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Gravitational self-force on a point particle around a Schwarzschild black hole

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Extreme Mass Ratio Inspiral (EMRI)

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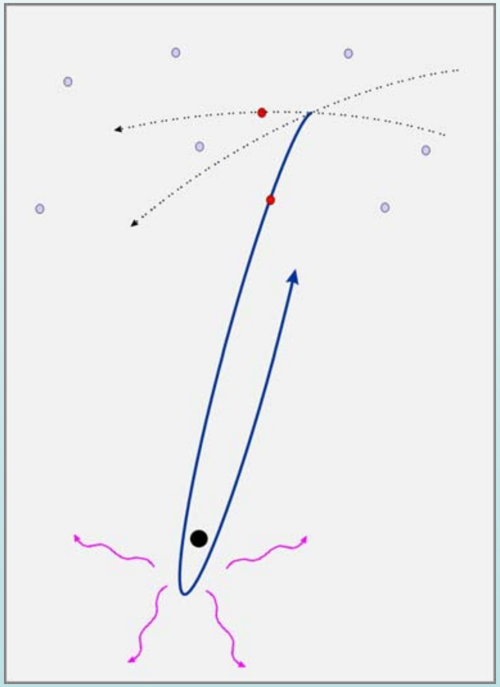
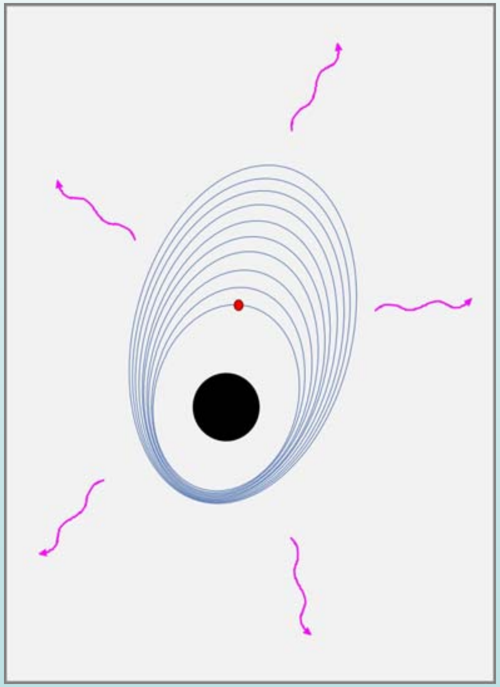
Compact object captured by a supermassive BH in the galactic centre

Mass ratio: $\mu / M \approx 10^{-5} - 10^{-7}$

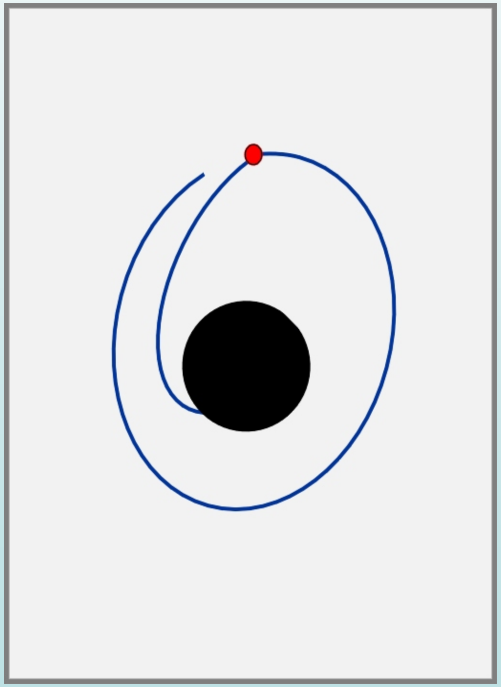
10-1000 detectable events / (1-yr LISA obs.) are expected.



Captured by central BH due to two body relaxation →



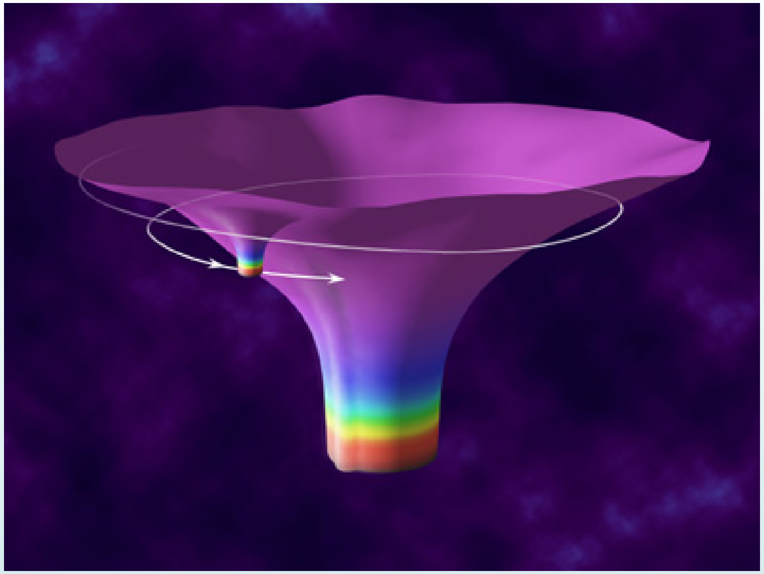
Inspiral, radiating GW →



Plunge to BH

Black Hole Perturbation

We deal with EMRI system by using black hole perturbation regime.



