Second-order gauge-invariant perturbation theory and its applications

(Short review of my poster presentation)

Some details can be seen in my poster

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References :

K.N. Prog. Theor. Phys., **110** (2003), 723. (gr-qc/0303039).
K.N. Prog. Theor. Phys., **113** (2005), 413. (gr-qc/0410024).
K.N. Phys. Rev. D **74** (2006), 101301R. (gr-qc/0605107).
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K.N. preprint (arXiv:0804.3840[gr-qc])

(+ α)

I. Introduction

- The second order perturbation theory in general relativity has very wide physical motivation.

  – Cosmological perturbation theory
    - Expansion law of inhomogeneous universe (back reaction effect, averaging problem)
    - Non-Gaussianity in CMB (beyond WMAP)

  – Black hole perturbations
    - Radiation reaction effects due to the gravitational wave emission.
    - Close limit approximation of black hole - black hole collision (Gleiser, et.al (1996))

  – Perturbation of a star (Neutron star)
    - Rotation – pulsation coupling (Kojima 1997)

There are many physical situations to which higher order perturbation theory should be applied.

However, general relativistic perturbation theory requires more delicate treatments of “gauges”.

It is worthwhile to formulate the higher order gauge invariant perturbation theory from general point of view.

- In this poster presentation, we show …
  - General framework of the second-order gauge-invariant perturbation theory.

    (K.N. PTP, **110** (2003), 723; ibid, **113** (2005), 413.)

  - Applications
    - Second-order cosmological perturbations

      (K.N. PRD, **74** (2006), 101301R; PTP,**117** (2007), 17; arXiv:0711.0996[gr-qc]; arXiv:0804.3840[gr-qc].)
    - Toward to the application to the radiation reaction. (+α)

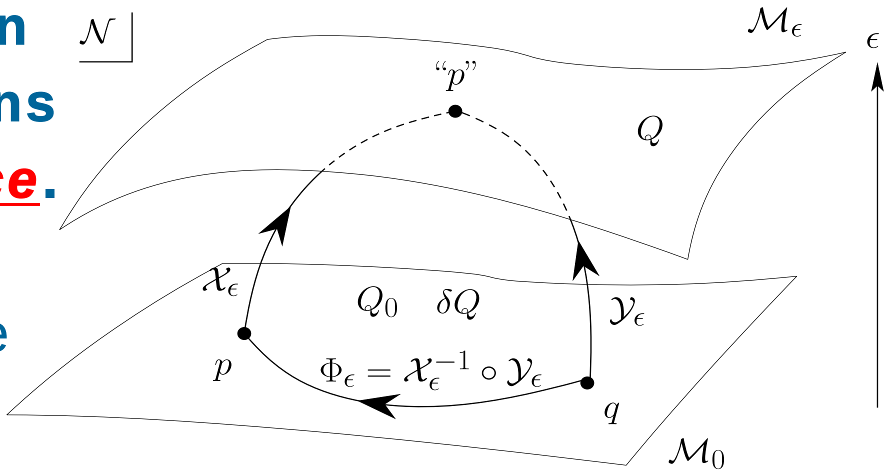
      (This part is not complete.)

II. Gauge degree of freedom in perturbations

(Stewart and Walker, PRSL **A341** (1974), 49.)

■ "***Gauge degree of freedom***" **in general relativistic perturbations arises due to *general covariance*.**

Physical spacetime (PS)



Background spacetime (BGS)

○ In any perturbation theories, we always treat two spacetimes :

– *Physical Spacetime (PS);*

– *Background Spacetime (BGS).*

○ In perturbation theories, we always write equations like

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p)$$

Through this equation, we always identify the points on these two spacetimes.

This identification is called "***gauge choice***" in perturbation theory.

Gauge transformation rules of each order

- **Expansion of gauge choices :**

We assume that each gauge choice is an exponential map.

$$Q_{\mathcal{X}} = \mathcal{X}_\epsilon^* Q = Q + \epsilon \pounds_u Q + \frac{1}{2}\epsilon^2 \pounds_u^2 Q + O(\epsilon^3)$$

$$Q_{\mathcal{Y}} = \mathcal{Y}_\epsilon^* Q = Q + \epsilon \pounds_v Q + \frac{1}{2}\epsilon^2 \pounds_v^2 Q + O(\epsilon^3)$$

$$\Phi_\epsilon^* Q = (\mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon)^* Q = Q + \epsilon \pounds_{\xi_1} Q + \frac{1}{2}\epsilon^2 \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right) Q + O(\epsilon^3)$$

(Sonego and Bruni, CMP, **193** (1998), 209.)

