Proposal for a Self-Consistent Wave-Generation Formalism

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Outline

- Review of the EFE for a point-particle
- An iterative approach to solving the EFE over long timescales
- Using Green's identity and matched asymptotic expansion to generate solutions at each iteration
- Work in progress: proving that the first-order solution is that of a point-particle

Review part 1: solution of the EFE with a point-particle source

- A point particle with mass m travels on a worldline γ in a background spacetime with a local radius of curvature $\mathcal{R}$. It produces a metric perturbation $h \sim \varepsilon$, where $\varepsilon = m/\inf\{\mathcal{R}\}$.

- The EFE is expanded in powers of h:

$$G^{(1)}[h] + G^{(2)}[h] + \ldots = 8\pi T^{(1)}[\gamma] + 8\pi T^{(2)}[h,\gamma] + \ldots$$

- Solve the 1st-order EFE $G^{(1)}[h] = 8\pi T^{(1)}[\gamma]$

  → The Bianchi identity $G^{(1)\alpha\beta}{}_{;\beta} = 0$ implies that γ is a geodesic for all times

  … but h carries off energy and angular momentum.

- The reason for the inconsistency: the worldline γ would obviously depend on m in an exact solution, and $T^{(n)}$ is a functional of γ.

  → We cannot equate $G^{(n)}$ with $T^{(n)}$.

Review part 2: a consistent solution valid only on short timescales

- To equate $G^{(1)}$ with $T^{(1)}$, we must expand γ as $\gamma = \gamma_0 + \varepsilon\gamma_1 + \ldots$

  → The true 1st-order EFE is $G^{(1)}[h] = 8\pi T^{(1)}[\gamma_0]$

  → γ_0 is a geodesic, but the 2nd-order EFE would determine a correction γ_1.

- There is no inconsistency here, but the expansion of γ is valid only on short timescales $t \sim \mathcal{R}$.

  → A strict power series expansion of the EFE is valid only on short timescales.

- So although the 1st-order EFE $G^{(1)}[h] = 8\pi T^{(1)}[\gamma]$ implies γ is a geodesic, the equation itself is valid only on the short timescales for which $\gamma \cong \gamma_0$.

- We need a way of correcting the motion on long timescales.

  → Rather than expanding the worldline in powers of ε, we need to expand the acceleration.

Review part 3: incorporating the self-force

- Solve the 1st-order EFE $G^{(1)}[h] = 8\pi T^{(1)}[\gamma]$
- Substitute the regular part of the solution into the "full" Bianchi identity
  $G^{\alpha\beta}{}_{;\beta} = 0 \quad \Rightarrow \quad {}^{(g+h)}\nabla_\beta T[g+h,\gamma]^{\alpha\beta} = 0$ ← consistency condition for 2nd-order perturbation theory
  $\Rightarrow$ MiSaTaQuWa force
- But how do we implement this long-term correction to the motion?

  - Option 1: use a "mixed" EFE $G^{(1)}[h] = 8\pi T^{(1)}[\gamma_{\text{corrected}}]$

    → The metric perturbation is not actually a solution to the EFE.

  - Option 2: use a method of variation of constants to make a smooth transition from geodesic to geodesic, without approximation.

    → This would involve major difficulties with the tail term.

  - Option 3: perform a two-timescale expansion of the EFE; an analogue of the "mixed" EFE should appear from the leading-order terms.

    → This presumably breaks the covariance of the formalism.

Our option: an n^{th}-order solution valid on long timescales

- We extend the *entire* solution, not just the motion, to higher order.
- We do not solve the EFE order-by-order, but write it as

$$G^{(1)}[h]=8\pi T^{(1)}[\gamma]+8\pi T^{(2)}[h,\gamma]-G^{(2)}[h]-G^{(3)}[h]-\ldots$$

higher-order Einstein tensor terms are treated as sources

- We decompose this equation into two sub-problems:
  (1) The inhomogenous wave equation: $\nabla_\mu\nabla^\mu\bar{h}^{\alpha\beta}+2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}^{\mu\nu}=sources$
  (2) The Lorenz gauge condition: $\bar{h}^{\alpha\beta}{}_{;\beta}=0$

