Mass (and Angular Momentum) in General Relativity

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Scheme

Outline

Problems in defining a gravitational mass

Problem: mass/energy of an *extended system* in General Relativity (GR)

- Dealing with **matter**, integrate appropriate components of the stress-energy tensor $T_{\mu\nu}$ in the relevant spatial volume V.
- More generically, to account for the **energy in the gravitational field**, one could try to follow a similar procedure...

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But...

- **Equivalence Principle**: absence of gravitational energy **density**
  Point-like free falling particles do not "feel" gravitational fields (no analogue of Electromagnetic Poynting vector and energy density of EM field).

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- **Equivalence Principle**: absence of gravitational energy **density**
  Point-like free falling particles do not "feel" gravitational fields (no analogue of Electromagnetic Poynting vector and energy density of EM field).
- **Absence of background rigid structures**.
  Physical parameters in Physics are often defined in terms of *some kind* of rigid structure, e.g.:
  - Conserved quantities under some symmetry.
  - Inertial families of observers (*kinematical symmetries*): particles as representations of the Poincaré group...
  - ...

  All fields are dynamical in GR: no a priori "rigid structure" available.

Need of additional structure

But we must deal with massive extended objects...:

- Relativistic astrophysics: relativistic binaries (mergers), grav. collapse...
- Black Holes in General Relativity and approaches to *Quantum Gravity*.
- Mathematical relativity: positive-defined quantities to be tracked along geometric flows or to define appropriate variational principles.
- ...

Specific problems suggest concrete solutions:

- Low velocities and weak self-gravity: Post-Newtonian approaches.
- Perturbation theory around a known exact solution.
- Study of isolated systems.
- Black holes in certain physical regimes.
- ...

Moral

"Need" of some kind of additional *structure*.

Need of additional structure

Here we focus on:

1. Mass (and angular momentum) of Isolated systems.
2. Mass (and angular momentum) of black holes in quasi-equilibrium in otherwise dynamical spacetimes.

Outline

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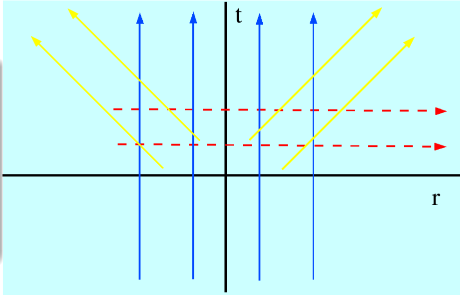
Isolated systems in GR: Asymptotic Flatness

[following esentially Gourgoulhon 07]

Flat curvature far apart from the source

- Different manners of "getting far".



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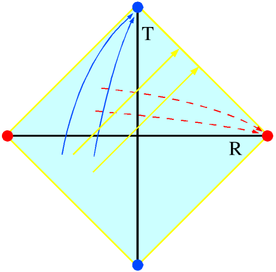

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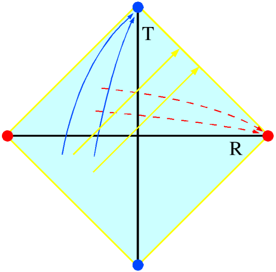


Conformally compatified picture: asymptotic simplicity and asymptotic flatness

A smooth space-time $(\mathcal{M}, g)$ is *asymptotically simple* if there exists another smooth Lorentz manifold $(\tilde{\mathcal{M}}, \tilde{g})$ such that:

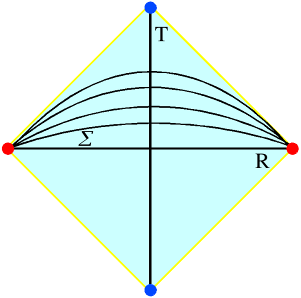
i) $\mathcal{M}$ is an open submanifold $\tilde{\mathcal{M}}$ with smooth boundary $\partial\mathcal{M} = \mathscr{I}$,

ii) There is a smooth scalar field Ω on $\tilde{\mathcal{M}}$, such that $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on $\mathcal{M}$, and so that $\Omega = 0$, $d\Omega \neq 0$ on $\mathscr{I}$.

iii) Every null geodesic in $\mathcal{M}$ acquires a future and a past endpoint on $\mathscr{I}$.

An asymptotically simple spacetime is called *asymptotically flat* if, in addition, $R_{\mu\nu} = 0$ in a neighbourhood of $\mathscr{I}$.

An interlude: 3+1 spacetime decompositions



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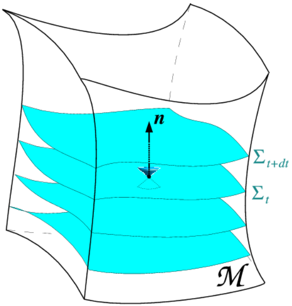

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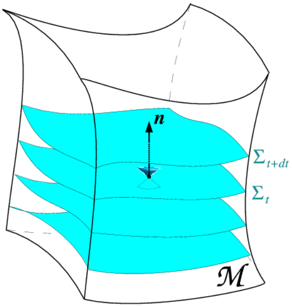
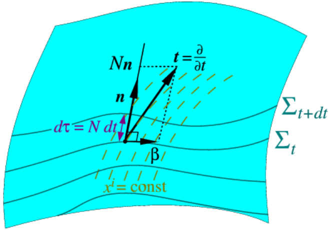


$\{\Sigma_t\}$	3+1 slicing of spacetime
n^μ	timelike unit normal to Σ_t
$t^\mu = N n^\mu + \beta^\mu$	evolution vector
N	lapse function
β^μ	shift vector
$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$	spatial 3-metric
$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$	extrinsic curvature



In particular, $K_{ij} = \frac{1}{2N}\left(\gamma_{ik} D_j \beta^k + \gamma_{jk} D_i \beta^k - \dot{\gamma}_{ij}\right)$.

