Motion in Loop Quantum Gravity

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Overview

Introduction : LQG in a nutshell

1. The basis of Loop Quantum Gravity

- *Ashtekar variables : Gravity "looks" like Yang-Mills*
- *Quantum geometry : states and geometrical operators*
- *The problem of the Hamiltonian constraint*

2. The motion in a quantum geometry: a model

- *3D gravity : a topological theory with no gravitational waves*
- *Quantum gravity makes space-time non-commutative*
- *Effects of the non-commutativity on the motion of a field*

Discussion and perspectives

▷ A self-gravitating Quantum Field Theory ?

▷ LQG compared to other approaches of Quantum Gravity...

What is Loop Quantum Gravity ?

In a nutshell...

The Fundamental characteristics : LQG is supposed to be

- ▷ Canonical quantization of 4D Gravity : $\mathcal{M} = \Sigma \times \mathbb{R}$
- ▷ Non-perturbative quantization : question of renormalization avoided
- ▷ Background independent quantization : no background metric needed

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The main results : LQG is supposed to provide

- ▷ Structure of Quantum geometry at Planck scale : discretness
- ▷ Microscopic explanation of Black holes thermodynamics : $S = A/4$
- ▷ Resolution of cosmological singularity : LQ Cosmology

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- ▷ Consistent coupling to other interactions
- ▷ Free parameters to be understood : Immirzi ambiguity

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LQG offers a very nice, still under construction but fascinating framework to Quantum Gravity : worth being studied...

The ADM formulation of gravity ('61)

Lagrangian formulation : M a 4D manifold

- Einstein-Hilbert action : functional of the metric g

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Hamiltonian formulation : $M = \Sigma \times \mathbb{R}$

▷ ADM variables : $ds^2 = N^2 dt^2 - (N^a dt + h_{ab} dx^b)(N^a dt + h_{ac} dx^c)$

▷ ADM action : (h, π) conjugate variables

$$S_{ADM}[h, \pi; N, N^a] = \int dt \int d^3x (\dot{h}\pi + N^a H_a[h, \pi] + NH[h, \pi])$$

▷ Constraints $H = 0 = H_a$ generate the diffeomorphisms

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What about the quantization ?

- Path integral : non-renormalizability
- Canonical : too complicated constraints !

Ashtekar variables : constraints are "polynomials" ('86)

First order Lagrangian

▷ The dynamical variables are the Cartan data :
a tetrad $e^I = e^I_\mu dx^\mu$ such that $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$
a $so(3,1)$ spin-connection $\omega = \omega^i R_i + \omega^{0i} B_i$; $F(\omega)$ its curvature

▷ Einstein-Palatini-Holst action : depends on the free parameter $\gamma \neq 0$

$$S_P[e, A] = \int e^I \wedge e^J \wedge (\star F_{IJ}(\omega) - \frac{1}{\gamma} F_{IJ}(\omega))$$

▷ For any value of γ, equations of motion are the same as EH ones

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Canonical analysis with the choice $e^0 = 0$

▷ Ashtekar variables : $E^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c$ and $A^i_a = \omega^i_a + \gamma \omega^{0i}_a$

$$\{A^i_a(x), E^b_j(y)\} = (8\pi\gamma G)\delta^b_a \delta^i_j \delta^3(x,y)$$

▷ The constraints : they generate the symmetries

$$\mathcal{G}_i = D_a E^a_i, \quad H_a = F^i_{ab} E^b_i, \quad H = (F^{ij}_{ab} + (\gamma^2+1) K^i_{[a} K^j_{b]}) E^a_i E^b_j$$

The space of Kinematical states ('95)

Very simple quantization strategy

- ▷ The classical variables (A, E) become operators
- ▷ Choice of polarization : states are $\Psi(A)$ and schematically

$$(A_a^i(x) \triangleright \Psi)(A) = A_a^i(x)\Psi(A), \quad (\frac{E_i^a(x)}{8\pi\gamma G} \triangleright \Psi)(A) = -i\hbar \frac{\delta\Psi(A)}{\delta A_a^i(x)}$$

- ▷ Solutions of the constraints : $\mathcal{G} = 0$, $H_a = 0$ and $H = 0$
- ▷ Construct observables to "recover" the geometry : area, volume...