- We solve the wave equation iteratively:

follows post-Minkowski method

(0^{th} order) $h^{(0)}=0$

$$\Downarrow$$

(1^{st} order) $G^{(1)}[h^{(1)}]=8\pi T^{(1)}[\gamma] \;\Rightarrow\; h^{(1)}[\gamma]\sim m$

$$\Downarrow$$

(2^{nd} order) $G^{(1)}[h^{(2)}]=8\pi T^{(1)}[\gamma]+8\pi T^{(2)}[h^{(1)},\gamma]-G^{(2)}[h^{(1)}] \;\Rightarrow\; h^{(2)}[\gamma]\sim m\,\&\,m^2$

$$\vdots$$

- At each order, the solution to the wave equation is expressed as a functional of an undetermined worldline:

$$(1^{\text{st}})\quad \bar{h}^{(1)}[\gamma]^{\alpha\beta} = 8\pi\int G^{\alpha\beta}{}_{\mu'\nu'}T^{(1)\mu'\nu'}[\gamma]dV'$$

$$(2^{\text{nd}})\quad \bar{h}^{(2)}[\gamma]^{\alpha\beta} = 8\pi\int G^{\alpha\beta}{}_{\mu'\nu'}\left(T^{(1)}[\gamma]^{\mu'\nu'} + T^{(2)}[h^{(1)},\gamma]^{\mu'\nu'} - \frac{1}{8\pi}G^{(2)}[h^{(1)}]^{\mu'\nu'}\right)dV'$$

- Imposing the Lorenz gauge condition at n^{th} order determines the motion and guarantees that $h^{(n)}$ solves the n^{th}-order EFE:

$$(1^{\text{st}})\quad \bar{h}^{(1)\alpha\beta}{}_{;\beta} = 8\pi\int G^{\alpha}{}_{\mu'}T^{(1)}[\gamma]^{\mu'\nu'}{}_{;\nu'}dV' \quad\Rightarrow\quad a^{\alpha} = 0$$

true *only* if Lorenz gauge is imposed at first order

$$(2^{\text{nd}})\quad \bar{h}^{(2)\alpha\beta}{}_{;\beta} = 8\pi\int G^{\alpha}{}_{\mu'}\left(T^{(1)}[\gamma]^{\mu'\nu'}{}_{;\nu'} + T^{(2)}[h^{(1)},\gamma]^{\mu'\nu'}{}_{;\nu'} - \frac{1}{8\pi}G^{(2)}[h^{(1)}]^{\mu'\nu'}{}_{;\nu'}\right)dV'$$

$$\Rightarrow\quad a^{\alpha} = -\tfrac{1}{2}\left(g^{\alpha\beta} - u^{\alpha}u^{\beta}\right)\left(2h^{(1)}_{\beta\gamma;\delta} - h^{(1)}_{\delta\gamma;\beta}\right)u^{\gamma}u^{\delta}$$

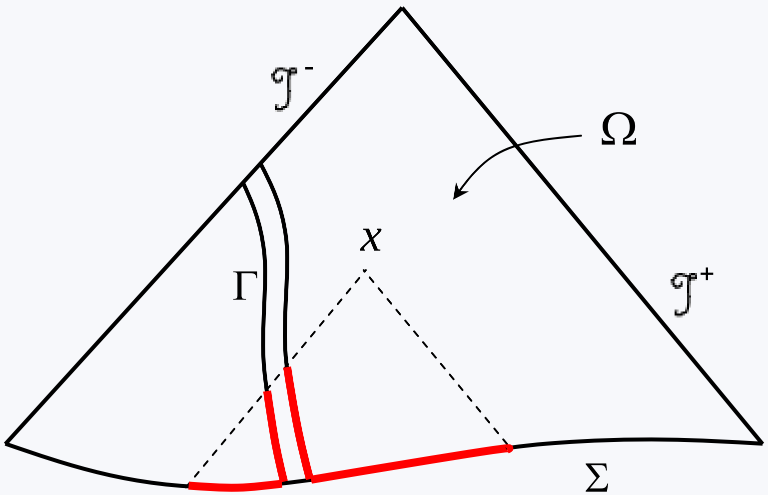
the “correct” expression

- But *the Lorenz gauge is imposed* only *at n^{th} order*

  → $h^{(n\text{-}1)}$ is a solution to the lower-order wave equation, but it is *not* required to be a solution to the lower-order EFE.

