uniquely fixed by any $\{\xi^a, \nabla_{[a}\xi_{b]}\}$ on Γ : $\{P_\xi | \forall \xi^a\} \Leftrightarrow \{p_a, S_{ab}\}$
- Always satisfy $\mathcal{L}_\xi g_{ab}|_\Gamma = \nabla_a \mathcal{L}_\xi g_{bc}|_\Gamma = 0$. This means that

$$\frac{d}{ds} P_\xi = (\dot{p}^a - \frac{1}{2} S^{bc} \dot{\gamma}^d R_{bcd}{}^a) \xi_a + \frac{1}{2} (\dot{S}^{ab} - 2 p^{[a} \dot{\gamma}^{b]}) \nabla_{[a}\xi_{b]}$$

- Details can be found in AIH, arXiv:0805.4259

Scalar fields

- Field equation: $\Box \Phi = -4\pi\rho$
- Stress-energy conservation: $\nabla_b T^b{}_a = \rho \nabla_a \Phi$
- Define singular and radiative Green functions
  1. Split up the retarded field: $G^{ret} = G^R + G^S$
  2. One part should allow pair averaging: $G^S(x, x') = G^S(x', x)$
  3. The rest should vary slowly: $\Box G^R = 0$
- Very useful to also set $G^S(x, x') = 0$ when its arguments are timelike-separated. All of this recovers the Detweiler-Whiting Green functions.

Laws of motion

Using stress-energy conservation, the final (exact) result is that

“Self-momentum”

External force

Extended body self-force

$$\frac{d}{ds}(P_{»}+E_{»}) = \int_{§(s)} \left[\frac{1}{2}T^{ab}L_{»}g_{ab} + ½L_{»}(©^{ext} + ©^{R}) + \frac{1}{2}\int_{W} ½½L_{»}G^{S}dV' \right] t^{a}dS_{a}$$

“Bare” momentum

Gravitational force

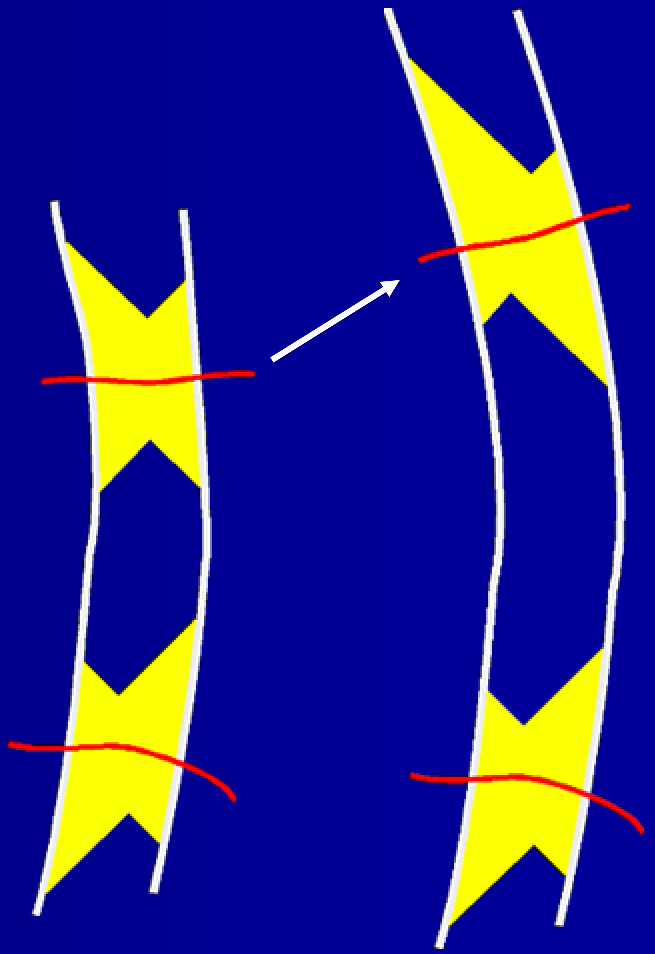
$\backslash\, »_{a}T^{bc} i\, ^{a}_{bc}$ "