Background geometry

+

Perturbation induced by a particle

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots$$

<u>Linearised Einstein equation</u>

$$\delta G^{(1)}_{\mu\nu}[h^{(1)}_{\mu\nu}] = 8\pi G T^{(1)}_{\mu\nu}$$ → Energy-momentum tensor for geodesic

$$\delta G^{(1)}_{\mu\nu}[h^{(2)}_{\mu\nu}] = 8\pi G T^{(2)}_{\mu\nu} + \delta G^{(2)}_{\mu\nu}[h^{(1)}_{\mu\nu}, h^{(1)}_{\mu\nu}]$$

$$\vdots$$

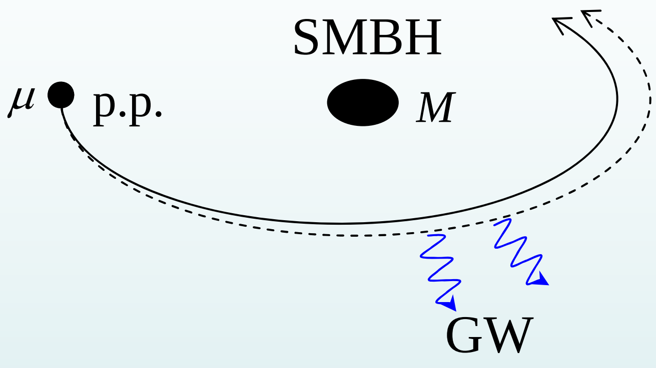
$\delta G^{(2)}_{\mu\nu}[h^{(1)}_{\mu\nu}, h^{(1)}_{\mu\nu}]$ → Non-linear effect

$8\pi G T^{(2)}_{\mu\nu}$ → Self-force effect

Self Force Effect

- Test particle case

$$\mu\left(\frac{d^2 z^\mu}{d\tau^2}+\Gamma^{(0)\,\mu}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\right)=0$$



- Taking account of the self field

$$\mu\left(\frac{d^2 z^\mu}{d\tau^2}+\Gamma^{(0)\,\mu}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\right)=F^\mu$$

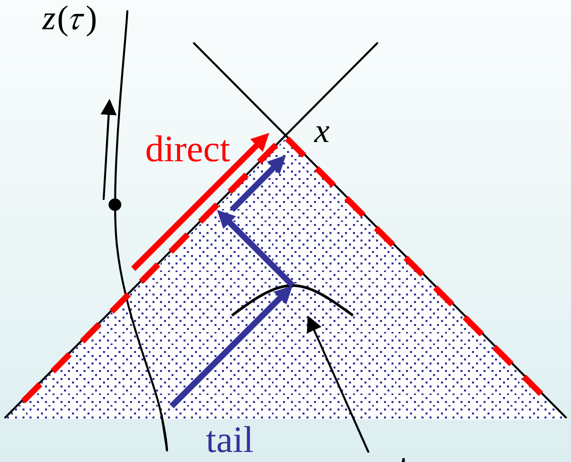
Self force $\propto \partial h$

Correspond to the shift from the background geodesic

But, metric perturbation (h) diverges at the location of the particle…

Formal Expression of SF

[Mino, Sasaki & Tanaka (1997), Quinn & Wald (1997)]



Decomposition of MP in the Lorenz (L) gauge

$$h_{\mu\nu}^{\text{full}}(x) = h_{\mu\nu}^{\text{tail}}(x) + h_{\mu\nu}^{\text{dir}}(x)$$

direct part : supported on the light cone
same singularity as the full field

tail part : supported inside the light cone
regular at the particle's location

Regularized self-force

$$F^{\mu}(\tau) = \lim_{x \to z} F^{\mu}[h_{\alpha\beta}^{\text{tail}}(x)] = -\frac{\mu}{2}(g^{\mu\nu} + u^{\mu}u^{\nu})(2h_{\nu\alpha;\beta}^{\text{tail,L}} - h_{\alpha\beta;\nu}^{\text{tail,L}})u^{\alpha}u^{\beta}$$