- The above prescription cannot be implemented in practice because the point-particle solution is singular at the position of the particle.

- How do we construct a regular solution?



D'Eath's Method

Surround the particle with a small world-tube Γ and seek a solution to the EFE in a vacuum region Ω outside this tube.

→ Singular source terms on a worldline are replaced by regular boundary data on a worldtube.

- The surfaces $\mathcal{I}^{\pm}$ are null, so they do not influence the field in Ω.

  → In a black hole spacetime, one of them can be pushed in to the event horizon, the other out to $\mathscr{I}^+$.

- The initial surface Σ can be pushed to past ∞ for simplicity, or it can be carried throughout the calculation to create an initial value formulation.

Structure of solutions

- The first-order wave equation is $G^{(1)}[h^{(1)}]=0$ in Ω

  → The first-order solution depends only on boundary data.

$$h^{(1)}[\Gamma]^{\alpha\beta}(x)=\int_{\Gamma\cup\Sigma}\left(G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\nabla^{\mu'}h^{(1)\gamma'\delta'}(x')-h^{(1)\gamma'\delta'}(x')\nabla^{\mu'}G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\right)dS_{\mu'}$$

- The second-order wave equation is $G^{(1)}[h^{(2)}]=-G^{(2)}[h^{(1)}]$ in Ω

  → The second-order solution is sourced by the first-order solution.

$$h^{(2)}[\Gamma]^{\alpha\beta}(x)=\int_{\Gamma\cup\Sigma}\left(G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\nabla^{\mu'}h^{(2)\gamma'\delta'}(x')-h^{(2)\gamma'\delta'}(x')\nabla^{\mu'}G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\right)dS_{\mu'}$$
$$-\int_{\Omega}G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')G^{(2)\gamma'\delta'}[h^{(1)}(x')]dV'$$

homogeneous solution

non-homogeneous solution

- All that is required is consistent boundary values.

Determining boundary values

- Assume the “particle” is a Schwarzschild black hole. Near the black hole $r \sim m \sim \varepsilon\mathcal{R}$, so the expansion around a background breaks down.

  → We use *matched asymptotic expansion.*

- We situate the tube Γ such that

  - inside Γ, $g_{\mathrm{I}} = g_{\mathrm{Sch}} + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots$ (perturbed Schwarzschild)
  - outside Γ, $g_{\mathrm{E}} = g_{\mathrm{BG}} + h$ (perturbed background)
  - on Γ, both approximations are valid. (buffer region)

- On the scale of the small black hole, the tube is a boundary “near” asymptotic infinity.
- On the scale of the external spacetime, the tube has vanishingly small diameter.

More specifically…

- As seen from its interior, Γ is a surface $r_I = \tilde{r}$ in the limit $m/r_\mathrm{I} \to 0$.

$$\Rightarrow g_I\big|_\Gamma \sim \underbrace{\eta \,\&\, \frac{m}{\tilde{r}} \,\&\, \frac{m^2}{\tilde{r}^2}}_{\lim\limits_{m/r\to 0} g_{Sch}} \,\&\, \underbrace{\frac{\tilde{r}}{\mathcal{R}} \,\&\, \frac{m}{\mathcal{R}} \,\&\, \frac{m^2}{\tilde{r}\mathcal{R}}}_{\lim\limits_{m/r\to 0} \varepsilon H_1} \,\&\, \underbrace{\frac{\tilde{r}^2}{\mathcal{R}^2} \,\&\, \frac{m\tilde{r}}{\mathcal{R}^2} \,\&\, \frac{m^2}{\mathcal{R}^2}}_{\lim\limits_{m/r\to 0} \varepsilon^2 H_2} \,\&\, \ldots$$

- As seen from its exterior, Γ is a surface $r_E = \tilde{r}$ in the limit $r_\mathrm{E}/\mathcal{R} \to 0$.