Asymptotic Euclidean spatial slices

Σ_t is *asymptotically* Euclidean (*flat*) if there exists a Riemannian "background" metric f_{ij} such that:

i) f_{ij} is flat, except possibly on a compact domain $\mathcal{B}$ of Σ_t.

ii) There exists a coordinate system $(x^i) = (x, y, z)$ such that outside $\mathcal{B}$, $f_{ij} = \mathrm{diag}(1, 1, 1)$ ("Cartesian-type coordinates") and the variable $r := \sqrt{x^2 + y^2 + z^2}$ can take arbitrarily large values on Σ_t.

iii) When $r \to +\infty$,

$$\gamma_{ij} = f_{ij} + O(r^{-1}),$$
$$\frac{\partial \gamma_{ij}}{\partial x^k} = O(r^{-2});$$
$$K_{ij} = O(r^{-2}),$$
$$\frac{\partial K_{ij}}{\partial x^k} = O(r^{-3}).$$

Given an asymptotically flat spacetime foliated by asymptotically Euclidean slices $\{\Sigma_t\}$, the "region" $r \to +\infty$ is called *spatial infinity* and is denoted i^0.

Asymptotic symmetries

Set of diffeomorphisms $(x^\alpha) = (t, x^i) \to (x'^\alpha) = (t', x'^i)$ preserving i)-iii):

$$x'^\alpha = \Lambda^\alpha{}_\mu x^\mu + c^\alpha(\theta, \varphi) + O(r^{-1}) \quad ,$$

with $\Lambda^\alpha{}_\beta$ is a Lorentz matrix and the c^α's are four functions of the angles (θ, φ) related to the coordinates $(x^i) = (x, y, z)$ by the standard formulæ:

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta.$$

Consequences:

- Not a canonical *Poincaré* asymptotic symmetry.
- *Supertranslations*: $c^\alpha(\theta, \varphi) \neq \mathrm{const}$ and $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta$ ("angle-dependent translations").
- Infinite-dimensional symmetry (related to *Spi group* [Ashtekar & Hansen 78,80]).
- Analogue situation at $\mathscr{I}$: Bondi-Metzger-Sachs (BMS) asymptotic symmetry.

Outline

Hamiltonian framework

Hilbert-Einstein action in spacetime $\mathcal{M}$ with (timelike) boundary hypersurface $\partial\mathcal{M}$:

$$S = \int_{\mathcal{M}} {}^4R\sqrt{-g}\, d^4x + 2\oint_{\partial\mathcal{M}} (Y - Y_0)\sqrt{h}\, d^3y,$$

with Y (Y_0) the trace of extrinsic curvature of $\partial\mathcal{M}$ in $(\mathcal{M}, g_{\mu\nu})$ (resp. $(\mathcal{M}, \eta_{\mu\nu})$).
We consider a $3+1$ slicing, write

$$\mathcal{S}_t := \partial\mathcal{M} \cap \Sigma_t.$$

and define a Lagrangian density on $(\gamma_{ij}, \dot{\gamma}_{ij}; N, \beta)$. Construct a Hamiltonian through a Legendre transformation $(\gamma_{ij}, \dot{\gamma}_{ij}) \to (\gamma_{ij}, \Pi^{ij} \equiv \frac{\delta L}{\delta\dot{\gamma}_{ij}})$:

$$H = -\int_{\Sigma_t^{\rm int}} \left(NC_0 - 2\beta^i C_i\right)\sqrt{\gamma} d^3x - 2\oint_{\mathcal{S}_t} \left[N(\kappa - \kappa_0) + \beta^i (K_{ij} - K\gamma_{ij})s^j\right]\sqrt{q}\, d^2y$$

where κ (and κ_0) is the trace of the extrinsic curvature of $\mathcal{S}_t$ embedded in (Σ_t, γ_{ij}) (resp. embedded in (Σ_t, f_{ij})), s^i normal to $\mathcal{S}_t$ in Σ_t and:

$$\begin{aligned} C_0 &:= R + K^2 - K_{ij}K^{ij}, \\ C_i &:= D_j K^j{}_i - D_i K \end{aligned}$$

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$$H_{\text{solution}} = -2\oint_{\mathcal{S}_t} \left[N(\kappa - \kappa_0) + \beta^i(K_{ij} - K\gamma_{ij})s^j\right]\sqrt{q}\, d^2y.$$

Remark:

- H as *generator of diffeomorphisms*: vanishing for *gauge transformations* (not moving points of the phase space).
- In spacetimes with boundaries not all the diffeomorphisms are gauge transformations: implications for the quantum theory (residual degrees of freedom: diffeomorphisms not preserving boundary conditions).

ADM mass [Arnowitt, Deser & Misner 60,62]

Total energy contained in Σ_t: value of $H_{\rm solution}$ on a surface $\mathcal{S}_t$ at spatial infinity (i.e. for $r \to +\infty$) and coordinates (t, x^i) associated with some asymptotically inertial observer (i.e. $N = 1$, $\beta^i = 0$):

$$M_{\rm ADM} := -\frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} (\kappa - \kappa_0) \sqrt{q}\, d^2 y.$$

Evaluating κ and κ_0:

$$M_{\rm ADM} = \frac{1}{16\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} \left[\mathcal{D}^j \gamma_{ij} - \mathcal{D}_i (f^{kl} \gamma_{kl}) \right] s^i \sqrt{q}\, d^2 y,$$

In Cartesian-type coordinates (x^i) in the definition of asymptotic flatness:

$$M_{\rm ADM} = \frac{1}{16\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} \left(\frac{\partial \gamma_{ij}}{\partial x^j} - \frac{\partial \gamma_{jj}}{\partial x^i} \right) s^i \sqrt{q}\, d^2 y.$$

Remark:

Asymptotic flatness fall-off conditions makes the integral finite.