In practical, we perform the subtraction:

$$F^{\alpha}(\tau) = \lim_{x \to z(\tau)} (F^{\alpha}[h_{\mu\nu}^{\text{full}}(x)] - F^{\alpha}[h_{\mu\nu}^{\text{dir}}(x)])$$

Mode Sum Scheme

[Barack, Mino, Nakano, Ori & Sasaki (2002)]

We perform the subtraction by multipole modes:

$$F^{\alpha}(\tau) = \sum_{\ell} \lim_{x \to z(\tau)} \left(F_{\ell}^{\alpha}[h_{\mu\nu}^{\text{full}}(x)] - F_{\ell}^{\alpha}[h_{\mu\nu}^{\text{dir}}(x)]\right)$$

Advantage

- Full and direct parts in each mode are finite.
- Direct part is given analytically.

$$\lim_{x \to z} F_{\ell}^{\alpha}[h_{\mu\nu}^{\text{dir}}(x)] = A_{\pm}^{\alpha}(l + \tfrac{1}{2}) + B^{\alpha} + D_{\ell}^{\alpha}$$

$$A_{\pm}^{\alpha}, B^{\alpha}: \text{ known analytically,} \quad \sum_{\ell=0}^{\infty} D_{\ell}^{\alpha} = 0$$

To implement 'mode-sum' scheme, we need to obtain the metric perturbation for given orbit.

Field Equations

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[Barack & Lousto (2005)]

$$-\partial_t^2 \bar{h}^{(i)}_{lm} + \partial^2_{r_*} \bar{h}^{(i)}_{lm} + \mathcal{M}^{(i)}_{(j)} \bar{h}^{(j)}_{lm} = S^{(i)}_{lm} \quad (i = 1, \cdots, 10)$$

(1+1)-dim. 7 (even) + 3 (odd) coupled equations

- Expand the MP in terms of tensor harmonics

$$\bar{h}_{\mu\nu}(x) = \sum_{\ell m} \bar{h}^{(i),\ell m}(t, r) \mathbf{Y}^{(i),\ell m}_{\mu\nu}(\theta, \phi) \qquad (i = 1, \ldots, 10)$$

→ Reduce the problem into (1+1)-dimension

- Work in the time domain

→ No need for integration in frequency

- Take the Lorenz gauge condition

→ Avoid the gauge problem
Equations are hyperbolic
MP is continuous at the particle's location

Numerical Scheme

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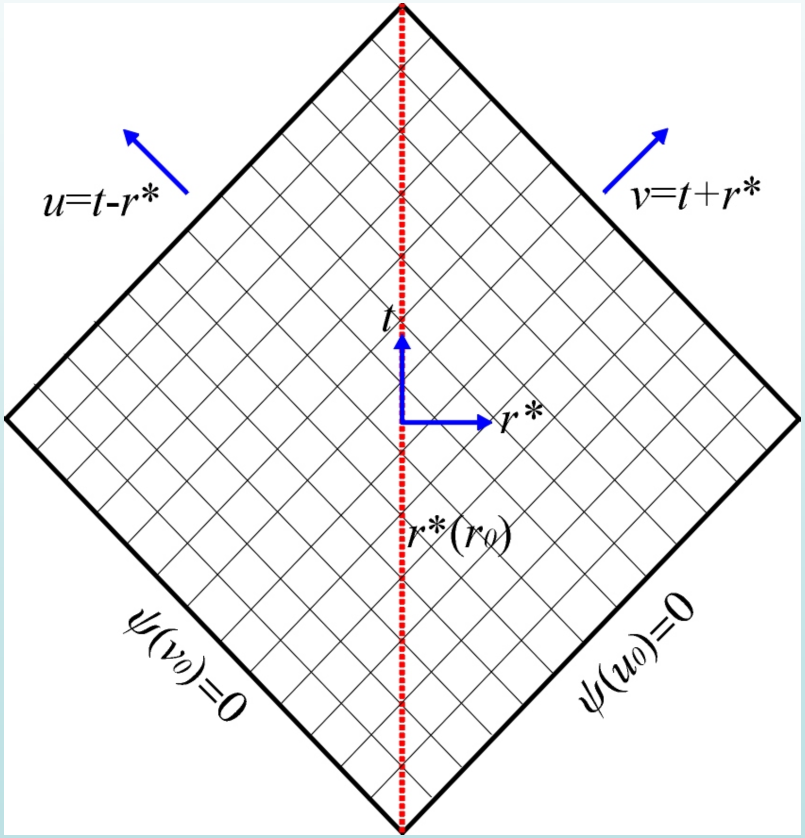
We solve the field equations in double null, uniform grid space by using a finite differential scheme.