-------> $\xi_1 = u - v, \quad \xi_2 = [u, v]$

- **Expansion of the variable :** $Q = Q_0 + \epsilon Q_1 + \frac{1}{2}\epsilon^2 Q_2 + O(\epsilon^3)$

- **Order by order gauge transformation rules :**

$$Q_{1\mathcal{Y}} - Q_{1\mathcal{X}} = \pounds_{\xi_1} Q_{0\mathcal{X}}$$

$$Q_{2\mathcal{Y}} - Q_{2\mathcal{X}} = 2\pounds_{\xi_1} Q_{1\mathcal{X}} + \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right) Q_{0\mathcal{X}}$$

Inspecting these gauge transformation rules, we develop second-order gauge-invariant perturbation theory.

III. Gauge invariant variables

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$

  Our general framework of the second-order gauge invariant perturbation theory is based on a single assumption.

- **linear order (assumption) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X_a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$$ {}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X_a - {}_{\mathcal{X}}X_a = \left(\xi_{(1)}\right)_a $$

under the gauge transformation $\Phi_\epsilon := \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

This is correct in cosmological perturbations **(see my poster)**.

Second order :

Once we accept the above assumption for the linear order metric perturbation h_{ab} , we can always decompose the second order metric perturbations l_{ab} as follows :

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Z - \pounds_X^2\right) g_{ab},$$

where $\mathcal{L}_{ab}$ is gauge invariant part and Z_a is gauge variant part.

Under the gauge transformation $\Phi_\epsilon := \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$ the vector field Z_a is transformed as

$$_{\mathcal{Y}}Z^a - _{\mathcal{X}}Z^a = \xi_{(2)}^a + [\xi_{(1)}, X]^a$$

- Components of gauge invariant variable $\mathcal{L}_{ab}$ in cosmological perturbations :

$$\mathcal{L}_{\eta\eta} = -2a^2 \overset{(2)}{\Phi}, \quad \mathcal{L}_{i\eta} = a^2 \overset{(2)}{\nu}_i, \quad \mathcal{L}_{ij} = -2a^2 \overset{(2)}{\Psi} \gamma_{ij} + a^2 \overset{(2)}{\chi}_{ij}$$

Perturbations of an arbitrary matter field Q :

Using gauge variant part of the metric perturbation of each order, gauge invariant variables for an arbitrary fields Q other than metric are defined by

- First order perturbation of Q :

$$^{(1)}\mathcal{Q} := {}^{(1)}Q - \pounds_X {}^{(0)}Q$$

- Second order perturbation of Q :

$$^{(2)}\mathcal{Q} := {}^{(2)}Q - 2\pounds_X {}^{(1)}Q - \left\{\pounds_Z - \pounds_X^2\right\} {}^{(0)}Q$$

These implies that each order perturbation of an arbitrary field is always decomposed as

$$^{(1)}Q = \boxed{^{(1)}\mathcal{Q}} + \boxed{\pounds_X {}^{(0)}Q}$$

$$^{(2)}Q = \boxed{^{(2)}\mathcal{Q}} + \boxed{2\pounds_X {}^{(1)}Q + \left\{\pounds_Z - \pounds_X^2\right\} {}^{(0)}Q}$$

(green box) : gauge invariant part

(yellow box) : gauge variant part

Energy momentum tensor (perfect fluid)

$$\bar{T}_a{}^b = (\bar\rho + \bar p)\bar u_a g^{bc}\bar u_c + \bar p \delta_a{}^b$$
$$= T_a{}^b + \epsilon \, {}^{(1)}T_a{}^b + \frac{1}{2}\epsilon^2 \, {}^{(2)}T_a{}^b,$$

$$\bar\rho = \rho + \epsilon \overset{(1)}{\rho} + \frac{1}{2}\epsilon^2 \overset{(2)}{\rho}$$
$$\bar p = p + \epsilon \overset{(1)}{p} + \frac{1}{2}\epsilon^2 \overset{(2)}{p}$$
$$\bar u_a = u_a + \epsilon \overset{(1)}{(u_a)} + \frac{1}{2}\epsilon^2 \overset{(2)}{(u_a)}$$