ADM Linear Momentum

- ADM momentum: conserved quantity associated with spatial translations. In the Cartesian-type coordinates (x^i) three directions for translations at spatial infinity, $(\partial_i)_{i\in\{1,2,3\}}$: $N = 0$ and $\beta^i = 1$:

$$\boxed{P_i := \frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} \left(K_{jk} - K\gamma_{jk}\right) (\partial_i)^j \, s^k \sqrt{q} \, d^2y}, \qquad i \in \{1,2,3\}.$$

- ADM 4-momentum:

$$\boxed{P^{\rm ADM}_\alpha := (-M_{\rm ADM}, P_1, P_2, P_3)}$$

transforms under those $(x^\alpha) = (t, x^i) \to (x'^\alpha) = (t', x'^i)$ preserving spatial asymptotic flatness properties $i) - iii)$, as:

$$P'^{\rm ADM}_\alpha = (\Lambda^{-1})^\mu{}_\alpha \, P^{\rm ADM}_\mu.$$

Remark:

Correct transformation under (vector linear representation of) the Poincaré group.

Angular Momentum at spatial infinity

Standard approach to Angular Momentum: conserved quantity under rotations. A first attempt: evaluate $H_{\rm solution}$ with $N = 0$ and β^i given by rotational Killing vector ϕ of the asymptotic flat metric f_{ij} (as in P_i, but with $(\partial_i)^j \longrightarrow (\phi_i)^j$):

$$\begin{aligned} \boldsymbol{\phi}_x &= -z\boldsymbol{\partial}_y + y\boldsymbol{\partial}_z \\ \boldsymbol{\phi}_y &= -x\boldsymbol{\partial}_z + z\boldsymbol{\partial}_x \\ \boldsymbol{\phi}_z &= -y\boldsymbol{\partial}_x + x\boldsymbol{\partial}_y. \end{aligned}$$

$$J_i := \frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} \left(K_{jk} - K\gamma_{jk}\right) (\phi_i)^j \, s^k \sqrt{q} \, d^2y, \qquad i \in \{1, 2, 3\}.$$

Problems:

- Supertranslations ambiguity: J_i do not transform properly under infinite-dimensional asymptotic symmetries (non-trivial commutator of rotations and translations in Poincaré).
- Fall-off conditions: not enough by themselves to guarantee finite J_i.

Angular Momentum at spatial infinity

$$J_i := \frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} (K_{jk} - K\gamma_{jk}) (\phi_i)^j s^k \sqrt{q}\, d^2 y, \qquad i \in \{1, 2, 3\}.$$

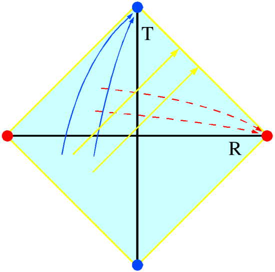
Removing ambiguities:

- Define a subclass of coordinate systems and transformations for which J_i transforms as a vector, by imposing more restrictive fall-off conditions [e.g. York 79, in terms of a confomal metric $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$]:

$$\frac{\partial \tilde{\gamma}_{ij}}{\partial x^j} = O(r^{-3}),$$
$$K = O(r^{-3}).$$

- These are *asymptotic gauge conditions*.
- Other prescriptions in the literature.
- Strictly speaking, no "ADM angular momentum".

A brief (vague) note on null infinity $\mathscr{I}$



Generic issues:

- Bondi (Bondi-Sachs,Trautman-Bondi) mass $M_{\rm Bondi}$: positive quantity defined on $\mathscr{I}^+$, monotonically decreasing from $M_{\rm ADM}$.
- Linear momentum $P^i_{\rm Bondi}$ properly defined.
- Ambiguities in the identification of Poincaré group: BMS group.
- Framework providing rigorous understanding of gravitational wave emission and energy carried out from the system.
- Subtle issues on matching with i^0 and i^+ still to be assessed.

Outline

Komar quantities [Komar 59]

Let us consider a Killing vector field (infinitesimal isometry) k^μ in $\mathcal{M}$ and a closed 2-surface $\mathcal{S}_t$. We define the Komar quantity $k_{\rm K}$ as,

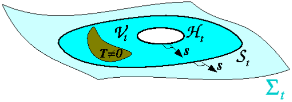
$$k_{\rm K} := -\frac{1}{8\pi}\oint_{\mathcal{S}_t} \nabla^\mu k^\nu \, dS_{\mu\nu},$$

with

$$dS_{\mu\nu} = (s_\mu n_\nu - n_\mu s_\nu)\sqrt{q}\, d^2y.$$

Then, the Komar quantity is conserved in the sense that it does not depend on $\mathcal{S}_t$ as long as one stays outside matter:

$$k_{\rm K} = 2\int_{\mathcal{V}_t}\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) n^\mu k^\nu \sqrt{\gamma}\, d^3x + k_{\rm K}^{\mathcal{H}}.$$



Remark:

- $k_{\rm K}$ coordinate independent.
- Slicing $\{\Sigma_t\}$ not needed.