Working grid space

Use double null coordinates:

$$v = t + r_*, \quad u = t - r_*$$

Take uniform grid: $\Delta v = \Delta u = h$



Solving differential equation

Use finite difference scheme

Initial condition

Put the initial condition as:

$$\bar{h}_{lm}^{(i)} = 0 \text{ at } v = v_0 \text{ and } u = u_0$$

Progress of Our Project

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Previous work [Barack-Sago (07)]

Numerics : 2nd order finite difference scheme
+ Richardson extrapolation

Temporal and radial force for circular orbits had been derived.

This work

1. Comparison to results in RW gauge [circular]
2. Loss of energy and angular momentum [eccentric]
3. Shift of ISCO [eccentric] (Preliminary result)

For eccentric orbit, we change the numerical scheme to 4th order finite difference scheme [Lousto (05), Haas (07)].

Comparison to RW Results (I)

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For circular orbit, two "gauge-invariants" exist: [Ref: Detweiler ('08)]

Ω : Frequency

u^t : Time function (t-comp. of 4-velocity)

These values are gauge independent within certain gauge class.

$\dot{\xi}^\alpha = (\partial_t + \Omega \partial_\varphi)\xi^\alpha = 0$ (helical symmetry)

Gauge-invariant radius

$$R_\Omega \equiv (M/\Omega^2)^{1/3} = r_0\left[1 + \frac{2}{3}\alpha + \frac{r_0(r_0 - 3M)}{3\mu M}F_r\right] \qquad \alpha = \frac{\mu}{\sqrt{r_0(r_0 - 3M)}}$$

Correction in time function

$$\Delta\tilde{u}^t = \sqrt{\frac{R_\Omega}{R_\Omega - 3M}}\left[\alpha\frac{R_\Omega - 2M}{R_\Omega - 3M} + \frac{1}{2}u^\mu u^\nu h^R_{\mu\nu}\right]$$

Comparison to RW Results (II)

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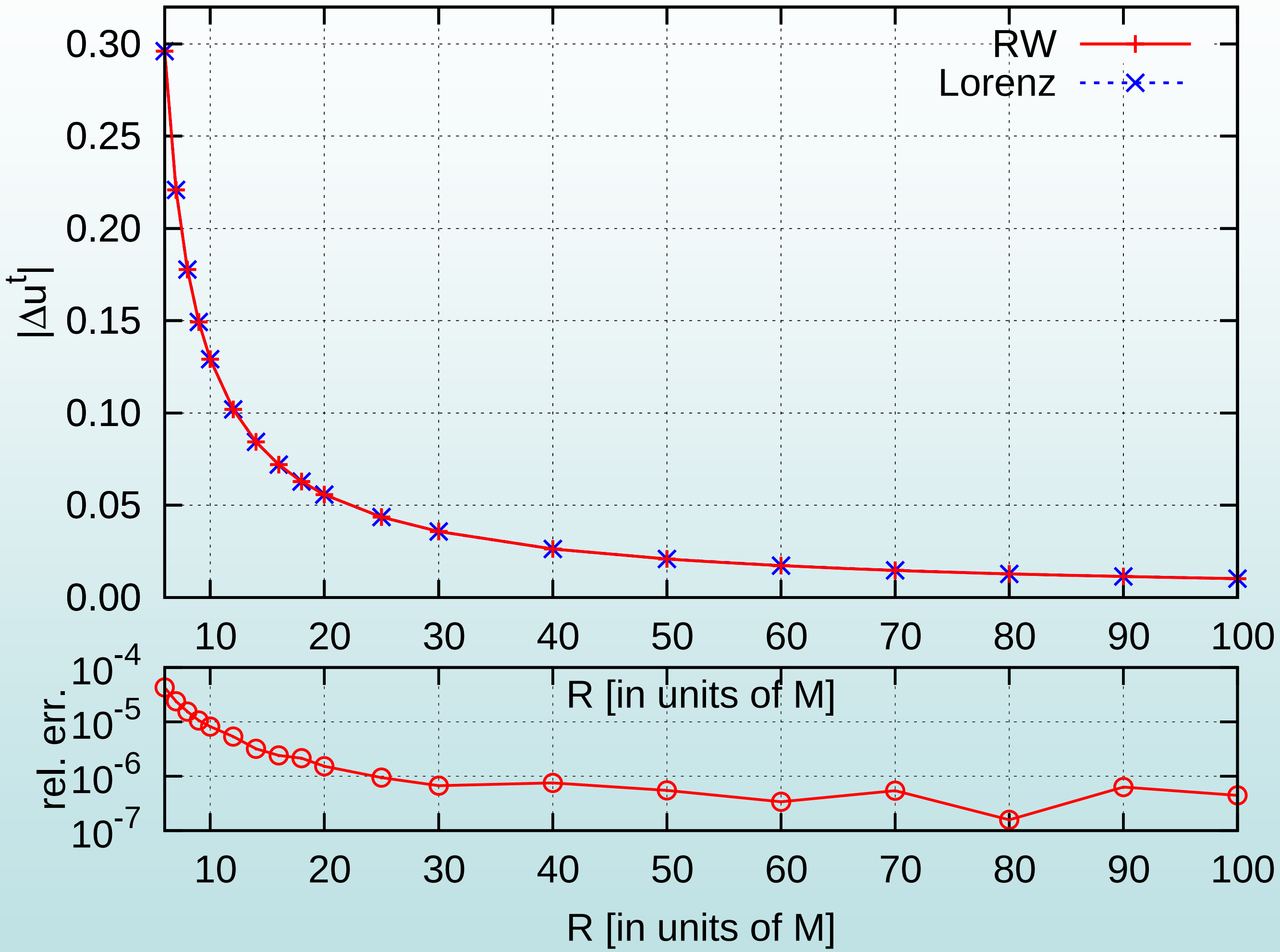