- First order gauge invariant variables

$$\overset{(1)}{\mathcal{E}} := \overset{(1)}{\rho} - \pounds_X \rho, \quad \overset{(1)}{\mathcal{P}} := \overset{(1)}{p} - \pounds_X p, \quad \overset{(1)}{\mathcal{U}}_a := \overset{(1)}{(u_a)} - \pounds_X u_a$$

- Second order gauge invariant variables

$$\overset{(2)}{\mathcal{E}} := \overset{(2)}{\rho} - 2\pounds_X \overset{(1)}{\rho} - \left(\pounds_Z - \pounds_X^2\right)\rho$$
$$\overset{(2)}{\mathcal{P}} := \overset{(2)}{p} - 2\pounds_X \overset{(1)}{p} - \left(\pounds_Z - \pounds_X^2\right)p$$
$$\overset{(2)}{\mathcal{U}}_a := \overset{(2)}{(u_a)} - 2\pounds_X \overset{(1)}{(u_a)} - \left(\pounds_Z - \pounds_X^2\right)u_a$$

Perturbations of Einstein tensor and Energy momentum tensor

- First order :

$$ {}^{(1)}\bar{G}_a{}^b = {}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] + £_X G_a{}^b , $$

$$ {}^{(1)}T_a{}^b = \left(\overset{(1)}{\mathcal{E}} + \overset{(1)}{\mathcal{P}} \right) + \overset{(1)}{\mathcal{P}} \delta_a{}^b + (\rho + p) \left(\overset{(1)}{\mathcal{U}}_a u^b - g^{bd}\mathcal{H}_{dc}u^c u_a + g^{bc} \overset{(1)}{\mathcal{U}}_c u_a \right) + £_X T_a{}^b $$

- Second order :

$$ {}^{(2)}\bar{G}_a{}^b = {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] + 2£_X {}^{(1)}\bar{G}_a{}^b + \left\{ £_Z - £_X^2 \right\} G_a{}^b , $$

$$ \begin{aligned} {}^{(2)}T_a{}^b = {} & (\rho + p)u_a \left(2\mathcal{H}^{bc}\mathcal{H}_{cd}u^d - \mathcal{L}^{bd}u_d - 2\mathcal{H}^{bf} \overset{(1)}{\mathcal{U}}_f + g^{cb} \overset{(2)}{\mathcal{U}}_c \right) + 2(\rho + p) \overset{(1)}{\mathcal{U}}_a \left(-\mathcal{H}^{bc}u_c + g^{bc} \overset{(1)}{\mathcal{U}}_c \right) \\ & + (\rho + p) \overset{(2)}{\mathcal{U}}_a u^b - 2 \left(\overset{(1)}{\mathcal{E}} + \overset{(1)}{\mathcal{P}} \right) u_a \mathcal{H}^{bc} u_c + 2 \left(\overset{(1)}{\mathcal{E}} + \overset{(1)}{\mathcal{P}} \right) u_a g^{bc} \overset{(1)}{\mathcal{U}}_c \\ & + 2 \left(\overset{(1)}{\mathcal{E}} + \overset{(1)}{\mathcal{P}} \right) \overset{(1)}{\mathcal{U}}_a u^b + \left(\overset{(2)}{\mathcal{E}} + \overset{(2)}{\mathcal{P}} \right) u_a u^b + \overset{(2)}{\mathcal{P}} \delta_a{}^b \\ & + 2£_X {}^{(1)}T_a{}^b + \left(£_Z - £_X^2 \right) T_a{}^b \end{aligned} $$

(green box) : gauge invariant part

(yellow box) : gauge variant part

IV. Gauge Invariant Einstein equations

- We impose the Einstein equation of each order,

$$ {}^{(p)}G_a{}^b = 8\pi G \ {}^{(p)}T_a{}^b, \quad p = 0, 1, 2. $$

Then, the Einstein equation of each order is necessarily given in terms of gauge invariant variables :

- **linear order :** $\ {}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] = 8\pi G \ {}^{(1)}\mathcal{T}_a{}^b,$

- **second order :**

$$ {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H}, \mathcal{H}] = 8\pi G \ {}^{(2)}\mathcal{T}_a{}^b. $$

We do not have to care about gauge degree of freedom at least in the level where we concentrate only on Einstein equations.