Komar mass

For **stationary** spacetimes, k^μ timelike Killing vector. If asymptotically flat, unique by normalization to $k^\mu k_\mu = -1$ at spatial infinity. Then:

$$\boxed{M_{\rm K} := -\frac{1}{8\pi}\oint_{\mathcal{S}_t} \nabla^\mu k^\nu \, dS_{\mu\nu}},$$

Given a 3-slicing $\{\Sigma_t\}$ and choosing $(\boldsymbol{\partial}_t)^\mu = k^\mu$:

$$\boxed{M_{\rm K} = \frac{1}{4\pi}\oint_{\mathcal{S}_t} \left(s^i D_i N - K_{ij} s^i \beta^j\right) \sqrt{q}\, d^2y}.$$

Remark:

For foliations $\{\Sigma_t\}$ whose unit normal vector n^μ coincide with the timelike Killing vector k^μ at spatial infinity (i.e. $N \to 1$ and $\boldsymbol{\beta} \to 0$ [Beig, Ashtekar, Magnon-Ashtekar]):

$$\boxed{M_{\rm K} = M_{\rm ADM}}.$$

Relevant for hellical symmetry for quasi-circular binaries [Detweiler, Meudon group...].

Komar angular momentum

Let us consider an **axisymmetric** spacetime with axial Killing vector ϕ^μ. That is, ϕ^μ is a spacelike Killing vector whose action on $\mathcal{M}$ has compact orbits, two stationary points (the poles), and normalized such that its natural affine parameter moves in $[0, 2\pi)$. Then, the Komar angular momentum is given by:

$$\boxed{J_{\rm K} := \frac{1}{16\pi} \oint_{\mathcal{S}_t} \nabla^\mu \phi^\nu \, dS_{\mu\nu}}.$$

Adopting a 3-slicing adapted to the axisymmetry ($n^\mu \phi_\mu = 0$), we have:

$$\boxed{J_{\rm K} = \frac{1}{8\pi} \oint_{\mathcal{S}_t} K_{ij} s^i \phi^j \sqrt{q} \, d^2 y}.$$

Remark:

No ambiguity in the Komar angular momentum, in contrast with the ADM one.

Outline

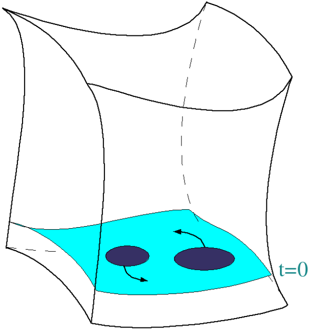
A presentation of the problem: Cauchy evolutions

First: encoding the Physics

- Need of *telling* which system we have at the *initial time*.
- *Codify* initial masses, velocities, angular momenta, radiation content, orbital parameters...

Then: extraction of Physics

- Calculating *final physical parameters*.
- *"Follow up"* of the dynamical process.



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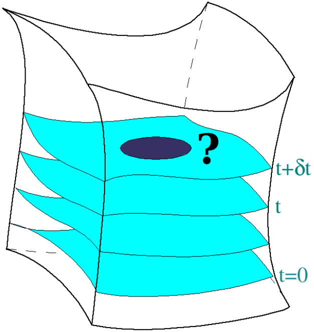

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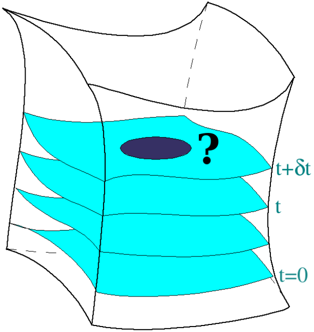


All this involves the association of physical parameters to bounded regions...

Quasi-local masses (and angular momenta)

Outline

Classical definition of a black hole

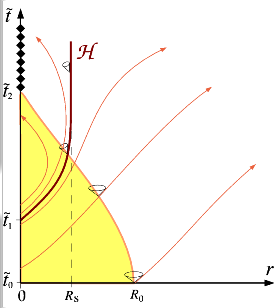
Classical Black Hole

Region of "no escape" to infinity: $\mathcal{B} := \mathcal{M} - J^-(\mathscr{I}^+)$

- $\mathcal{M}$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

Event horizon

Boundary of the Black Hole region: $\mathcal{H} := \dot{J}^-(\mathscr{I}^+)$ (boundary of $J^-(\mathscr{I}^+)$)



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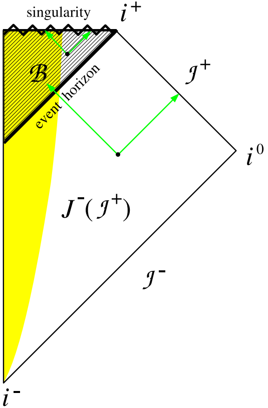

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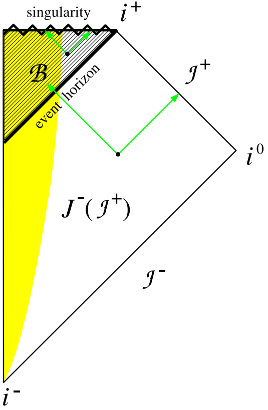
Problems:

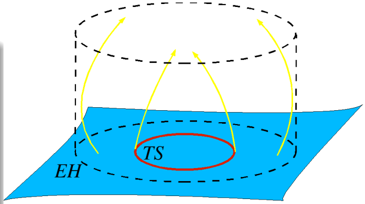
- Global (*teleological*) concept, full spacetime history knowledge to locate the *event horizon*.
- Difficulties for associating mass and angular momentum in non-stationary situations.



Quasi-local black holes [Hayward, Ashtekar, Krishnan...]

Isolated and Dynamical/Trapping Horizons

- **Event horizon**: no information to *infinity*.
- **Trapped surfaces**: no information *"outwards"*. Black Hole as the *trapped region*. Quasi-local notion.