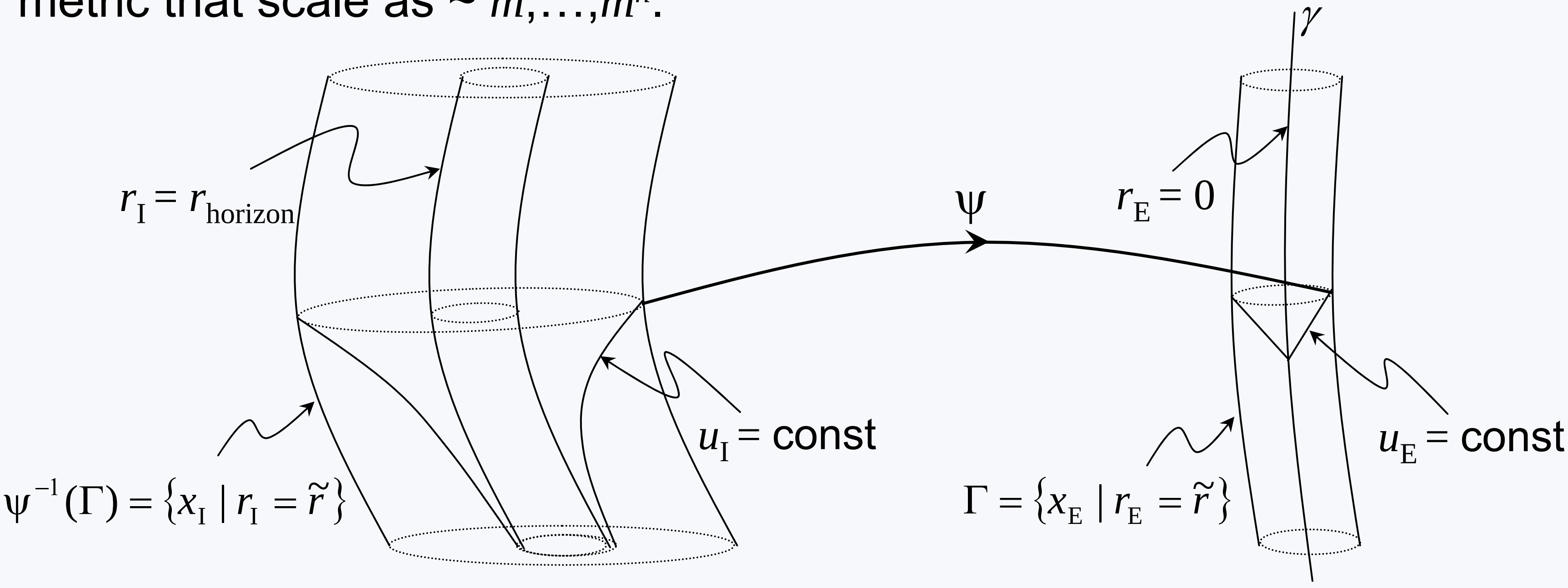
$$\Rightarrow g_E\big|_\Gamma \sim \underbrace{\eta \,\&\, \frac{\tilde{r}}{\mathcal{R}} \,\&\, \frac{\tilde{r}^2}{\mathcal{R}^2}}_{\lim\limits_{r/\mathcal{R}\to 0} g_{BG}} \,\&\, \underbrace{\frac{m}{\tilde{r}} \,\&\, \frac{m}{\mathcal{R}} \,\&\, \frac{m\tilde{r}}{\mathcal{R}^2}}_{\lim\limits_{r/\mathcal{R}\to 0} h^{(1)}} \,\&\, \underbrace{\frac{m^2}{\tilde{r}^2} \,\&\, \frac{m^2}{\tilde{r}\mathcal{R}} \,\&\, \frac{m^2}{\mathcal{R}^2}}_{\lim\limits_{r/\mathcal{R}\to 0} (h^{(2)} - h^{(1)})} \,\&\, \ldots$$

- Γ is situated such that $\tilde{r}/\mathcal{R} = \varepsilon^p$ and $m/\tilde{r} = \varepsilon^{1-p}$. The relative scaling of the two is chosen such that each term in the solution is much larger than the neglected terms.