2^{nd} order Einstein equations (cosmological, scalar modes)

- components of perturbation of the fluid four-velocity

$$\overset{(2)}{\mathcal{U}}_a := \overset{(2)}{\mathcal{U}}_\eta \, (d\eta)_a + a \left(D_i \overset{(2)}{v} + \overset{(2)}{\mathcal{V}}_i \right) (dx^i)_a \, , \qquad D^i \overset{(2)}{\mathcal{V}}_i = 0 \, ,$$

$$\overset{(2)}{\mathcal{U}}_\eta = a \left\{ \left(\overset{(1)}{\Phi} \right)^2 - \overset{(2)}{\Phi} - \left(D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i - \overset{(1)}{\nu}_i \right) \left(D^i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^i - \overset{(1)}{\nu}^i \right) \right\} .$$

- energy density perturbation

$$4\pi G a^2 \overset{(2)}{\mathcal{E}} = \left(-3\mathcal{H}\partial_\eta + \Delta + 3K - 3\mathcal{H}^2 \right) \overset{(2)}{\Phi} - \Gamma + \frac{3}{2} \left(\Delta^{-1} D^i D_j \Gamma_i{}^j - \frac{1}{3} \Gamma_k{}^k \right)$$
$$- \frac{9}{2} \mathcal{H} \partial_\eta \left(\Delta + 3K \right)^{-1} \left(\Delta^{-1} D^i D_j \Gamma_i{}^j - \frac{1}{3} \Gamma_k{}^k \right) .$$

- pressure perturbation

$$4\pi G a^2 \overset{(2)}{\mathcal{P}} = \left(\partial_\eta^2 + 3\mathcal{H}\partial_\eta - K + 2\partial_\eta \mathcal{H} + \mathcal{H}^2 \right) \overset{(2)}{\Phi} - \frac{1}{2} \Delta^{-1} D^i D_j \Gamma_i{}^j$$
$$+ \frac{3}{2} \left(\partial_\eta^2 + 2\mathcal{H}\partial_\eta \right) \left(\Delta + 3K \right)^{-1} \left(\Delta^{-1} D^i D_j \Gamma_i{}^j - \frac{1}{3} \Gamma_k{}^k \right) .$$

- velocity perturbation

$$8\pi G a^2 (\epsilon + p) D_i \overset{(2)}{v} = -2\partial_\eta D_i \overset{(2)}{\Psi} - 2\mathcal{H} D_i \overset{(2)}{\Phi} + D_i \Delta^{-1} D^k \Gamma_k \, .$$

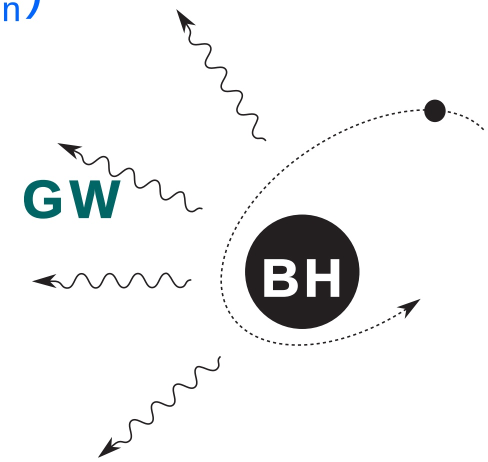
- traceless part of the spatial component of Einstein equation

$$\overset{(2)}{\Psi} - \overset{(2)}{\Phi} = \frac{3}{2} \left(\Delta + 3K \right)^{-1} \left(\Delta^{-1} D^i D_j \Gamma_i{}^j - \frac{1}{3} \Gamma_k{}^k \right) .$$

V. Toward the application to the radiation reaction problem via black hole perturbations

The capture of solar-mass compact objects by massive black holes (galactic centers $10^6 M_{sun}$)

・・・ one of the promising sources of gravitational waves for LISA. (http://lisa.jpl.nasa.gov/)