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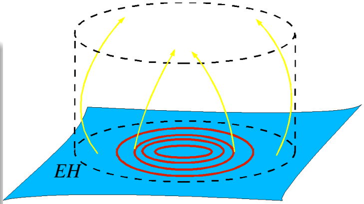

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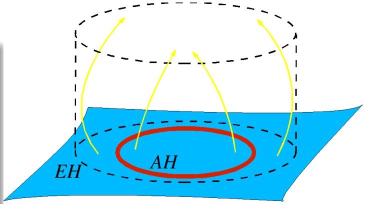

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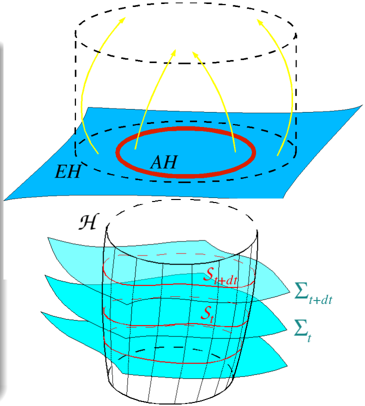
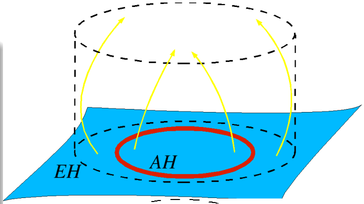
- **Isolated and Dynamical/Trapping Horizons**: world-tube of apparent horizons (with some additional geometric conditions).

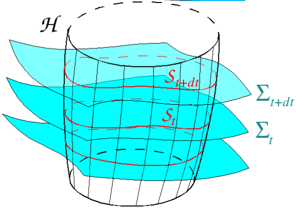


Quasi-local black holes [Hayward, Ashtekar, Krishnan...]

Isolated and Dynamical/Trapping Horizons

- **Event horizon**: no information to *infinity*.
- **Trapped surfaces**: no information *"outwards"*. Black Hole as the *trapped region*. Quasi-local notion.
- **Apparent horizons**: outermost (marginally) trapped surface.
- **Isolated and Dynamical/Trapping Horizons**: world-tube of apparent horizons (with some additional geometric conditions).





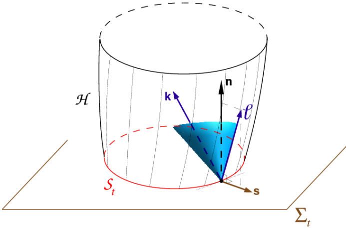
Properties and Applications of **Isolated and Dynamical/Trapping Horizons**

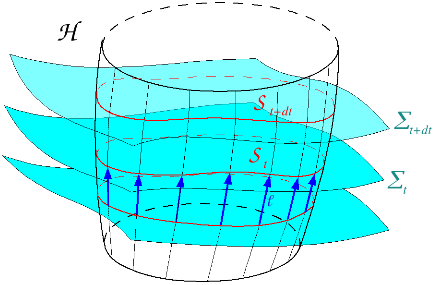
- Geometric object that can be located along a Cauchy evolution.
- It provides expressions for **mass** and **angular momentum**.
- It offers an approach to a geometric description of black hole deformations through mass and angular momentum (source) **multipoles**.

The Black Hole quasi-equilibrium case: Isolated Horizons

Basic notion: apparent horizon $\mathcal{S}_t$

s^μ	unit normal vector to $\mathcal{S}_t$, in Σ_t
ℓ^μ	outgoing null vector
k^μ	ingoing null vector $(k^\mu \ell_\mu = -1)$
$q_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu$	induced metric on $\mathcal{S}_t$
$\theta_{(\ell)} \equiv q^{\mu\nu} \nabla_\mu \ell_\nu = 0$	Vanishing (outgoing) expansion (apparent horizon condition)





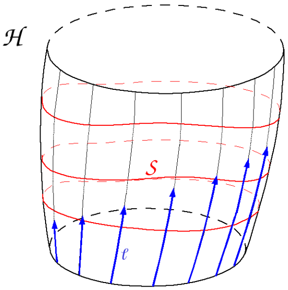
World-tube $\mathcal{H}$ of apparent horizons $\mathcal{S}_t$
$\mathcal{S}_t$ constant area $\Rightarrow \mathcal{H}$ null hypersurface
$\mathcal{H}$ generated by ℓ^μ: outgoing null vector

Given the induced slicing $\{\mathcal{S}_t\} \Longrightarrow$
Natural evolution vector on $\mathcal{H}$:

$$\ell = N \cdot (n^\mu + s^\mu)$$

(ℓ Lie draggs the surfaces $\mathcal{S}_t$)

Isolated horizons: geometric definition



[Ashtekar and Krishnan, Liv.Rev.Rel **7**, 10 (2004)]

Non-expanding horizon

- Null-hypersurface $\mathcal{H} \approx S^2 \times \mathbb{R}$ sliced by marginally (outer) trapped surfaces $\mathcal{S}$: $\theta_{(\ell)} = 0$.
  Raychaudhuri equation $\Rightarrow \sigma_{(\ell)} = 0$
- Einstein equations satisfied on $\mathcal{H}$
- $-T^{\mu}{}_{\nu}\ell^{\nu}$ future directed

Well defined connection $\hat{\nabla}$, induced by the spacetime ∇:
Geometry of the null hypersurface $\mathcal{H}$ charaterized by $(q_{\mu\nu}, \hat{\nabla})$

- Some components of $\hat{\nabla}$ define an intrinsic 1-form ω on $\mathcal{H}$:
$$\hat{\nabla}_{\mu}\ell^{\nu} = \omega_{\mu}\ell^{\nu}$$
- Notion of surface gravity: $\hat{\nabla}_{\ell}\ell^{\mu} = \kappa_{(\ell)}\ell^{\mu} \quad \Leftrightarrow \quad \boxed{\kappa_{(\ell)} = \ell^{\mu}\omega_{\mu}}$

Isolated horizons: hierarchical structure

Physical idea: dynamical spacetime with a black hole in equilibrium.
Isolated Horizon hierarchy: increasing level of equilibrium.

