Self-force and wave generation

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Themes

A second-order formulation of perturbation theory justifies the relaxed treatment of the first-order perturbative problem.

Second-order perturbation theory should not be attempted with a point particle (Adam's talk).

The arguments presented here are based on the close analogy with the structure of post-Minkowski theory.

Linearized theory

The Einstein field equations are linearized about flat spacetime.

They break down into two sets:

1. A wave equation for potentials $h^{\alpha\beta}$,

$$\Box h^{\alpha\beta} = -16\pi T^{\alpha\beta}[\eta, z]$$

2. A gauge condition on the potentials,

$$\partial_\beta h^{\alpha\beta} = 0$$

The metric is then $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$.

The energy-momentum tensor is a functional of the Minkowski metric and a set of world lines $z_A(\tau)$.

The gauge condition is in a one-to-one correspondence with the conservation equation:

$$\partial_\beta h^{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \partial_\beta T^{\alpha\beta}[\eta, z] = 0$$

Conservation implies that each world line is straight: $z = v\tau$.

In linearized theory, the bodies cannot interact gravitationally.

After solving the wave equation only, the potentials are functionals of arbitrary world lines: $h^{\alpha\beta} = h^{\alpha\beta}(x, z)$.

After imposing the gauge condition the world lines become straight: $h^{\alpha\beta} = h^{\alpha\beta}(x, z = v\tau)$.

Relaxed linearized theory

We next examine a relaxed formulation in which the gauge condition and conservation equations are modified:

$$0 = \partial_\beta h^{\alpha\beta} \quad \longrightarrow \quad 0 = \nabla_\beta[g] h^{\alpha\beta} = \partial_\beta h^{\alpha\beta} + O(h\partial h)$$

$$0 = \partial_\beta T^{\alpha\beta}[\eta, z] \quad \longrightarrow \quad 0 = \nabla_\beta[g] T^{\alpha\beta}[\eta, z] = \partial_\beta T^{\alpha\beta} + O(T\partial h)$$

The wave equation is unchanged, and the metric is still $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$.

In this formulation the world lines are no longer straight: the particles interact gravitationally.

Relaxed linearized theory: near zon

In the near zone the formulation is acceptable. The metric is

$$ds^2 = -(1-2U)\,dt^2 + (1+2U)(dx^2 + dy^2 + dz^2)$$

and the motion is determined by $a^a = \partial^a U$.

It is understood that in this relaxed formulation, U and $z(\tau)$ are to be determined self-consistently.

It would be inappropriate (moronic!) to solve the wave equation with $T^{\alpha\beta}[\eta, v\tau]$ to get $h^{\alpha\beta}(x, v\tau)$, and then to impose the equations of motion to let $v\tau \to z(\tau)$.

This procedure would work for short times (small deviations from straight lines), but the relaxed formulation promises a self-consistent treatment that holds for long times.

Relaxed linearized theory: wave zon

The relaxed formulation of linearized theory is unacceptable in the wave zone. The solution to the wave equation is

$$\begin{aligned} h^{ab} &= 4\int \frac{T^{ab}(t-|\boldsymbol{x}-\boldsymbol{x'}|,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,dV' \\ &\simeq \frac{4}{r}\int T^{ab}(t-r+\boldsymbol{n}\cdot\boldsymbol{x'},\boldsymbol{x'})\,dV' \\ &\simeq \frac{4}{r}\int T^{ab}(t-r,\boldsymbol{x'})\,dV' \end{aligned}$$

With $T^{ab} = \sum_A m_A v_A^a v_A^b \delta(\boldsymbol{x}-\boldsymbol{z}_A)$, this is

$$h^{ab} = \frac{4}{r}\sum_A m_A v_A^a v_A^b$$

This is not equal to the expected answer,

$$h^{ab} = \frac{2}{r}\ddot{Q}^{ab}(t - r)$$

with $Q^{ab} = \sum_A m_A z_A^a z_A^b$ and

$$\ddot{Q}^{ab} = \sum_A m_A \Big(2v_A^a v_A^b + z_A^a a_A^b + a_A^a z_A^b\Big)$$

The acceleration terms are missing.