- The black hole perturbation (mass M)
- Perturbation parameter $\mu/M \sim 10^{-6}$
- Energy momentum tensor $T^{\mu\nu} = \mu \int d\tau \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau}$

Perturbations in radiation reaction problem

To discuss the radiation reaction effect by black hole perturbations we consider the following order counting:

- Energy momentum tensor :

$$T_{ab} = 0$$

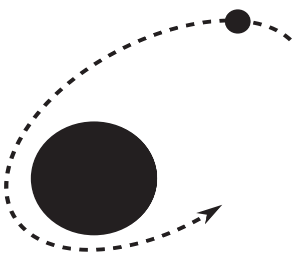
(Background)

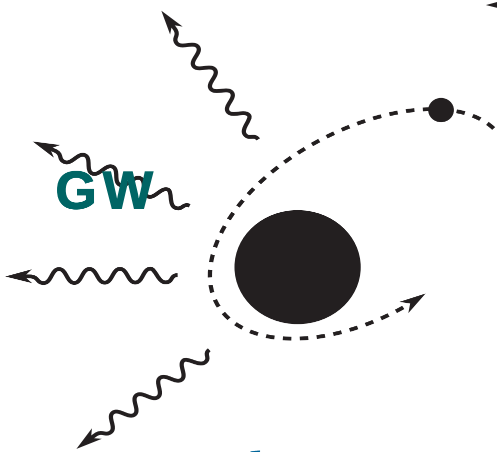
$$\epsilon^{(1)}T_{ab}$$

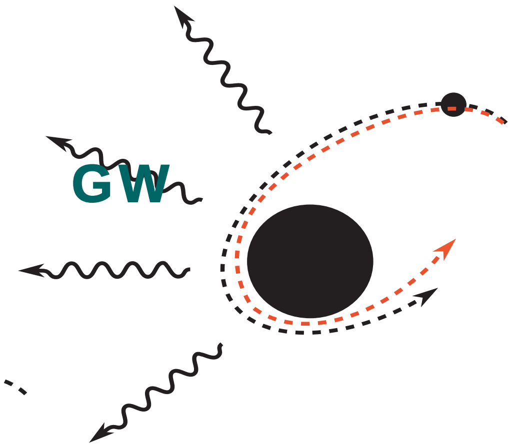
(Point particle , geodesic)

$$\epsilon^{(1)}T_{ab} + \frac{1}{2}\epsilon^{2(2)}T_{ab}$$

(geodesic + its deviation)







- metric perturbations :

$$g_{ab}$$

(Background metric)

$$g_{ab} + \epsilon h_{ab}$$

(Background + GW emission)

In terms of gauge invariant variables …

Energy momentum tensor and equation of motion of each order is given as follows :

First order :

- Gauge invariant energy momentum tensor :

$$ {}^{(1)}\mathcal{T}_b{}^a := {}^{(1)}T_b{}^a - \pounds_X T_b{}^a = {}^{(1)}T_b{}^a $$

(energy momentum tensor for a point particle)

- Eq. of motion in gauge invariant form :

$$ \begin{aligned} {}^{(1)}(\bar{\nabla}_a \bar{T}_b{}^a) &= \nabla_a {}^{(1)}\mathcal{T}_b{}^a + H_{ca}{}^a\left[{}^{(1)}\mathcal{H}\right] T_b{}^c - H_{ba}{}^c\left[{}^{(1)}\mathcal{H}\right] T_c{}^a \\ &\quad + \pounds_X \nabla_a T_b{}^a \\ &= \nabla_a {}^{(1)}\mathcal{T}_b{}^a = 0 \end{aligned} $$

This equation gives the geodesic equation around the black hole.

Second order :

- Gauge invariant energy momentum tensor :

$$ {}^{(2)}\mathcal{T}_b{}^a := {}^{(2)}T_b{}^a - 2\pounds_X {}^{(1)}T_b{}^a - \left\{\pounds_Y - \pounds_X^2\right\} T_a{}^b $$
$$ = {}^{(2)}T_b{}^a - 2\pounds_X {}^{(1)}T_b{}^a $$

(This describes deviations from geodesic motions.)