- Non-Expanding Horizon (NEH): $\mathcal{L}_\ell \, q_{\mu\nu} = 0$
  (minimal constraint on the geometry)
- Weakly Isolated Horizon (WIH): $\mathcal{L}_\ell \, \omega_\mu = 0$
  Dependent on ℓ due to the rescaling behaviour:
  $$\ell \to \ell' = \alpha \ell \implies \omega \to \omega + \hat{\nabla}\alpha$$
  Restriction of ℓ to a WIH-equivalence class: $\ell \sim \ell'$ iff $\ell' = \mathrm{const} \cdot \ell$
  WIH = NEH + WIH-equivalence class of null normals
  (not a restriction on the null geometry!)
- (Strongly) Isolated Horizon: $[\mathcal{L}_\ell, \hat{\nabla}] = 0$
  (strongest equilibrium condition on the geometry)

Geometrical consequences

NEH

$$\left.\begin{array}{r}\theta_{(\ell)} = 0 \\ \sigma_{(\ell)} = 0\end{array}\right\} \Longrightarrow \mathcal{L}_\ell q_{\mu\nu} = 0$$

In addition, for the components of the Weyl tensor:

$$d\omega = \mathrm{Im}\Psi_2\,{}^2\epsilon \;\; ; \;\; \Psi_0 = 0 = \Psi_1$$

WIH

$$\mathcal{L}_\ell \omega = 0 \Leftrightarrow \hat{\nabla}\kappa_{(\ell)} = 0$$ (*zeroth law of BH mechanics*)

If $\kappa_{(\ell)} \neq \mathrm{const}$, then $\ell' = \alpha\ell$, with $\mathrm{const} = \nabla_\ell \alpha + \alpha\kappa_\ell$, has const $\kappa_{(\ell')}$.

Therefore, a WIH is not a restriction on a NEH.

It is rather a condition on the null normal $\ell \Leftrightarrow$ the 3+1 slicing

WIH-compatible slicings

IH

Mass and angular momentum *multipole moments* characterizing the horizon $\mathcal{H}$

Physical Parameters: Symmetries

Approach in the Isolated Horizon approach

Physical parameter: conserved quantity under a *symmetry* transformation (*canonical transformation* on the solution (phase) space of Einstein equation)

Underlying symmetry notion:

WIH-symmetries

A vector field $\boldsymbol{W}$ on a WIH $(\mathcal{H}, [\ell])$ is a WIH-symmetry iff:

$$\mathcal{L}_{\boldsymbol{W}}\boldsymbol{\ell} = \mathrm{const}\cdot\boldsymbol{\ell}, \quad \mathcal{L}_{\boldsymbol{W}}\boldsymbol{q} = 0 \quad \text{and} \quad \mathcal{L}_{\boldsymbol{W}}\boldsymbol{\omega} = 0$$

General form of $\boldsymbol{W}$:

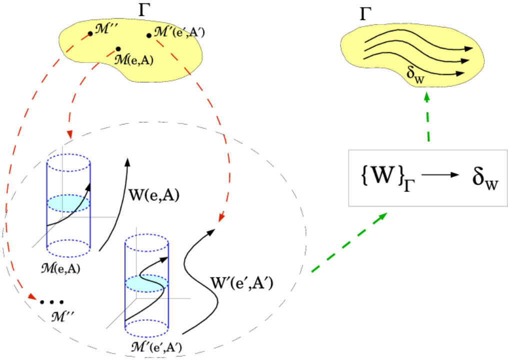
$$\boldsymbol{W} = c_{\boldsymbol{W}}\boldsymbol{\ell} + b_{\boldsymbol{W}}\boldsymbol{S}$$

where $c_{\boldsymbol{W}}$ and $b_{\boldsymbol{W}}$ are constants and $\boldsymbol{S}$ is a symmetry of $\mathcal{S}_t$.

Physical Parameters: symplectic (Hamiltonian) analysis

Procedure

1) Construction of the phase space Γ (each point a spacetime $\mathcal{M}$)
2) In particular the symplectic form (closed 2-form).
3) Extension of $\boldsymbol{W}$ on $\mathcal{H}$ to infinitesimal diffeomorphism on each $\mathcal{M}$ $\rightarrow$ family $\{\boldsymbol{W}\}_\Gamma$
4) $\{\boldsymbol{W}\}_\Gamma \rightarrow$ canonical transformation $\delta_{\boldsymbol{W}}$ on Γ ($\delta_{\boldsymbol{W}}$ preserves the symplectic form$\Leftrightarrow$ existence of a Hamiltonian function $H_{\boldsymbol{W}}$ for $\delta_{\boldsymbol{W}}$).
5) Physical parameter: conserved $H_{\boldsymbol{W}}$ under $\delta_{\boldsymbol{W}}$