For a Newtonian binary system in circular orbit,

$$h_\times = -\frac{2m_{\rm reduced}}{r}\Big(v^2 + \underbrace{\frac{m_{\rm total}}{b}}_{\text{missing}}\Big)\cos\theta\sin[\Omega(t-r)-\phi]$$

Because $v^2 = m/b$ in Newtonian dynamics, the waveform returned by the relaxed formulation of linearized theory is wrong by a factor of 2.

Fluxes calculated from this would be wrong by a factor of 4.

Post-Minkowski theory

The relaxed formulation of linearized theory was adopted without proper justification.

It fails as a wave-generation formalism.

Both deficiencies can be cured in a post-linear formulation.

The Landau-Lifshitz formulation of the Einstein field equations is convenient for this purpose.

The equations break down into two sets:

1. A wave equation for potentials $h^{\alpha\beta}$,

$$\Box h^{\alpha\beta} = -16\pi\tau^{\alpha\beta}$$

$$\tau^{\alpha\beta} = (-g)(T^{\alpha\beta}[g, z] + t^{\alpha\beta})$$

   with $t^{\alpha\beta} \sim \partial h \partial h + h\partial^2 h +$ higher orders.

2. A gauge condition on the potentials

$$\partial_\beta h^{\alpha\beta} = 0$$

The potentials are related to the metric by $\sqrt{-g} g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$.

The gauge condition is in a one-to-one correspondence with the conservation equation:

$$\partial_\beta h^{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \partial_\beta \tau^{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \nabla_\beta[g] T^{\alpha\beta}[g, z] = 0$$

This formulation is exact.

It is implemented by iterations.

Iterations

One iterates the wave equation only, leaving the world lines arbitrary until the very end when the gauge condition is imposed:

First iteration:

$$h = h_0 = 0$$

$$\tau_0 = T[\eta, z]$$

$$\Box h = \tau \Longrightarrow h = h_1[z] \Longrightarrow g = g_1[h_1]$$

Second iteration:

$$h = h_1[z]$$

$$\tau_1 = T[g_1, z] + (\partial h_1[z])^2 + \cdots$$

$$\Box h = \tau \Longrightarrow h = h_2[z] \Longrightarrow g = g_2[h_2]$$

Third iteration:

$$h = h_2[z]$$

$$\tau_2 = T[g_2, z] + (\partial h_2[z])^2 + \cdots$$

$$\Box h = \tau \Longrightarrow h = h_3[z] \Longrightarrow g = g_3[h_3]$$

and so on.

Finally, after n iterations, the motion is determined:

$$\partial h_n = 0 \Longrightarrow \partial \tau_{n-1} = 0 \Longrightarrow z(\tau)$$

There is no need to fix the motion until the very last step.

Second iteration

After two iterations we find that the gauge condition produces $a^a = \partial^a U$, the Newtonian equations of motion; the relaxation step is now justified.

And during the second iteration of the wave equation we find

$$\tau^{ab} = \sum_A m_A v_A^a v_A^b \delta(\boldsymbol{x} - \boldsymbol{z}_A) - 4\partial^a U \partial^b U + 2\delta^{ab}\partial_c U \partial^c U$$

Then

$$h^{ab} \simeq \frac{4}{r}\int \tau^{ab}(t-r, \boldsymbol{x'})\, dV' = \frac{2}{r}\ddot{Q}^{ab}(t-r)$$

This is the correct answer, and the field contribution to τ^{ab} is not small.

Post-linear perturbation theory

We formulate a high-order perturbation theory about a background metric $g_{\alpha\beta}$.

The Einstein field equations break down into two sets:

1. A wave equation for potentials $h^{\alpha\beta}$,

$$\Box h^{\alpha\beta} + 2R^{\alpha\ \beta}_{\ \gamma\ \delta} h^{\gamma\delta} = -16\pi\tau^{\alpha\beta}$$

   with $\tau = T[g+h, z] + \partial h \partial h + h\partial^2 h +$ higher orders.

2. A gauge condition on the potentials

$$\nabla_\beta[g] h^{\alpha\beta} = 0$$

The perturbed metric is $\hat{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2} h g_{\alpha\beta}$.