- Eq. of motion in gauge invariant form :

$$ {}^{(2)}\!\left(\bar{\nabla}_a \bar{T}_b{}^a\right) = \nabla_a {}^{(2)}\mathcal{T}_b{}^a $$
$$ - \left(2H_{cad}\left[{}^{(1)}\mathcal{H}\right] {}^{(1)}\mathcal{H}^{da} - H_{ca}{}^a\left[{}^{2)}\mathcal{H}\right]\right) T_b{}^c $$
$$ + \left(2H_{bad}\left[{}^{(1)}\mathcal{H}\right] {}^{(1)}\mathcal{H}^{dc} - H_{ba}{}^c\left[{}^{(2)}\mathcal{H}\right]\right) T_c{}^a $$
$$ -2H_{ba}{}^c\left[{}^{(1)}\mathcal{H}\right] {}^{(1)}\mathcal{T}_c{}^a + 2H_{ca}{}^a\left[{}^{(1)}\mathcal{H}\right] {}^{(1)}\mathcal{T}_b{}^c $$
$$ +2\pounds_{{}^{(1)}X} {}^{(1)}\!\left(\bar{\nabla}_a \bar{T}_b{}^a\right) + \left\{\pounds_{{}^{(2)}X} - \pounds^2_{{}^{(1)}X}\right\} \left(\nabla_a T_b{}^a\right) $$
$$ = \nabla_a {}^{(2)}\mathcal{T}_b{}^a - 2H_{ba}{}^c\left[{}^{(1)}\mathcal{H}\right] {}^{(1)}\mathcal{T}_c{}^a + 2H_{ca}{}^a\left[{}^{(1)}\mathcal{H}\right] {}^{(1)}\mathcal{T}_b{}^c = 0 $$

These terms correspond to the self-force.

- **Eq. for the deviation from geodesic should be given in a gauge invariant form,** although we do not consider the “regularization”, yet.

However, to evaluate this self-force completely, there are many problems which should be clarified.

- Gauge invariant treatments of perturbations
  - Schwarzschild case ... Problems in the treatments of l=0,1 modes.
    - No gauge invariant variables in l=0,1 modes. (in many ref.)

      <---->

      We have defined gauge-invariant variables for the perturbations with FRW background (special case of spherically symmetric spacetimes) and these also include spherical modes.

      How should we understand these consistently.
  - Kerr case ... ??? (Newman-Penrose formulation)
- Treatments of a point particle <---> "regularization"
  - We should clarify the gauge invariant treatments of a point particle or the regularization (or extraction of tail part) of the metric in the gauge invariant manner.
- A systematic higher order perturbative expansion like the post-Newtonian expansion is possible ??? (It might be a dream)

VI. Summary

We have shown the framework of the general relativistic second order perturbations from general point of view, which is developed in [K.N., PTP **110** (2003), 723; ibid, **113** (2005), 413.].

- **We have verified the framework in the above references is applicable to cosmological perturbations.**
  - **We have derived the second order Einstein equations in terms of gauge invariant variables defined along this general framework.** [K.N., PRD**74** (2007), 101301. gr-qc/0605108]

In this framework, we do not specify anything about the background spacetime nor the physical meaning of the infinitesimal parameter for perturbations.

- (I hope) This framework will be applicable to any theory in which general covariance is imposed.

This framework will have very many applications.

List of application candidates (1)

- Second-order cosmological perturbation theory (in progress)
  - Ignoring the first order vector- and tensor-modes
    - Single perfect fluid system. (OK)
    - Single scalar field system. (OK)
  - Extension of our formulation to include the first order vector- and tensor-modes.
    - Single perfect fluid system (OK)
    - Single scalar field system (OK)
  - Extensions to imperfect fluid system (in progress)
  - Extensions to the multi-fields system
  - Extensions to the Einstein-Boltzmann system
  - Nonlinear effects in CMB physics
- Radiation reaction Problem based on the black hole perturbation theory (Just planning).

List of application candidates (2)

- The correspondence between observables in experiments (observation) and gauge invariant variables defined here.
  - Ex. The relation between gauge invariant variables and phase difference in the laser interferometer for GW detection.
- Post-Minkowski expansion alternative to post Newtonian expansion (post-Minkowski description of a binary system).
  - The second-order perturbation of the Einstein tensor is already given !!!
  - But we have to specify the energy momentum tensor of a binary system.