- The boundary values for $h^{(n)}$ are found by matching them to the parts of the internal solution that scale as ~ $m, \ldots, m^n$.

More geometrically…

- The approximations g_I and g_E are constructed on manifolds $\mathcal{M}_\mathrm{I}$ and $\mathcal{M}_\mathrm{E}$, corresponding to Schwarzschild and the external background.
- The worldline γ exists in $\mathcal{M}_\mathrm{E}$, the small black hole in $\mathcal{M}_\mathrm{I}$.
- Points in the buffer regions of $\mathcal{M}_\mathrm{I}$ and $\mathcal{M}_\mathrm{E}$ are related by a map $\psi:\mathcal{M}_\mathrm{I} \to \mathcal{M}_\mathrm{E}$. Matching determines ψ as well as the internal and external solutions.
- The boundary values for $h^{(n)}$ are pushed-forward from parts of the internal metric that scale as ~ $m,\ldots,m^n$.



Coordinate systems

Internal coordinates (defined outside the event horizon in $\mathcal{M}_I$):

- The radial coordinate r_I is an affine parameter on outgoing light rays.
- The retarded time $u_I \sim t\text{-}r^*$ is constant on each outgoing light cone.

External coordinates (defined in the normal neighborhood of γ):

- The radial coordinate r_E is the background luminosity distance from a worldline γ.
- The retarded time $u_I \sim t\text{-}r$ is constant on each outgoing light cone and is equal to proper time (as defined in the external background) on γ.

Coordinate transformation:

- The map ψ is partially fixed by, e.g., relating internal and external light cones: $u_E = u_I + 2m\ln(r_I/2m\text{-}1)$ + corrections
- It is completely fixed by specifying internal and external gauges.

First-order matching

- D'Eath sketched the following proof that, at first order, the external solution is that of a point-particle:

(1) In retarded coordinates, the first-order solution can be written as

$$h^{(1)\alpha\beta}(x) \sim -\int_{\Gamma\cup\Sigma}\left(G\cdot\nabla^{\mu'}h^{(1)} - h^{(1)}\cdot\nabla^{\mu'}G\right)\tilde{r}^2\partial_{\mu'}r'\,du'\,d\Omega'$$

(2) As $m/r \to 0$, the leading Schwarzschild term gives $h^{(1)\gamma'\delta'} \sim \frac{m}{\tilde{r}}u^{\gamma'}u^{\delta'}$

(3) As the tube shrinks, the derivative of the perturbation dominates:

$$\nabla^{\mu'}h^{(1)\gamma'\delta'}(x') \sim \frac{m}{\tilde{r}^2}u^{\gamma'}u^{\delta'}\partial^{\mu'}r' >> h^{(1)\gamma'\delta'}$$

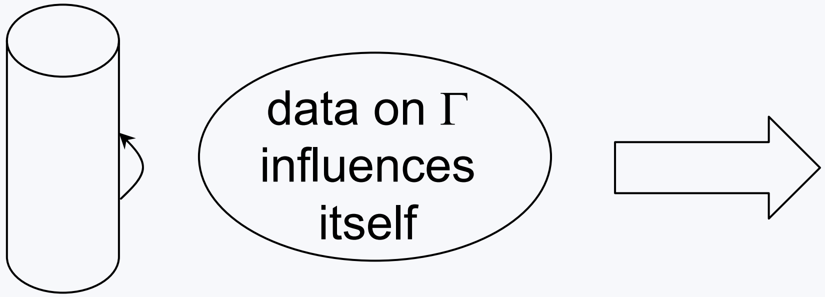
(4) The tube shrinks with ε, so the solution is necessarily that of a point-particle:

$$h^{(1)\alpha\beta}(x) \sim \int_{\Gamma\cup\Sigma} G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\frac{m}{\tilde{r}^2}u^{\gamma'}u^{\delta'}\tilde{r}^2\,du'\,d\Omega'$$

$$\sim m\int_{\gamma} G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\,u^{\gamma'}u^{\delta'}\,du'$$