The gauge condition is in a one-to-one correspondence with the conservation equation:

$$\nabla_\beta[g] h^{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \nabla_\beta[g] \tau^{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \nabla_\beta[\hat{g}] T^{\alpha\beta}[\hat{g}, z] = 0$$

According to this last equation, the world line is a geodesic of the perturbed spacetime.

The motion is self-forced.

The equations are solved by iterations.

Iterations, formal scheme

Only the wave equation is iterated, for arbitrary $z(\tau)$:

First iteration:

$$h = h_0 = 0$$
$$\tau_0 = T[g, z]$$
$$\Box h = \tau \Longrightarrow h = h_1[z]$$

Second iteration:

$$h = h_1[z]$$
$$\tau_1 = T[g + h_1, z] + (\partial h_1[z])^2 + h_1[z]\partial^2 h_1[z] + \cdots$$
$$\Box h = \tau \Longrightarrow h = h_2[z]$$

After the last (second) iteration, the gauge condition/conservation equation is imposed:

$$\partial h_2 = 0 \Longrightarrow \partial \tau_1 = 0 \Longrightarrow z = \text{geod}\{g + h_1[z]\}$$

The motion is self-forced, and $h_2[z]$ gives the correct waveforms, consistent with the first-order self-force.

$h_1[z]$ and $z(\tau)$ are determined self-consistently — this is the relaxed first-order scheme.

Iterations, practical scheme

A practical scheme, equivalent to the above, is as follows.

First iteration:

$$h = h_0 = 0$$

$$\tau_0 = T[g, z]$$

$$\begin{cases} \Box h = \tau \Longrightarrow h = h_1[z] \\ z = \text{geod}\{g + h_1[z]\} \end{cases}$$

Second iteration:

$$h = h_1[z]$$

$$\tau_1 = T[g + h_1, z] + (\partial h_1[z])^2 + h_1[z]\partial^2 h_1[z] + \cdots$$

$$\Box h = \tau \Longrightarrow h = h_2[z]$$

$h_1[z]$ and $z(\tau)$ are determined self-consistently directly within the first iteration — this is again the relaxed formulation of the first-order problem.

There is then no need to impose the gauge condition/conservation equation after the second iteration.

The waveforms are still correctly obtained from $h_2[z]$.

$h_2[z]$ versus $h_1[z]$

$h_2[z]$ and $h_1[z]$ and sourced by the same motion $z(\tau)$, which is a solution to the self-consistent MiSaTaQuWa equation.

They differ essentially by the additional terms $(\partial h_1[z])^2 + h_1[z]\partial^2 h_1[z]$ in the source.

Relative to $T[z]$, these are smaller by a factor of order m/M.

They give rise to $O(m/M)$ corrections to the wave's amplitude and phase; these can be ignored for many purposes.

The second-order formulation provides a justification for the relaxed first-order problem, and it produces an error estimate.

Factor of 2?

In the post-Minkowski context the waveform was proportional to

$$\underbrace{v^2}_{\text{from } T[z]} + \underbrace{\frac{m_{\rm total}}{b}}_{\text{from } (\partial h)^2}$$

In the second-order perturbation theory context this becomes

$$\underbrace{v^2 + \frac{M}{b}}_{\text{from } T[z]} + \underbrace{\frac{m}{b}}_{\text{from } (\partial h)^2}$$

The factor of 2 has become a relative factor of order m/M.

[Thanks to Bob for saving me from major embarrassment.]

Conclusion

In the self-force problem, the first-order perturbation $h_1[z]$ and the world line $z = \text{geod}\{g + h_1[z]\}$ must be obtained self-consistently.

This is a statement of the MiSaTaQuWa equation.

This procedure is justified by the second-order formulation.

Two iterations of the wave equation return $h_2[z]$, which differs from $h_1[z]$ by unimportant terms of relative order m/M.

Question: Is the second-order self-force required to follow the inspiral over a radiation-reaction timescale? (Hinderer-Flanagan)

The arguments presented here were schematic and formal.

In his talk, Adam Pound will describe his progress at formulating the second-order problem in well-defined, concrete terms.

In particular, he will describe how to formulate the problem for a black hole instead of a point particle.