    In particular, we have to treat a two-point particle system and some regularization procedures are necessary to treat this system.
- … etc.

There are many applications to which our formulation should be applied.

I want to clarify these problems step by step.

おわり
（ＥＮＤ）

II. "Gauge" in general relativity

(R.K. Sachs (1964).)

- There are two kinds of "gauge" in general relativity.
  - The concepts of these two "gauge" are closely related to the general covariance.
  - "General covariance" :

    There is no preferred coordinate system in nature.

- **The first kind "gauge"** is a coordinate system on a single spacetime manifold.

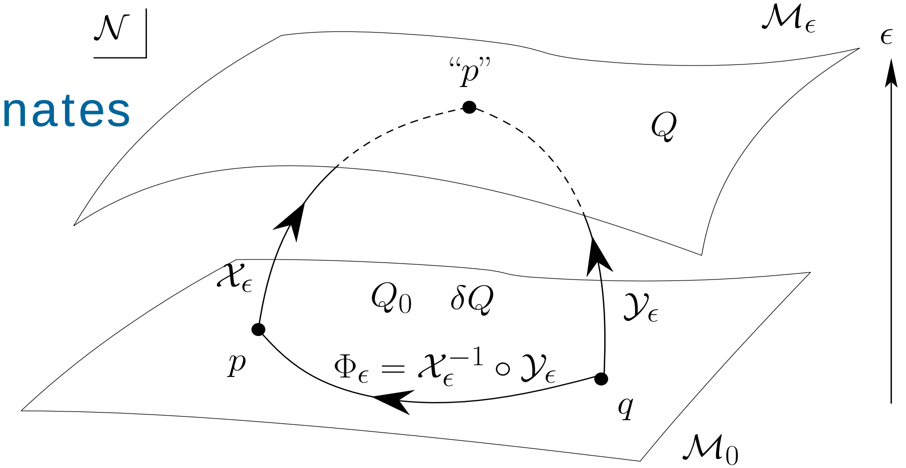
- **The second kind "gauge"** appears in the perturbation theory.

  This is a point identification between the physical spacetime and the background spacetime.
  - To explain this second kind "gauge", we have to remind what we are doing in perturbation theory.

The gauge choice is not unique by virtue of general covariance.

General covariance :

- "There is no preferred coordinates in nature" (intuitively).



Gauge transformation :

- The change of the point identification map.

- **Different gauge choice :** $\mathcal{X}_\epsilon$, $\mathcal{Y}_\epsilon$
- **Representation of physical variable :**

$$Q_{\mathcal{X}} := \mathcal{X}_\epsilon^* Q \ , \quad Q_{\mathcal{Y}} := \mathcal{Y}_\epsilon^* Q$$

- **Gauge transformation :** $\mathcal{X} \to \mathcal{Y}$

$$\Phi_\epsilon := \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon \ , \quad Q_{\mathcal{Y}} = \Phi_\epsilon^* Q_{\mathcal{X}}$$

Source terms in the 2nd-order Einstein eq. (for exam ...)

$$\Gamma := +8\pi G a^2 (\epsilon + p) D_i \overset{(1)}{v} D^i \overset{(1)}{v} - 3 D_k \overset{(1)}{\Phi} D^k \overset{(1)}{\Phi} - 8 \overset{(1)}{\Phi} \Delta \overset{(1)}{\Phi}$$

$$-3 \left(\partial_\eta \overset{(1)}{\Phi}\right)^2 - 12K \left(\overset{(1)}{\Phi}\right)^2 - 12\mathcal{H}^2 \left(\overset{(1)}{\Phi}\right)^2$$

$$-4 \left(\partial_\eta D_i \overset{(1)}{\Phi} + \mathcal{H} D_i \overset{(1)}{\Phi}\right) \overset{(1)}{\mathcal{V}^i} - 2\mathcal{H} D_k \overset{(1)}{\Phi} \overset{(1)}{\nu^k}$$

$$+8\pi G a^2 (\epsilon + p) \overset{(1)}{\mathcal{V}_i} \overset{(1)}{\mathcal{V}^i} + \frac{1}{2} D_k \overset{(1)}{\nu_l} D^{(k} \overset{(1)}{\nu^{l)}} + 3\mathcal{H}^2 \overset{(1)}{\nu^k} \overset{(1)}{\nu_k}$$

