# Constructing the self-force

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# The problem

A point particle with [scalar charge, electric charge, mass] moves on a word line $\gamma$ in a curved spacetime.

The particle carries a [scalar, electromagnetic, gravitational] field.

Because of its interaction with the spacetime curvature, the field acts back on the particle and influences its motion; it produces a self-force.

How does one calculate the self-force?

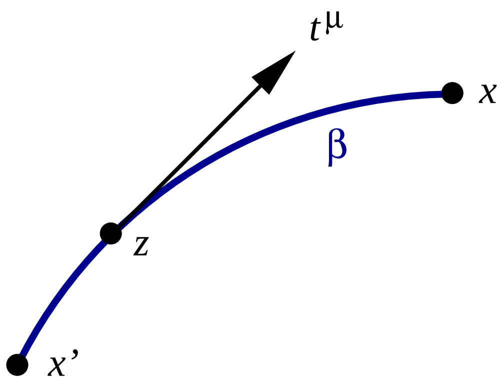
How does one deal with the **singular nature** of the field on the particle's word line?

In this lecture we will focus on the **scalar self-force**; computations for the electromagnetic and gravitational cases are similar.

# Geometrical elements

Let $x$ and $x'$ be two points in spacetime, sufficiently close that there is a unique geodesic segment $\beta$ linking them.

**Synge's world function** is




$$\sigma(x,x') = \frac{1}{2}(\text{squared geodesic distance})$$

$\sigma(x,x') < 0$ for timelike separation, and $\sigma(x,x') > 0$ for spacelike separation.

The vector $-\sigma^{\alpha'} := -\nabla^{\alpha'}\sigma(x,x')$ at $x'$ is tangent to $\beta$ and points away from $x'$.

Its squared length is equal to $2\sigma$ and it can be thought of as a separation vector between $x$ and $x'$.

The **parallel propagator** $g^{\alpha}{}_{\alpha'}(x,x')$ takes a vector at $x'$ and moves it to $x$ by parallel transport:

$$A^{\alpha}(x) = g^{\alpha}{}_{\alpha'}(x,x')A^{\alpha'}(x')$$

The operation can also be defined on dual vectors and tensors.

An arbitrary tensor $A^{\alpha\beta}$ at $x$ (or a bitensor defined at each point) can be expressed as a Taylor expansion in powers of $-\sigma^{\alpha'}$:

$$A^{\alpha\beta} = g^{\alpha}{}_{\alpha'}g^{\beta}{}_{\beta'}\Big[A^{\alpha'\beta'} - A^{\alpha'\beta'}{}_{\gamma'}\sigma^{\gamma'} + \frac{1}{2}A^{\alpha'\beta'}{}_{\gamma'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \cdots\Big]$$

The expansion coefficients are tensors at $x'$.

# Coordinate systems

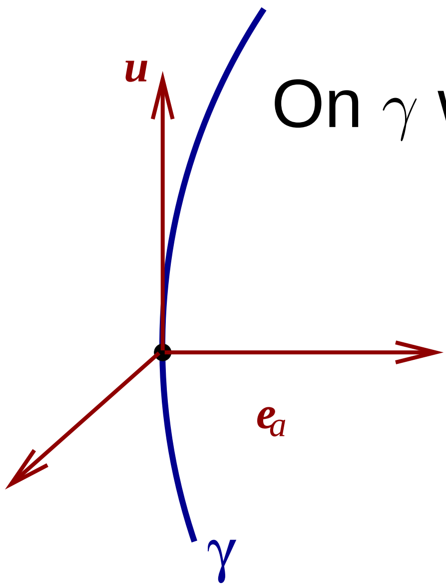
It is useful to construct coordinate systems in a neighbourhood of the particle's world line.

Calculations are best carried out using covariant methods, but the results are best presented in terms of coordinates.

This can be achieved by providing covariant definitions for the coordinates.

We consider three coordinates systems: Fermi, Thorne-Hartle-Zhang, and retarded.

Each coordinate system shares the following construction:




On $\gamma$ we erect a basis $(u^\mu, e_a^\mu)$ of orthonormal vectors.

These are transported on $\gamma$ so as to preserve their orthonormality property.

Relevant tensors are decomposed in this basis.