Relative errors are comparable with numerical accuracy.

Rate of Energy Loss (Eccentric orbits)

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t-component of local SF is related to the energy loss of the particle.

Energy balance formula (energy loss of particle) = (radiated energy)

$$\left\langle F_t / u_0^t \right\rangle = \left\langle \dot{E}_\infty \right\rangle + \left\langle \dot{E}_{\mathrm{EH}} \right\rangle \equiv \left\langle \dot{E}_{\mathrm{total}} \right\rangle$$

(p, e)=(7.50478, 0.188917) case

Our results	Cutler *et al*. ('94)	rel. err.
$\left\langle F_t / u_0^t \right\rangle = -3.16955 \times 10^{-4}$	$-\left\langle \dot{E}_\infty \right\rangle = -3.16804 \times 10^{-4}$	0.05%
$\left\langle F_\varphi / u_0^t \right\rangle = -5.96703 \times 10^{-3}$	$-\left\langle \dot{L}_\infty \right\rangle = -5.96562 \times 10^{-3}$	0.02%

[Estimated error = O(10^{-3})]

Here we ignore the contribution of the absorption to BH.

$$\frac{\left\langle \dot{E}_{EH} \right\rangle}{\left\langle \dot{E}_\infty \right\rangle} \approx \frac{\left\langle \dot{L}_{EH} \right\rangle}{\left\langle \dot{L}_\infty \right\rangle} \approx O(p^{-4}) \cong O(10^{-4})$$

Shift of ISCO

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Consider a small eccentricity case:

$$r_z(t_z) = r_0\left[1 + e\cos(\Omega_r t_z)\right]$$
$$F^r = F_0^r + e\left[F_{1c}^r \cos(\Omega_r t_z) + F_{1s}^r \sin(\Omega_r t_z)\right]$$
$$F_t = F_t^0 + e\left[F_t^{1c} \cos(\Omega_r t_z) + F_t^{1c} \sin(\Omega_r t_z)\right]$$

<u>SF correction of ISCO</u>

$$\Delta r_{isco} = \frac{18M}{\mu}\left[3MF_0^r + 6MF_{1c}^r + F_t^{1s}\right]_{r_0 \to 6M}$$

Roughly estimating from numerical results,

$$\frac{\Delta r_{isco}}{M} \approx 7.8\left(\frac{\mu}{M}\right), \quad \frac{\Delta\Omega_\phi}{\Omega_\phi} \approx -2.3\left(\frac{\mu}{M}\right)$$ (Preliminary)

For scalar case [Diaz-Rivera et al. '04]

$$\frac{\Delta r_{isco}}{q} \approx -0.122701\left(\frac{q}{\mu}\right), \quad \frac{\Delta\Omega_\phi}{\Omega_\phi} \approx 0.0291657\left(\frac{q^2}{\mu M}\right)$$

Summary

- Considered the local SF on a particle orbiting a Schwarzschild BH.

- Solved the field equations in the time domain (1+1)-dimension, under the Lorenz gauge condition.

- Derived the local force by using the mode sum scheme.

- Compared to RW gauge by evaluating some gauge-invariants. We found a good agreement between Lorenz and RW results.

- t / φ - components of local SF are balanced with the radiated energy and angular momentum.

- Rough estimation for the shift of ISCO by the local SF.