$$+ D_l D_k \overset{(1)}{\Phi} \overset{(1)}{\chi^{lk}}$$

$$-2\mathcal{H} D^k \overset{(1)}{\nu^l} \overset{(1)}{\chi_{kl}} - \frac{1}{2} D^k \overset{(1)}{\nu^l} \partial_\eta \overset{(1)}{\chi_{lk}}$$

$$+\frac{1}{8} \partial_\eta \overset{(1)}{\chi_{lk}} \partial_\eta \overset{(1)}{\chi^{kl}} + \mathcal{H} \overset{(1)}{\chi_{kl}} \partial_\eta \overset{(1)}{\chi^{lk}} - \frac{1}{8} D_k \overset{(1)}{\chi_{lm}} D^k \overset{(1)}{\chi^{ml}}$$

$$+\frac{1}{2} D_k \overset{(1)}{\chi_{lm}} D^{[l} \overset{(1)}{\chi^{k]m}} - \frac{1}{2} \overset{(1)}{\chi^{lm}} (\Delta - K) \overset{(1)}{\chi_{lm}}$$

Mode coupling :

(orange) : scalar-scalar	(yellow) : scalar-vector	(green) : scalar-tensor
(pink) : vector-vector	(blue) : vector-tensor	(purple) : tensor-tensor

Cosmological perturbations

- Background metric

$$g_{ab} = a^2(\eta)\left(-(d\eta)_a(d\eta)_b + \gamma_{ij}(dx^i)_a(dx^j)_b\right)$$

γ_{ij} : metric on maximally symmetric 3-space

- metric perturbation

$$\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$$

- decomposition of linear perturbation

$$h_{ab} = h_{\eta\eta}(d\eta)_a(d\eta)_b + 2h_{\eta i}(d\eta)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b$$

$$\begin{aligned} h_{\eta i} &= D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \\ h_{ij} &= a^2 h_{(L)}\gamma_{ij} + a^2 h_{(T)ij}, \quad h_{(T)}{}^i{}_i := \gamma^{ij} h_{(T)ij}, \\ h_{(T)ij} &= \left(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\right) h_{(TL)} + 2D_{(i}h_{(TV)j)} + h_{(TT)ij}, \\ D^i h_{(TV)i} &= 0, \quad D^i h_{(TT)ij} = 0. \end{aligned}$$

Uniqueness of this decomposition

---> Existence of Green functions Δ^{-1}, $(\Delta + 2K)^{-1}$, $(\Delta + 3K)^{-1}$

K : curvature constant associated with the metric γ_{ij}

Gauge variant and invariant variables of linear order metric perturbation.

- gauge variant variables : $X_a := X_\eta (d\eta)_a + X_i (dx^i)_a$

$$X_\eta := h_{(VL)} - \frac{1}{2} a^2 \partial_\eta h_{(TL)},$$

$$X_i := a^2 \left(h_{(TV)i} + \frac{1}{2} D_i h_{(TL)} \right),$$

where $\mathcal{Y} X_a - \mathcal{X} X_a = \left(\xi_{(1)} \right)_a$.

- gauge invariant variables :

$$\mathcal{H}_{\eta\eta} := -2a^2 \Phi := h_{\eta\eta} - 2 \left(\partial_\eta - \mathcal{H} \right) X_\eta$$

$$\mathcal{H}_{i\eta} := a^2 \nu_i := h_{i\eta} - D_i X_\eta - \left(\partial_\eta - 2\mathcal{H} \right) X_i$$

$$\mathcal{H}_{ij} := -2a^2 \Psi + a^2 \chi_{ij} := h_{ij} - 2D_{(i} X_{j)} + 2\mathcal{H} \gamma_{ij} X_\eta$$

$$D^i \nu_i = 0, \quad \gamma^{ij} \chi_{ij} = 0 = D^i \chi_{ij}$$ (J. Bardeen (1980))

where $\mathcal{Y} \mathcal{H}_{ab} - \mathcal{X} \mathcal{H}_{ab} = 0.$ $\quad \mathcal{H} = \partial_\eta a / a$

おまけ