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The kinematical states space $\mathcal{K}$

- ▷ Make use of techniques developped in gauge theory
- ▷ Quantum states are functions of holonomies $U_\lambda(A) \in SU(2)$ only
- ▷ Ψ defined by a graph Γ with L links and $f \in C(SU(2)^L)$

$$\Psi_{\Gamma,f}(A) = f(U_{\lambda_1}(A), \cdots, U_{\lambda_L}(A))$$

- ▷ Scalar product : $\langle \Psi_{\Gamma,f} | \Psi_{\Gamma',f'} \rangle = \delta_{\Gamma,\Gamma'} \int d\mu(U)\, \overline{f(U)} f'(U)$

Towards the construction of physical states

Imposing the constraints successively

$$\mathcal{K} \xrightarrow{\mathcal{G}_i} \mathcal{K}_0 \xrightarrow{H_a} \mathcal{K}_{diff} \xrightarrow{H} \mathcal{H}$$

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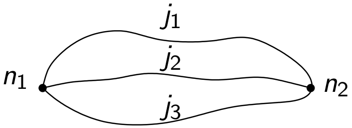
Internal gauge invariance : the space $\mathcal{K}_0$

▷ Spin-network : colored graph



: j_i are $SU(2)$ representations

: n_i are $SU(2)$ intertwiners

▷ Spin-network states $|S\rangle$ form an orthonormal basis of $\mathcal{K}_0$

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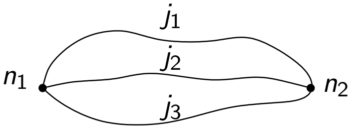

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Diffeomorphism invariance : the space $\mathcal{K}_{diff}$

▷ identify states related by a diffeomorphism

▷ States of $\mathcal{K}_{diff}$ are labelled by knots

▷ The Hilbert space $\mathcal{K}_{diff}$ is separable

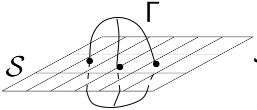
Geometrical operators

Area operator $\mathcal{A}(\mathcal{S})$ acting on $\mathcal{K}_0$

- ▷ Classical area of a surface $\mathcal{S}$: $\mathcal{A}(\mathcal{S}) \;=\; \int_{\mathcal{S}} \sqrt{n_a E^a_i n_b E^b_i}\, d^2\sigma$
- ▷ Quantum area operator : $\mathcal{S} = \cup_n^N \mathcal{S}_n$

$$\mathcal{A}(\mathcal{S}) \;=\; \lim_{N\to\infty} \sum_n \sqrt{E_i(\mathcal{S}_n) E_i(\mathcal{S}_n)} \quad \text{with} \quad E_i(\mathcal{S}_n) = \int_{\mathcal{S}_n} E_i$$

- ▷ Spectrum and Quanta of area



$$\mathcal{A}(\mathcal{S})|S\rangle \;=\; \frac{8\pi\gamma\hbar G}{c^3} \sum_{P\in\mathcal{S}\cap\Gamma} \sqrt{j_P(j_P+1)}|S\rangle$$

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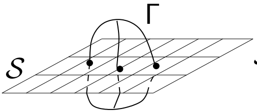

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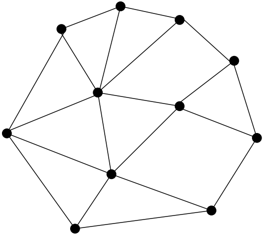
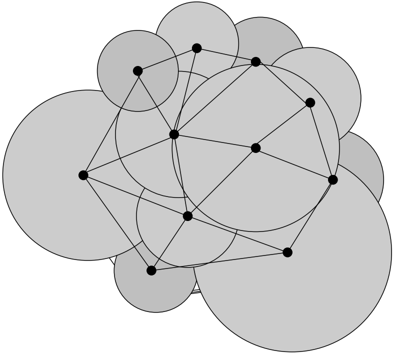
Volume operator $\mathcal{V}(\mathcal{R})$ acting on $\mathcal{K}_0$