Future Works

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<u>Self-force effect on orbits in Schwarzschild spacetime</u>

- Estimate corrections in periastron shift, ISCO shift
- Comparison with post-Newtonian results

<u>Waveform including the self force effect</u>

- Make a code to calculate the GW for given orbits
- Comparison with 'adiabatic' waveform

<u>Improvement the accuracy and efficiency</u>

- Introduce a mesh refinement method

<u>Extension to Kerr case</u>

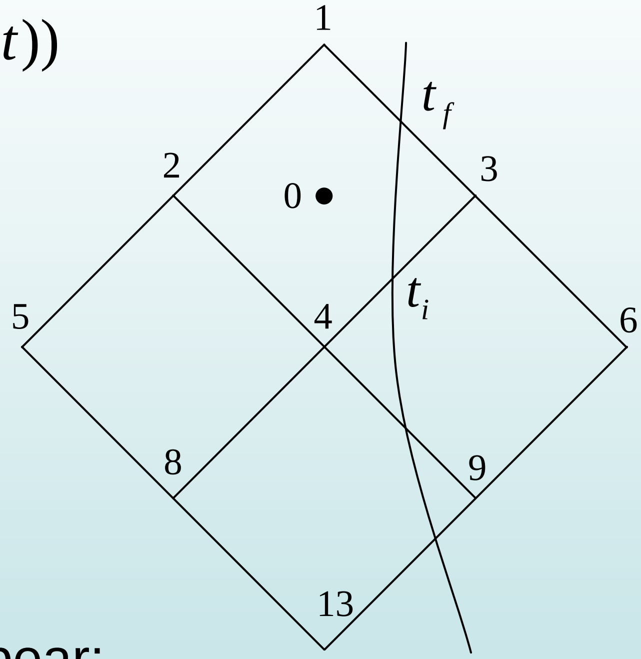
- Develop a time domain scheme for (2+1)-dimension
- Formulate the metric perturbation in Kerr.
- Improve a regularization method for the time domain scheme.

Appendix

2nd Order Finite Difference Scheme

Example (scalar field)

$$\left[\partial_v \partial_u + V_l(r)\right]\psi_{lm} = S_{lm}(t,r)\delta(r - r_p(t))$$



Integrating over a grid cell, we obtain:

$$\partial_v \partial_u \psi \longrightarrow \psi_1 - \psi_2 - \psi_3 + \psi_4$$

$$V(r)\psi \longrightarrow \tfrac{1}{2} h^2 V(r_0)(\psi_2 + \psi_3) + O(h^4 \text{ or } h^3)$$

$$S \longrightarrow \int_{t_i}^{t_f} dt S(r(t), \varphi(t))$$

For grav. case, r- and v- derivative terms appear:

$$V_r(r)\partial_r \psi, \quad V_v(r)\partial_v \psi$$

Integrating these terms up to O(h^2) requires more information.

Non-Stationarity error

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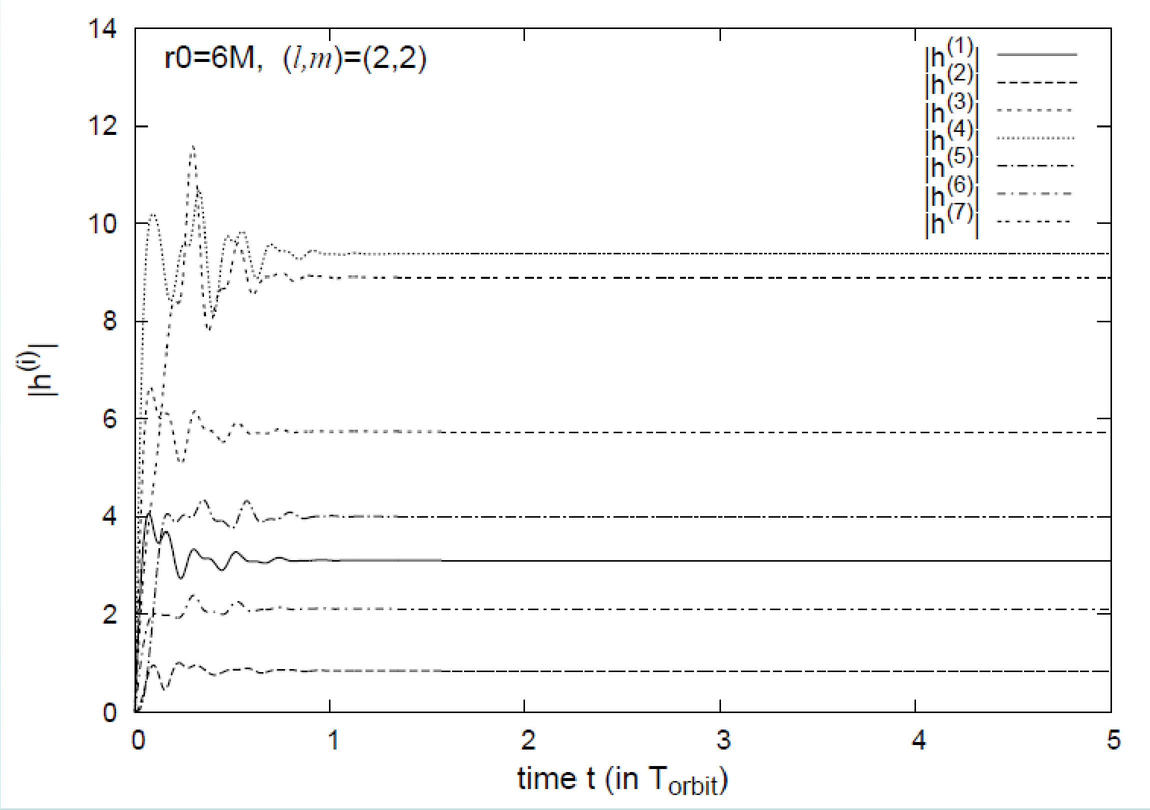
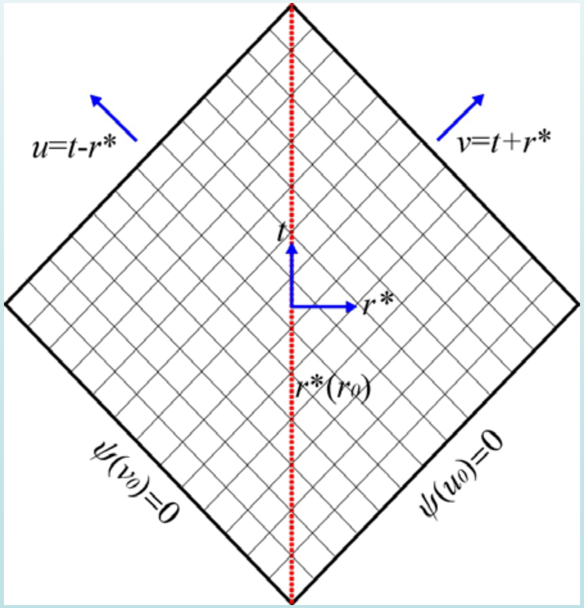
- Put the initial condition so that $h_{lm}^{(i)} = 0$ at $v = v_0$ and $u = u_0$