Physical Parameters: angular momentum

Angular momentum

ϕ^μ axial symmetry on $\mathcal{S}_t \to \delta_{\boldsymbol{\phi}}$ canonical transformation

$$J_{\mathcal{H}} = -\frac{1}{8\pi G}\int_{\mathcal{S}_t} \omega_\mu \phi^\mu \, {}^2\boldsymbol{\epsilon} = -\frac{1}{4\pi G}\int_{\mathcal{S}_t} f \mathrm{Im}\Psi_2 {}^2\boldsymbol{\epsilon}$$

with $\boldsymbol{\phi} = {}^2\vec{\boldsymbol{D}} f \cdot {}^2\boldsymbol{\epsilon}$ (since $\boldsymbol{\phi}$ is divergence-free)

$$J_{\mathcal{H}} = -\frac{1}{8\pi G}\int_{\mathcal{S}_t} \Omega_\mu \phi^\mu \, {}^2\boldsymbol{\epsilon} = \frac{1}{8\pi G}\int_{\mathcal{S}_t} \phi^\mu s^\nu K_{\mu\nu} \, {}^2\boldsymbol{\epsilon}$$

Remark:

Coincides with the expression for Komar angular momentum.

Physical Parameters: mass

Mass: *1st law of black hole thermodynamics*

Evolution vector $\boldsymbol{t} = \boldsymbol{\ell} + \Omega_{(\boldsymbol{t})}\boldsymbol{\phi}$.

1. Transformation $\delta_{\boldsymbol{t}}$ canonical iff $\exists\ E^t_{\mathcal{H}}$:

$$\delta E^t_{\mathcal{H}} = \frac{\kappa_{(\boldsymbol{t})}(a_{\mathcal{H}}, J_{\mathcal{H}})}{8\pi G}\,\delta a_{\mathcal{H}} + \Omega_{(\boldsymbol{t})}(a_{\mathcal{H}}, J_{\mathcal{H}})\,\delta J_{\mathcal{H}}$$

with $a_{\mathcal{H}} = \int_{\mathcal{S}_t} {}^2\boldsymbol{\epsilon} = 4\pi R^2_{\mathcal{H}}$ the area of $\mathcal{S}_t$.

2. Normalization of the energy function: stationary Kerr family $(a_{\mathcal{H}}, J_{\mathcal{H}})$

$$\begin{aligned}
M_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) &:= M_{\mathrm{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = \frac{\sqrt{R^4_{\mathcal{H}} + 4G^2 J^2_{\mathcal{H}}}}{2GR_{\mathcal{H}}}\,, \\
\kappa_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) &:= \kappa_{\mathrm{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = \frac{R^4_{\mathcal{H}} - 4G^2 J^2_{\mathcal{H}}}{2R^3_{\mathcal{H}}\sqrt{R^4_{\mathcal{H}} + 4GJ^2_{\mathcal{H}}}}\,, \\
\Omega_{\mathcal{H}}(R_{\mathcal{H}}, J_{\mathcal{H}}) &:= \Omega_{\mathrm{Kerr}}(R_{\mathcal{H}}, J_{\mathcal{H}}) = \frac{2GJ_{\mathcal{H}}}{R_{\mathcal{H}}\sqrt{R^4_{\mathcal{H}} + 4GJ^2_{\mathcal{H}}}}
\end{aligned}$$

Outline

Applications and caveats

Applications/need in multiple domains of Gravity

Numerical relativity (e.g. BH mergers), Quantum Gravity (e.g. BH entropy calculations), mathematical relativity (e.g. geometric flows and inequalities)...

Problem:

"Pletora" of proposals

Mass:

Approaches based on [see e.g. Brown & York 93]:

- Pseudo-tensor methods, leading to coordinate-dependent expressions.
- Identification of symmetries and construction of Noether charges [e.g. Wald, Wald & Iyer, Isolated Horizons here,...].
- Mathematical expressions from Cauchy data exhibiting physical properties associated with energy/mass [e.g. Hawking mass, Bartnik mass...].
- Action principle through a Hamilton-Jacobi-type analysis [e.g. Brown & York 93].
- ...

Applications and caveats

Angular momentum:

Key problem: identification of axial vector ϕ^μ. Different approaches:

- Axial symmetry [e.g. Isolated Horizons here].
- Prescriptions for a quasi-Killing vector (not respecting divergence-free character of ϕ^μ) [e.g. Dreyer et al. 03].
- Approximate Killing vector via a minimization variational prescription respecting the divergence-free character of ϕ^μ [e.g. Cook & Whiting 07].
- ϕ^μ consistent with the unique slicing of a Dynamical Trapping Horizon [Hayward 06].
- From a conformal decomposition of the metric [Korzynski 07].
- ...

Remark:

Divergence-free ϕ^μ: it suffices to guarantee the Komar mass expression to be well-defined on a closed 2-surface $\mathcal{S}_t$ (independece of a boost on the slice).

Applications and caveats

Recommended reference:

L.B. Szabados, *Quasi-local energy-momentum and angular momentum in GR: a review article*, Liv. Rev. Relat. **7**, 4 (2004), http://www.livingreviews.org/lrr-2004-4.

Outline

Outline

Positivity of ADM mass

Dominant energy condition:

For any timelike and future-directed vector v^μ, the vector $-T^\alpha{}_\mu v^\mu$ must be a future-directed timelike or null vector.
If v^μ is the 4-velocity of some observer, $-T^\alpha{}_\mu v^\mu$ is the energy-momentum density 4-vector as measured by the observer and the dominant energy condition means that this vector must be causal.

Positivity of mass Theorem: [Schoen & Yau 79,81, Witten 81].

If the matter content of an asymptotically flat spacetime satisfies the dominant energy condition, then:

$$\boxed{M_{\rm ADM} \geq 0}.$$

Furthermore, $M_{\rm ADM} = 0$ if and only if Σ_t is a hypersurface of Minkowski spacetime.