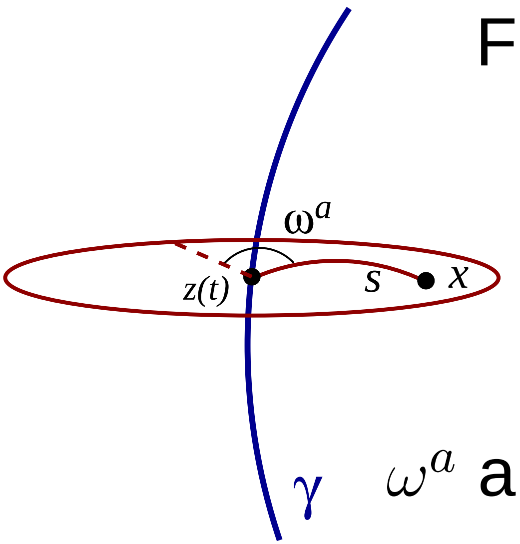
For example,

$$R_{0a0b}(\tau) := R_{\mu\lambda\nu\rho}\Big|_\gamma u^\mu e_a^\lambda u^\nu e_b^\rho$$

are frame components of the Riemann tensor.

# Fermi coordinates

The Fermi normal coordinates $(t, x^a = s\omega^a)$ are constructed as follows:




Find the unique spacelike geodesic that passes through $x$ and intersects $\gamma$ orthogonally; call the intersection point $z(t) \equiv \bar{x}$

$t$ is proper time at $\bar{x}$.
$s$ is proper distance from $\bar{x}$ to $x$
$\omega^a$ are direction cosines for the spacelike geodesic

More formally, the coordinates are defined by

$$x^a := -e^a_{\bar{\alpha}}(\bar{x})\sigma^{\bar{\alpha}}(x,\bar{x}), \qquad \sigma_{\bar{\alpha}}(x,\bar{x})u^{\bar{\alpha}}(\bar{x}) = 0, \qquad s^2 = 2\sigma(x,\bar{x})$$

Fermi normal coordinates are useful to define the **rest frame** of the moving particle.

# Thorne-Hartle-Zhang coordinates

The THZ coordinates $(t, y^a)$ are a variant of the Fermi normal coordinates, with

$$y^a = x^a - \frac{1}{6} s^2 \mathcal{E}^a_{\ b} x^b + \frac{1}{3} x^a \mathcal{E}_{bc} x^b x^c + O(s^4)$$

where $\mathcal{E}_{ab}(t) := C_{0a0b}(t) = R_{0a0b}(t)$.

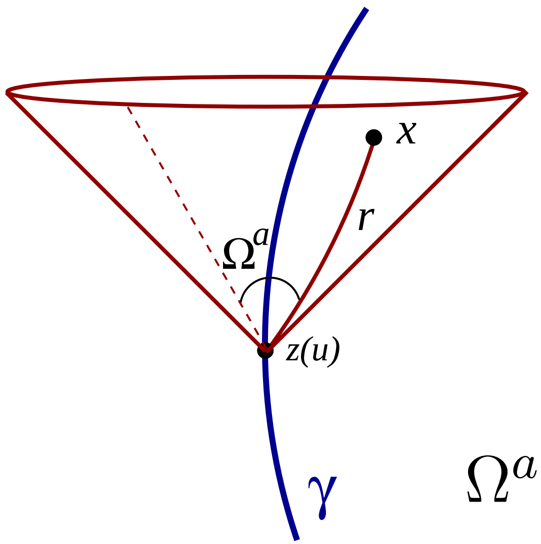
They are defined for Ricci-flat spacetimes only, and no covariant definition of $y^a$ is known.

The THZ coordinates are harmonic, in the sense that the metric satisfies $\partial_\beta(\sqrt{-g} g^{\alpha\beta}) = 0$.

The THZ coordinates enjoy the Florida winters.

# Retarded coordinates

The retarded coordinates $(u, x^a = r\Omega^a)$ are constructed as follows:




Find the unique null geodesic that passes through $x$ and intersects $\gamma$; call the intersection point $z(u) \equiv x'$

$u$ is proper time at $x'$
$r$ is affine-parameter distance from $x'$ to $x$
$\Omega^a$ are direction cosines for the null geodesic

More formally, the coordinates are defined by

$$x^a := -e^a_{\alpha'}(x')\sigma^{\alpha'}(x,x'), \qquad \sigma(x,x') = 0, \qquad r = \sigma_{\alpha'}(x,x')u^{\alpha'}(x')$$

Retarded coordinates give the simplest description of the [scalar, electromagnetic, gravitational] field near $\gamma$.

They naturally incorporate the **causal connection** between the field and the particle.

# Field equation and particle motion

A scalar charge $q$ moves on a world line $\gamma$ described by $x = z(\tau)$ and produces a scalar field $\Phi(x)$.

The scalar field obeys the wave equation

$$\Box\Phi(x) = -4\pi\mu(x) = -4\pi q \int_\gamma \delta_4(x, z)\, d\tau$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ and $\mu$ is the scalar-charge density.