- Matching calculations that justify the MiSaTaQuWa equation have typically assumed that the external solution is that of a point-particle.
- However, D'Eath's proof has a few missing details:

(1) Γ is a timelike surface, so data on it is not freely specifiable:



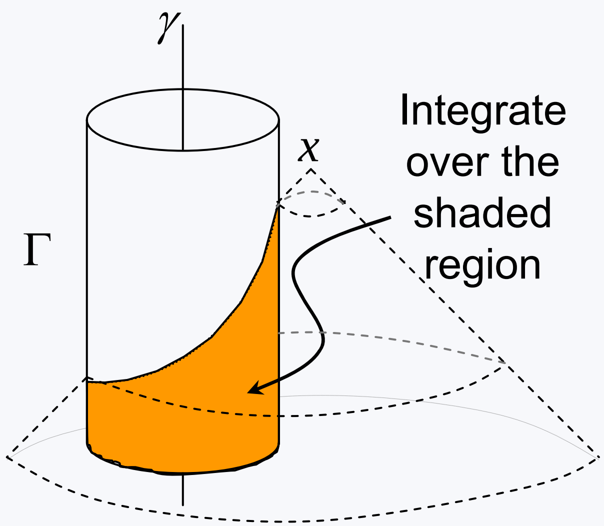
data on Γ must be a restriction of the external solution:

$$\lim_{x \to x' \in \Gamma} h(x) = h(x')$$

(2) The derivative of the Green's function is highly singular at $x=x'$, so it is not obvious that we can discard the term $h^{(1)\gamma'\delta'}(x')\nabla^{\mu'}G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')$

(3) It is not obvious that the angular integration becomes trivial as $r \to 0$

- To fill in these details, we perform an explicit calculation of the boundary integral



- We divide the integral into two parts: one over the normal neighborhood $\mathcal{N}(x)$,

$$h_N^{(1)\alpha\beta} = -\int_{\mathcal{N}(x)\cap\Gamma}\left[G^{\alpha\beta}{}_{\gamma'\delta'}\nabla^{\mu'}h^{(1)\gamma'\delta'} - h^{(1)\gamma'\delta'}\nabla^{\mu'}G^{\alpha\beta}{}_{\gamma'\delta'}\right]R^2\partial_{\mu'}r'\,du'\,d\Omega'$$

and one over its complement $\mathcal{N}^C(x)$,

$$h_{NC}^{(1)\alpha\beta} = -\int_{\mathcal{N}^C(x)\cap\Gamma}G^{\alpha\beta}{}_{\gamma'\delta'}\nabla^{\mu'}h^{(1)\gamma'\delta'}R^2\partial_{\mu'}r'\,du'\,d\Omega'$$

- Outside the normal neighbourhood, the term containing the derivative of the Green's function has been safely dropped, and tensors on Γ can be parallel-transported to points on γ.

$$\rightarrow \quad h_{NC}^{(1)\alpha\beta} = 4m\int_{\mathcal{N}^C(x)\cap\gamma}G^{\alpha\beta}{}_{\gamma'\delta'}u^{\gamma'}u^{\delta'}du' + O\left(\frac{mr}{\mathcal{R}^2}\right)$$

the portion of the tail integral outside $\mathcal{N}(x)$

- Inside the normal neighborhood, we can use the Hadamard decomposition of the Green's function:

$$G^{\alpha\beta}{}_{\gamma'\delta'} = U^{\alpha\beta}{}_{\gamma'\delta'}\delta(\sigma(x,x')) + V^{\alpha\beta}{}_{\gamma'\delta'}\theta(-\sigma(x,x'))$$

where $\sigma(x,x')$ is half the squared geodesic distance between x and x'.