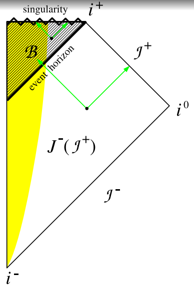
Difficult to overestimate the relevance of this theorem...

Outline

Weak Censorship Conjecture

Establishment's gravitational collapse picture:

1. Singularity theorems: when sufficient matter-energy density [e.g Penrose 65, Hawking & Penrose 70].
2. Weak Cosmic Censorship Conjecture: formation of an event horizon hiding the singularity.
3. Evolution towards a final stationary state.
4. Black Hole uniqueness theorems: final state is Kerr.



Heuristic chain of theorems and conjectures:

- Collapse gives rise to an event horizon $\mathcal{E}$ (assumes "conformal compactification picture" holds).
- Consider the Area $A^t_{\mathcal{E}}$ of the intersection $\mathcal{E}$ with a Cauchy slice Σ_t.
- Assume event horizon settles down to Kerr, with area: $A_{\rm Kerr} = 8\pi\left(M^2_{\rm Kerr} + \sqrt{M^4_{\rm Kerr} - J^2}\right) \leq 16\pi M^2_{\rm Kerr}$.
- Identify $M_{\rm Kerr}$ with final Bondi mass $M^{\infty}_{\rm Bondi} \leq M_{\rm ADM}$.

Then, $$A^t_{\mathcal{E}} \leq A_{\rm Kerr} \leq 16\pi M^2_{\rm Kerr} = 16\pi (M^{\infty}_{\rm Bondi})^2 \leq 16\pi M^2_{\rm ADM}.$$

Penrose inequality

Assuming *weak asymptotic predictability*, apparent horizons lay inside the event horizon. Then, we can formulate a version of Penrose inequality on Cauchy data:

Penrose's inequality (Penrose 73, Horowitz 84, ...)

Given the outermost marginally trapped surface $\mathcal{S}$ contained in Σ (with the dominant energy condition), it is **conjectured** that

$$A \leq 16\pi M_{\rm ADM}^2$$

where A is the minimal area enclosing the apparent horizon $\mathcal{S}$ [Ben-Dov 04]. Proved in the Riemannian case $K_{ij} = 0$ [Huisken & Ilmanen 97 , Bray 99].

Refinement of the positive mass theorem for BH spacetimes:

Defining the Black Hole irreducible mass, $M_{\rm irred} \equiv \sqrt{A/16\pi}$, Penrose inequality can be written as:

$$0 \leq M_{\rm irred} \leq M_{\rm ADM}$$

Penrose inequality has also been interpreted as an *isoperimetric inequality* for BH spacetimes [Gibbons 84, 97, Gibbons & Holzegel 06].

Outline

Black Hole extremality and geometric inequalities

For subextremal Kerr black holes, it is satisfied $J \leq M^2_{\rm Kerr}$.
More *generally* (but keeping axisymmetry):

Angular momentum-mass inequality (Theorem) [Dain 06]

For vacuum, maximal ($K = 0$), asymptotically flat, axisymmetric data:

$$|J| \leq M^2_{\rm ADM}$$

Inequality saturates only for a slice of extremal Kerr.

Does an analogue inequality holds in fully general cases?

Astrophysicists (we) are going to look *observationally* for that...

- Certainly, not *any* prescription holds: c.f. examples with $J/M^2_{\rm Komar} \geq 1$. Attempts of proving $8\pi|J| \leq A$ in certain regimes (with matter) [Petroff, Ansorg, Hennig, Pfister 05,08].
- **Problem**: Ambiguities in quasi-local masses and angular momentum...

  $$\Longrightarrow$$

  Alternative approach: characterization of (quasi-local) extremality by the absence of trapped surfaces [Booth & Fairhurst 07].

Outline

"Summary" :

- Gravitational energy in GR can be defined for extended spatial regions, and (generically) involves the introduction of some additional structure.
- Total ADM/Bondi quantities can be defined (positivity of total mass).
- For bounded regions there are inequivalent possibilites: "ambiguity" vs. "richness"...
- Study of properties of global an quasi-local notions of mass (and angular momentum) have extensive current applications in different areas of Gravity physics. "Interdisciplinary" field of research...
- BH astrophysical estimations of M and J, good enough using Kerr...
- But need to distinguish between appropriate approximations and conceptual full understanding.
- Moreover, "state of the art" of the *problem of motion* in full GR dependent on the subcommunity: "everyday calculations" on numerical relativity vs. "existence issues" in mathematical relativity [e.g. Choquet-Bruhat & Friedrich 06].

Some general references:

- L.B. Szabados, *Quasi-local energy-momentum and angular momentum in GR: a review article*, Liv. Rev. Relat. **7**, 4 (2004), `http://www.livingreviews.org/lrr-2004-4`.
- E.Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, lectures delivered at Institut Henri Poincaré in 2006, available as `gr-qc/0703035`.
- R.M. Wald, *General Relativity*, Chicago University Press, 1984.
- E. Poisson, *A relativist's toolkit*, Cambridge University Press, Cambridge (2004).
- B.Krishnan, *Fundamental properties and applications of quasi-local black hole horizons*, in `arXiv:0712.1575 [gr-qc]`, 2007.
- J.D. Brown & J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, Phys. Rev. D **47**(4), 1407–1419 (Feb 1993).
- J.L. Jaramillo, J.A. Valiente Kroon and E. Gourgoulhon, *From Geometry to Numerics: interdisciplinary aspects in mathematical and numerical relativity*, Class. Quant. Grav. **25**, 093001 (2008). [arXiv:0712.2332 [gr-qc]].