- Classical volume on a domain $\mathcal{R}$: $\mathcal{V}(\mathcal{R}) = \int_{\mathcal{R}} d^3x \sqrt{\frac{|\epsilon_{abc}\epsilon_{ijk} E^{ai} E^{bj} E^{ck}|}{3!}}$
- It acts on the nodes of $|S\rangle$: discrete spectrum

How looks space at the Planck scale

Space is fundamentaly discrete...





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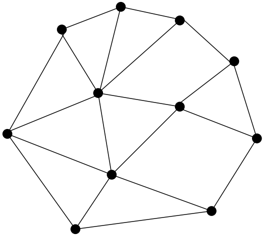

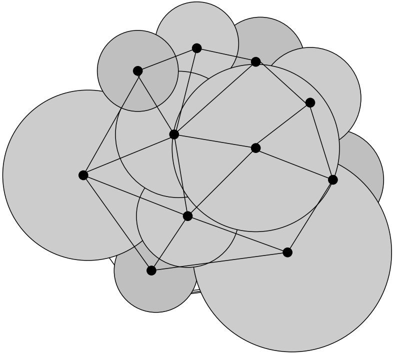

... and non-commutative !

The problem of the Hamiltonian constraint

Solving the Hamiltonian constraint is fundamental

- ▷ To find physical states : construct a projector P

$$|S\rangle_{phys} = P|S\rangle \text{ and } \langle S|S'\rangle_{phys} = \langle S|PS'\rangle$$

- ▷ To find physical observables and make robust physical predictions

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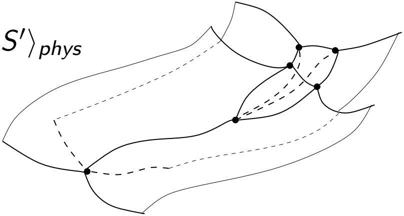
The different approaches to solve the problem

- ▷ Thiemann proposes a very tricky regularization
- ▷ Spin-Foam models : covariant quantization

$$\mathcal{A} = \langle S|S'\rangle_{phys}$$



3D gravity : a topological theory

Pure gravity : no gravitational waves

- ▷ Einstein equations are similar to 4D : $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$
- ▷ The Weyl tensor vanishes in 3D : $C_{\mu\nu\rho\lambda} = 0$
- ▷ Solutions are locally of constant curvature : no gravitational waves
- ▷ No local degrees of freedom : $\#(h_{ij}, \pi^{ij}) = 6$ and 3 constraints H, H_a
- ▷ First order gravity is a topological theory : Chern-Simons or BF

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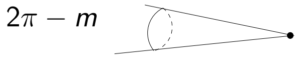
Coupling to point particles : momenta become group-like variables

- ▷ No Newton interactions between massive particles
- ▷ Massive particles create conical singularities : $m < 2\pi$



- ▷ Momenta are group-like variables : $\vec{p}$ associated to $g = \exp(ip^a J_a)$

Deformation of the Poincaré group

Quantum Gravity and deformed symmetry group

- ▷ The Poincaré group changed into Quantum Group : Drinfeld double
- ▷ First consequence : momenta are bounded by ℓ_P^{-1}
- ▷ Second consequence : modification of addition rule of momenta

$$\vec{p} \oplus \vec{q} = \sqrt{1 - \ell_P^2 p^2}\,\vec{q} + \sqrt{1 - \ell_P^2 q^2}\,\vec{p} + \ell_P \vec{p} \wedge \vec{q}$$

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The emergence of the non-commutative space