The spurious waves due to the imperfection of the initial data Appear in early stage of the evolution. But they damp rapidly so that it's negligible in late time of the evolution.





Outgoing boundary condition

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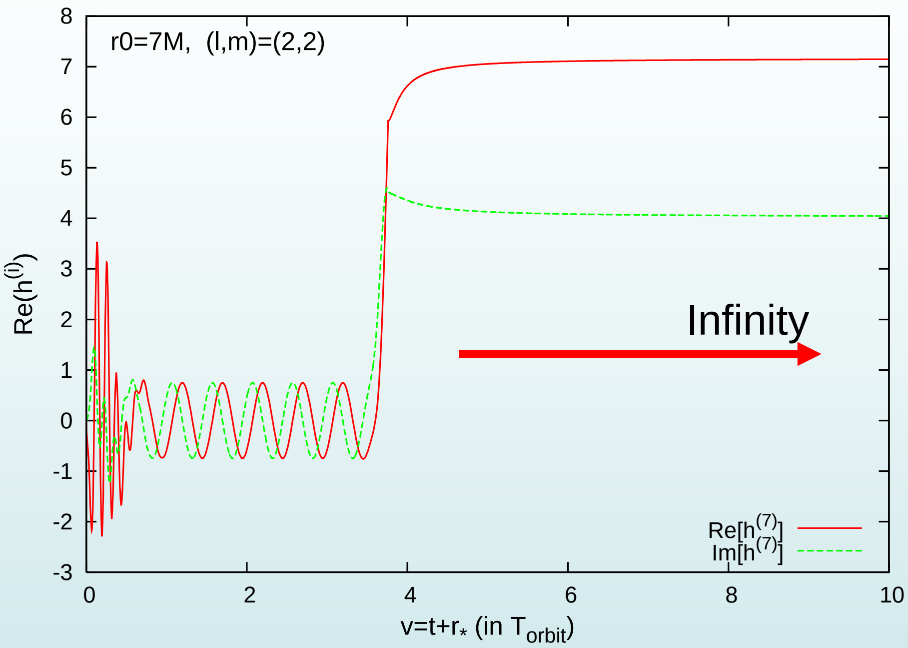
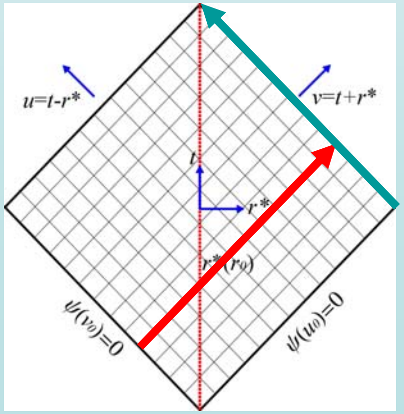
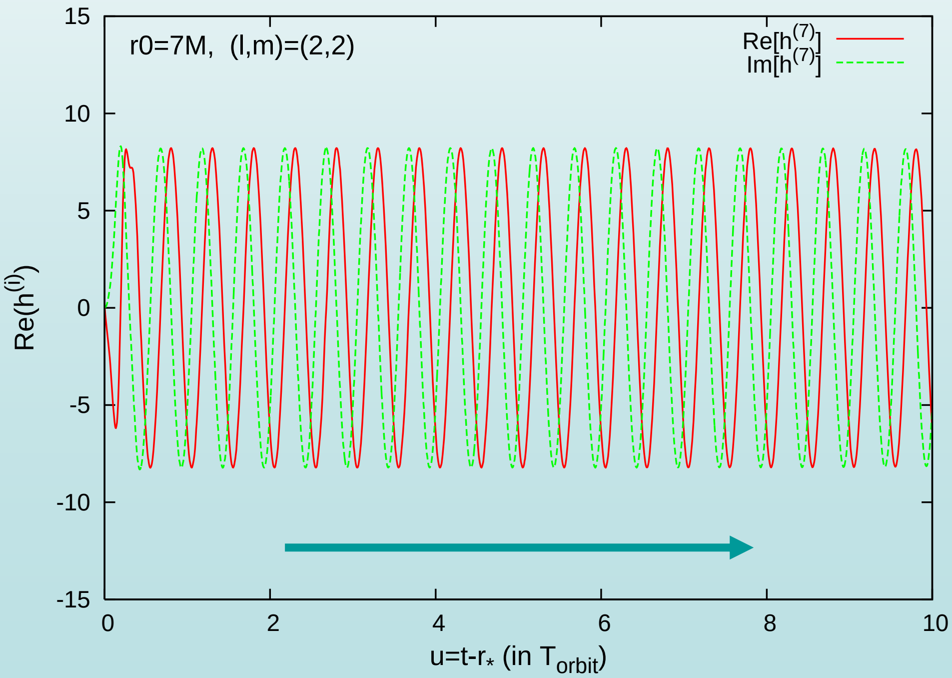