The scalar charge moves according to

$$m(\tau)a^\mu = q(g^{\mu\nu} + u^\mu u^\nu)\Phi_\nu$$

where $\Phi_\nu \equiv \nabla_\nu\Phi$ is evaluated on $\gamma$, and $m(\tau)$ is the particle's dynamical mass.

# Green's function

The wave equation is solved by means of a Green's function $G(x,x')$ that satisfies

$$\Box G(x,x') = -4\pi\delta_4(x,x')$$

The solution is

$$\Phi(x) = \int G(x,x')\mu(x')\sqrt{-g'}d^4x' = q\int_\gamma G(x,z)\,d\tau$$

We want the **retarded solution** to the wave equation.

This is obtained by selecting the retarded Green's function $G_{\rm ret}(x,x')$.

It is defined so that $G_{\rm ret}(x,x')=0$ when the field-point $x$ is in the past of the source-point $x'$.

# Hadamard decomposition

When $x$ is close to $x'$, the retarded Green's function can be expressed in the Hadamard form

$$G_{\mathrm{ret}}(x,x') = U(x,x')\delta_{\mathrm{future}}(\sigma) + V(x,x')\Theta_{\mathrm{future}}(-\sigma)$$

Seen as functions of $x$, the delta-function has support on the future light cone of $x'$, and the step-function has support inside the future light cone.

The biscalars $U(x,x')$ and $V(x,x')$ are smooth when $x \to x'$.

There exists an algorithm to calculate them as expansions in powers of $\sigma^{\alpha'}$:

$$U(x,x') = 1 + \frac{1}{12}R_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} + \cdots, \qquad V(x,x') = \frac{1}{12}R(x') + \cdots$$

# Alternate Green's functions

The **advanced** Green's function

$$G_{\text{adv}}(x,x') = U(x,x')\delta_{\text{past}}(\sigma) + V(x,x')\Theta_{\text{past}}(-\sigma)$$

is defined to be zero when $x$ is in the future of $x'$.

The **Detweiler-Whiting singular** Green's function

$$G_{\text{S}}(x,x') = \frac{1}{2}U(x,x')\delta(\sigma) - \frac{1}{2}V(x,x')\Theta(\sigma)$$

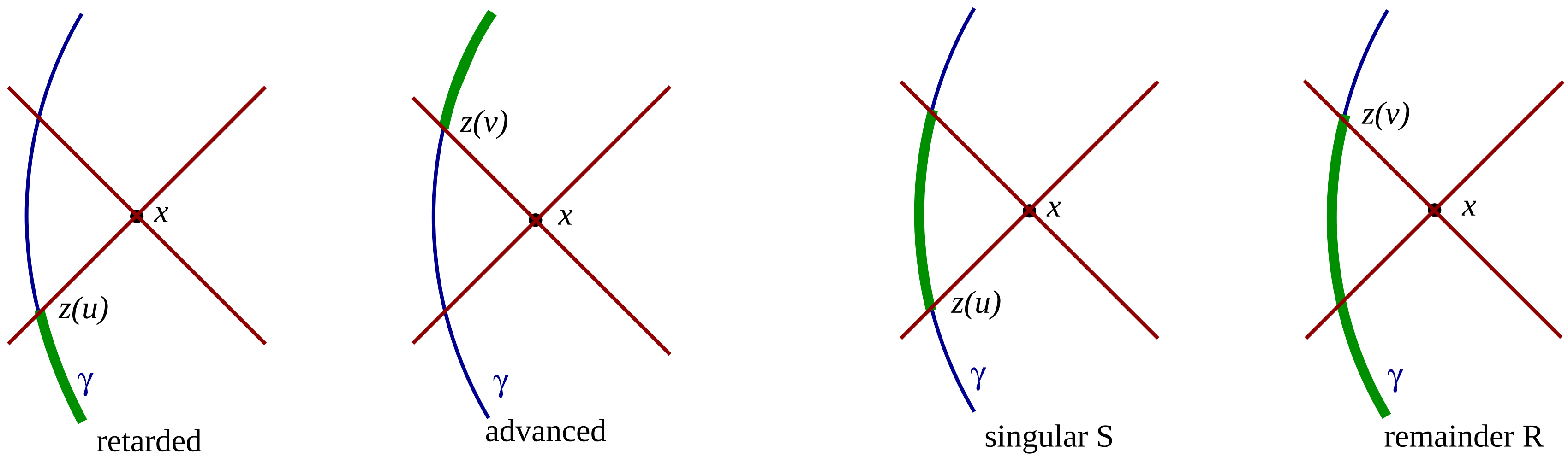
has support on and outside the (future and past) light cone.