- $V^{\alpha\beta}{}_{\delta'\gamma'}$ contributes the remainder of the usual tail integral:

$$4m \int_{\mathcal{N}(x)\cap\gamma} V^{\alpha\beta}{}_{\gamma'\delta'}\theta(u-u')u^{\gamma'}u^{\delta'}du' + O\left(\frac{mr}{\mathcal{R}^2}\right)$$

- $U^{\alpha\beta}{}_{\delta'\gamma'}$ contributes when x and x' are connected by a null geodesic

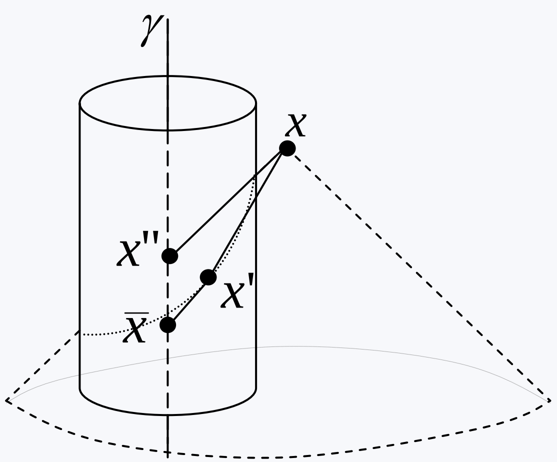
  → We need an expansion of $\sigma(x,x')$ in terms of known quantities.

- Since our coordinate system is based on null geodesics emanating from γ, we connect x and x' to γ with such geodesics.

  → Tensors at x' can be expanded around $\bar{x}$ and then around x'':

$$\sigma(x,x') = \sigma(x,\bar{x}) - \sigma_{\bar{\alpha}}(x,\bar{x})\sigma^{\bar{\alpha}}(x',\bar{x}) + \ldots$$

$$\sigma(x,\bar{x}) = \underbrace{\sigma(x,x'')}_{0} - \underbrace{\sigma_{\alpha''}(x,x'')u^{\alpha''}}_{r}(u''-\bar{u}) + \ldots$$



- We find that the integration over the $U^{\alpha\beta}{}_{\delta'\gamma'}$ terms contributes the following to $h^{(1)}$:

$$\frac{4m}{r}u^{\alpha}u^{\beta}+mf^{\alpha\beta}(a)$$

where $f(a)$ is linear in the acceleration a.

- Combining this with the $V^{\alpha\beta}{}_{\delta'\gamma'}$ terms, we find

$$\lim_{x\to x'\in\Gamma} h^{(1)\alpha\beta}(x)=\frac{4m}{\tilde{r}}u^{\alpha}u^{\beta}+h^{\alpha\beta}_{\text{tail}}+mf^{\alpha\beta}(a)$$

- Therefore, using the leading part of the internal Schwarzschild metric as boundary data does not provide a valid $O(\varepsilon)$ external solution.
- Also, part of $f(a)$ arises from the term $h^{(1)\gamma'\delta'}(x')\nabla^{\mu'}G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')$ in the boundary integral.

  → that term cannot be dropped, even in the limit $r\to 0$.

- We are currently analyzing the internal perturbation εH_1, and its push-forward via ψ, to show that the acceleration must be $O(m/\mathcal{R}^2)$ and the leading parts of the tail term are accounted for by $\psi_*(\varepsilon H_1)$.

Conclusions

- A wave-generation formalism that assumes the first-order solution is sourced by a geodesic will be valid only on very short timescales.
- The self-force can be incorporated into a wave-generation formalism in multiple ways.
- Our method iteratively solve the EFE to attain a solution that is valid at second order and over long timescales.
- We follow D’Eath’s approach, which uses Green’s identity and matched asymptotic expansion to generate solutions to the wave equation.

Work to be done

- Complete the first-order calculation.
- Proceed to second order.
- Evaluate the solution at null infinity to retrieve waveforms.