Typical scalar self-force

$$\frac{d}{ds}P_{»} = (p^{a} i \frac{1}{2}S^{bc\circ d}R_{bcd}{}^{a})»_{a} + \frac{1}{2}(S^{ab} i \ 2p^{[a\circ b]})r_{[a»b]}$$

Self-momentum

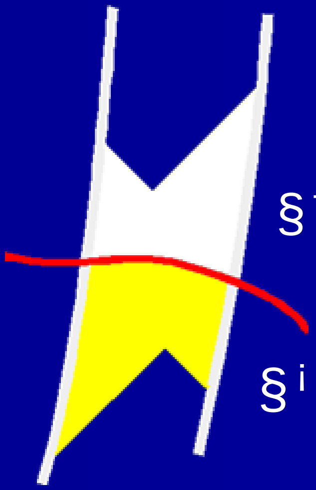
- Look at changes in $P_{»}$ over finite times. Find that there's a part of the self-force that only depends on regions right next to the bounding hypersurfaces.
- This can be associated directly with these hypersurfaces, and is naturally identified as the linear and angular momenta of the S-type self-field
- This only happens for DW self-fields.
- Would never have been completely found perturbatively or by considering only instantaneous rates of change.



Why call it a self-momentum?

- Explicitly,

$$E_{»} = \frac{1}{2}\left[\int_{§^{+}} ½ ⊥ {»} © ^{S}[§^{¡}]dV - \int_{§^{¡}} ½ ⊥ {»} © ^{S}[§^{+}]dV\right]$$



- If everything is time-symmetric,

$$E_{»} = ¡ \frac{1}{2}\int_{§} ½ © ^{S} »^{a} dS_{a}$$

- Static charge in flat spacetime:

$$p^{a}_{self} = \frac{1}{2} ∘^{a} \int d^3r\, d^3r' \frac{½ ½'}{|r ¡ r'|} \qquad S^{ab}_{self} = \int d^3r\, d^3r' \frac{r^{[a} ∘^{b]} ½ ½'}{|r ¡ r'|}$$

- The bare and self-momenta are not necessarily parallel in general.

- Different mass centers can arise when using $P_{»}$ or $P_{»} + E_{»}$:

Purely extended body self-force

$$\frac{1}{2}\int_{\partial\Sigma} \tau^{\alpha} dS_{\alpha} - \int_{\Omega} dV' \ldots \mathcal{L}_{\xi} G^{S}$$

- Measures the failure of Newton's 3rd law for the S-type field in "direction" ξ^{a}.
- Exactly vanishes if this vector field is Killing. This is true for all vector fields in Minkowski and de Sitter spacetimes.
- Smaller than the typical (radiative or regular) self-force for sufficiently small charge distributions.
- Possibly indicates that simple guesses for higher order terms in nonlinear field theories will not lead to anything general.

Generalized Detweiler-Whiting axiom

There exist linear and angular momenta $\hat{P}_\xi = P_\xi + E_\xi$ whose evolution is not affected by the DW singular component of the self-field.

- Exactly true for all charge distributions in Minkowski and de Sitter spacetimes
- Given any Killing vector K^a; this is also exact for $\hat{p}^a K_a + \frac{1}{2}\hat{S}^{ab}\nabla_{[a}K_{b]}$.
- True to leading order for shrinking charge distributions $\rho = \lambda^{-\alpha} A(r/\lambda; (t - t_0)/\lambda)$ in every spacetime.

Whether or not point particle-like equations of motion follow is another question. Corrections are often trivial if the center-of-mass can be defined using the effective momentum $\hat{P}_\xi$.

Conclusions

- A simple and elegant formalism was developed to help understand self-forces and self-torques.
- Nontrivial linear and angular self-momenta were derived without using any perturbation theory.
- Detweiler-Whiting axiom was derived (not guessed) very easily. Its correction has a very simple and explicit physical intepretation.
- Probably simple to generalize for more complicated field theories involving nonlinearities and gauge freedom.
- Will investigate the effects of L » G^S, and whether the mass center defined through the effective momenta makes sense.