The **regular** two-point function

$$G_{\text{R}}(x,x') = G_{\text{ret}}(x,x') - G_{\text{S}}(x,x')$$

satisfies the homogeneous wave equation: $\Box G_{\text{R}}(x,x') = 0$

# Solutions




The fields $\Phi_{\rm ret}(x)$, $\Phi_{\rm adv}(x)$, and $\Phi_{\rm S}(x)$ all satisfy the wave equation with source: $\Box\Phi = -4\pi\mu$.

They are all singular as $x$ approaches the world line.

The field $\Phi_{\rm R}(x)$ satisfies $\Box\Phi = 0$.

It is smooth on the world line.

# Coordinate expressions

In retarded coordinates $(u, x^a = r\Omega^a)$,

$$\begin{aligned}
\Phi_{\rm ret} &= \frac{q}{r} + q\int_{-\infty}^{u} V(x,z)\, d\tau + O(r^2) \\
\Phi_{\rm S} &= \frac{q}{r} - \frac{1}{3} q\dot{a}_a x^a + O(r^2) \\
\Phi_{\rm R} &= \frac{1}{3} q\dot{a}_a x^a + q\int_{-\infty}^{u} V(x,z)\, d\tau + O(r^2)
\end{aligned}$$

These equations are valid in Ricci-flat spacetimes.

In Fermi coordinates $(t, s\omega^a)$,

$$
\begin{aligned}
\Phi_{\rm ret} &= \frac{q}{s} - \frac{1}{2} q a_a \omega^a + qs\Big(\frac{1}{8}\dot{a}_0 + \frac{1}{3}\dot{a}_a\omega^a - \frac{1}{6}R_{0a0b}\omega^a\omega^b\Big) \\
&\quad + q\int_{-\infty}^{t} V(x,z)\, d\tau + O(s^2) \\
\Phi_{\rm S} &= \frac{q}{s} - \frac{1}{2} q a_a \omega^a + qs\Big(\frac{1}{8}\dot{a}_0 - \frac{1}{6}R_{0a0b}\omega^a\omega^b\Big) + O(s^2) \\
\Phi_{\rm R} &= \frac{1}{3} q\dot{a}_a x^a + q\int_{-\infty}^{t} V(x,z)\, d\tau + O(s^2)
\end{aligned}
$$

These equations are valid in Ricci-flat spacetimes.

# Field gradient

The preceding results imply

$$
\begin{aligned}
\nabla_a \Phi_{\rm S} &= -\frac{q}{s^2}\omega_a - \frac{q}{2s}(\delta_a^b - \omega^b\omega_a)a_b + \frac{q}{8}\dot{a}_0\,\omega_a \\
&\quad + \frac{q}{6}R_{0b0c}\,\omega_a\omega^b\omega^c - \frac{q}{3}R_{0a0b}\,\omega^b + O(s) \\
\langle \nabla_a \Phi_{\rm S} \rangle &= -\left(\frac{q}{3s}\right)a_a + O(s) \\
\nabla_a \Phi_{\rm R} &= \frac{1}{3}q\dot{a}_a + q\int_{-\infty}^{t} \nabla_a V(x,z)\,d\tau + O(s)
\end{aligned}
$$

In the second expression, the gradient of $\Phi_{\rm S}$ is averaged over a sphere $s =$ constant around the particle.

The gradient of the $\Phi_{\rm R}$ is smooth at $s = 0$; no averaging is necessary.

# Self-force

The gradient of $q\Phi_{\rm S}$, after spherical averaging, is equal to $-(\delta m)a^a$ (with $\delta m = q^2/3s$).

**The singular field produces no force on the particle;** it contributes to the particle's inertia.

Removing $\Phi_{\rm S}$ from $\Phi_{\rm ret}$ returns $\Phi_{\rm R}$, and the equations of motion are

$$(m + \delta m)a^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi_{\rm R}$$

After mass renormalization, this becomes

$$ma^\alpha = q^2(g^{\alpha\beta} + u^\alpha u^\beta)\Big(\frac{1}{3}\dot{a}_\beta + \int_{-\infty}^{\tau} \nabla_\beta V(x,z)\, d\tau'\Big)$$

This is Quinn's equation of motion for a scalar charge.