- The Space of momenta is curved and non-commutative :

$$\psi(p) \in (C(SU(2))^*, \circ)$$

- Product of plane waves : $\delta_{\vec{p}} \circ \delta_{\vec{q}} = \delta_{\vec{p} \oplus \vec{q}}$

Space-time is a Fuzzy space

The Fuzzy space representation

- ▷ We consider the Euclidean theory : $ISU(2) = SU(2) \ltimes \mathbb{R}^3$
- ▷ Space-time is obtained from $SU(2)$ harmonic analysis

$$\mathcal{F} : C(SU(2))^* \rightarrow \mathrm{Mat}(\mathbb{C}) = \oplus_{n=1}^{\infty} \mathrm{Mat}_n(\mathbb{C})$$

- ▷ Space-time is an union of concentric fuzzy spheres :
  Functions on space-time are matrices : $\hat{\Phi} = \oplus_I \hat{\Phi}_I$, $I \in \frac{1}{2}\mathbb{N}$
  To each $\hat{\Phi}$ is associated a (pair of) $\Phi(x)$ with a $\star$ product
  Coordinate functions : $\hat{x}_a = \ell_P \oplus_I D^I(J_a)$ satisfy

$$[\hat{x}_a, \hat{x}_b] = i\epsilon_{ab}{}^c \ell_P \hat{x}_c$$

Radius of the spheres : $\hat{R}^2 = \hat{x}_a\hat{x}_a = \ell_P^2 \oplus_I [I(I+1)]\mathbb{I}_I$

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Integro-Differential calculus

- Finite difference operators : $(\partial_a \hat{\Phi})_I = C_a^+ \hat{\Phi}_{I+\frac{1}{2}} + C_a^- \hat{\Phi}_{I-\frac{1}{2}}$
- Integral on the fuzzy space : $\mathrm{Tr}(\hat{\Phi}) = \sum_I (2I+1) \mathrm{tr}(\hat{\Phi}_I)$

An action for the scalar field

Free action : local action for $\Phi(x)$

- It is a matrix model for $\hat{\Phi}$

$$S[\hat{\Phi}] = \mathrm{Tr}(\partial_a \hat{\Phi} \partial_a \hat{\Phi} + m^2 \hat{\Phi}\hat{\Phi})$$

- Classical solutions are similar to standard ones
- The propagator is free of UV divergences :

$$G(x) = \frac{\ell_P^3}{4\pi x} \int_0^1 \frac{kdk}{\sqrt{1-k^2}} \frac{\sin(2\ell_P^{-1} xk)}{k^2 - (m\ell_P)^2 + i\epsilon} \Longrightarrow G(0) < \infty$$

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Interactions : deformed addition rule of momenta at vertex

- fundamentaly non-local in terms of $\Phi(x)$
- Physical processes like 3-points function modified

The model for a particle

Reduction to one-dimension

- Φ depends only on the variable t
- $\star$ product is commutative but non-local
- $\hat{\Phi}$ is a diagonal infinite dimensional matrix

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Classical solutions with a cubic interaction

$$S[\Phi] = \int dt(\partial_t \Phi \star \partial_t \Phi + \frac{\epsilon}{6} \Phi \star \Phi \star \Phi)$$

- Standard solution for $\Phi(0) = 0$ and $\dot{\Phi}(0) = v$

$$\Phi_{cl}(t) = vt - \epsilon \frac{v^2 t^4}{12} + \cdots$$

- Quantum gravity solutions

$$\Phi(t) = vt - \epsilon \frac{v^2 t^4}{12} - \epsilon \frac{\ell_P^2 t^2 v^2}{6} + \cdots$$

Conclusions and Perspectives

Loop Quantum Gravity compared to string theory

- ▷ Gravity is a fundamental theory
- ▷ No-extra structure needed (dimensions, particles)
- ▷ No unification of all interactions
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QFT in Loop Quantum Gravity

- ▷ Space is fundamentaly discrete : deformation of Poincaré group ?
- ▷ Resolution of UV divergences due to discretness ?
- ▷ Is it possible to measure the effects ?

References

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