# Derivations of the self-force

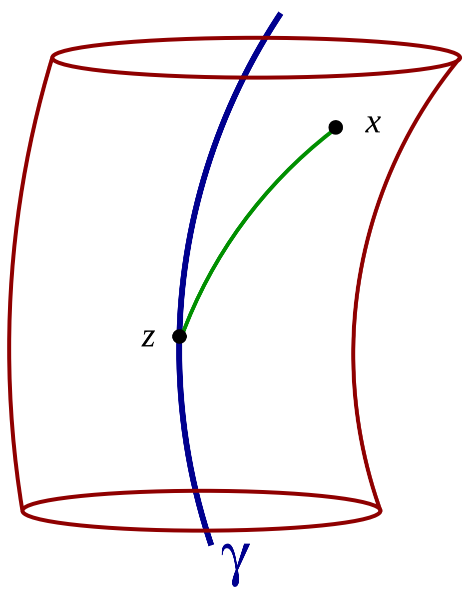
I next turn to a review of the various approaches that have been adopted to derive the equations of motion of a point particle.

The discussion covers the scalar, electromagnetic, and gravitational self-forces.

# DeWitt & Brehme

The approach of DeWitt & Brehme is based on energy-momentum conservation across a world tube surrounding the particle.

The conservation identity $\nabla_\beta T^{\alpha\beta} = 0$ cannot be turned into an integral statement. A trick must be used:




$$
\begin{aligned}
0 &= g^{\mu}{}_{\alpha}(z,x) \nabla_\beta T^{\alpha\beta}(x) \\
&= \nabla_\beta (g^{\mu}{}_{\alpha} T^{\alpha\beta}) - T^{\alpha\beta} \nabla_\beta g^{\mu}{}_{\alpha}
\end{aligned}
$$

Integration yields

$$0 = \int_{\text{wall}} g^{\mu}{}_{\alpha} T^{\alpha\beta}\, d\Sigma_\beta + \int_{\text{caps}} g^{\mu}{}_{\alpha} T^{\alpha\beta}\, d\Sigma_\beta - \int_{\text{interior}} T^{\alpha\beta} \nabla_\beta g^{\mu}{}_{\alpha}\, dV$$

The wall integration is well defined.

The caps integration diverges as $1/s$ and must be regularized.

The volume integration diverges as $\ln(s)$ and should also be regularized; a careful analysis of this term has never been produced.

DeWitt & Brehme simply discard this term, and they obtain the standard answer for the self-force.

# Quinn & Wald

Quinn & Wald adopt an axiomatic approach.

**Axiom 1:** Two particles move on world lines $\gamma$ and $\tilde{\gamma}$ in two different spacetimes. At points $z$ and $\tilde{z}$ their acceleration vectors have equal lengths. The neighbourhoods of $z$ and $\tilde{z}$, as well as the acceleration vectors, are identified in Fermi coordinates. Then

$$F^\alpha - \tilde{F}^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta) \lim_{s\to 0} \left\langle \nabla_\beta \Phi - \tilde{\nabla}_\beta \tilde{\Phi} \right\rangle$$

The limit is well defined, after the difference of field gradients is averaged over a sphere of radius $s$.

**Axiom 2:** $\tilde{F}^\alpha = 0$ in flat spacetime, for a particle with uniform acceleration.

The final result for $F^\alpha$ is the standard answer.

# Averaging method

In the derivation sketched in this lecture, we postulated that the **averaged retarded field** acts on the particle:

$$\begin{aligned} ma^\alpha &= q(g^{\alpha\beta} + u^\alpha u^\beta)\langle \nabla_\beta \Phi \rangle \\ &= q(g^{\alpha\beta} + u^\alpha u^\beta)\Big( \langle \nabla_\beta \Phi_{\rm S} \rangle + \nabla_\beta \Phi_{\rm R} \Big) \end{aligned}$$

This is equivalent to Axiom 1.

After computation of $\Phi_{\rm S}$ we obtained

$$(m + \delta m) a^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi_{\rm R}$$

This becomes the standard answer after mass renormalization.

Axiom 2 bypasses the need to renormalize the mass.

# Detweiler & Whiting

The approach of Detweiler & Whiting is based on an observation and a different choice of axiom.

**Observation:** The retarded field $\Phi$ can be decomposed uniquely into a singular piece $\Phi_{\rm S}$ and a regular remainder $\Phi_{\rm R}$.

**Axiom:** The singular field produces no force on the particle.

This approach again returns the standard answer,

$$ma^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi_{\rm R}$$

# Matched asymptotic expansions

All previous derivations rely on one or two axioms.

In the case of the gravitational self-force, one can do better by examining the motion of a small black hole in an external spacetime.

The metric of the black hole, tidally perturbed by the external universe, is matched to the metric of the external spacetime, perturbed by the black hole.

The motion of the black hole is determined by the requirement that the combined metric must be a valid solution to the Einstein field equations.

**No singularities! No axioms!**

This approach once more returns the standard answer for the gravitational self-force.