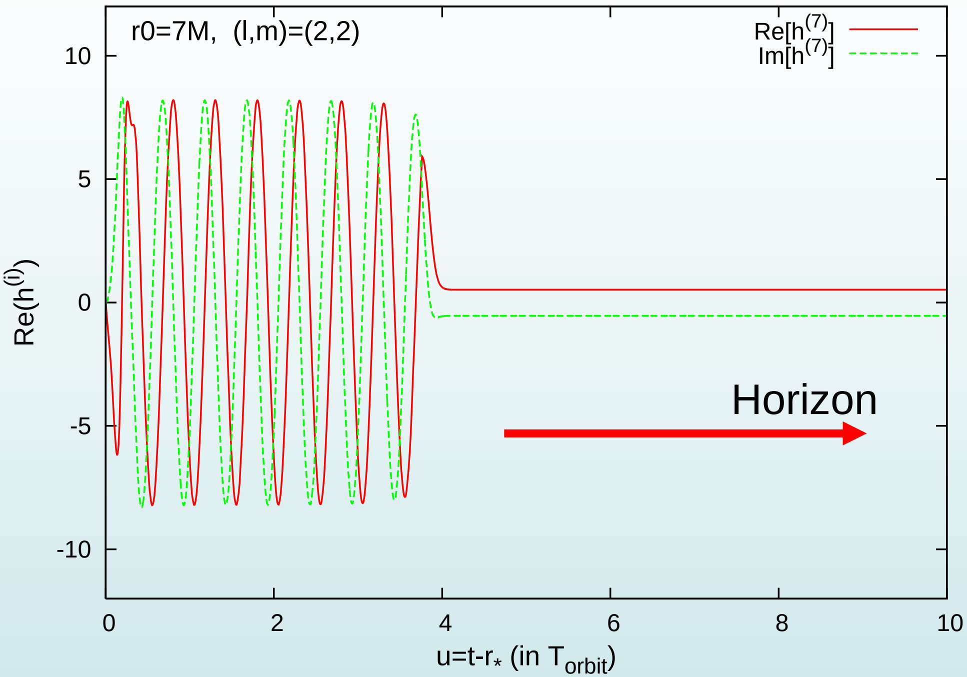
$$\psi \xrightarrow{v\to\infty} \psi(t - r_*)$$

Depend only on u at infinity.



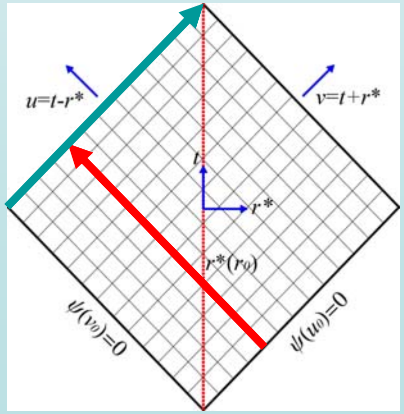


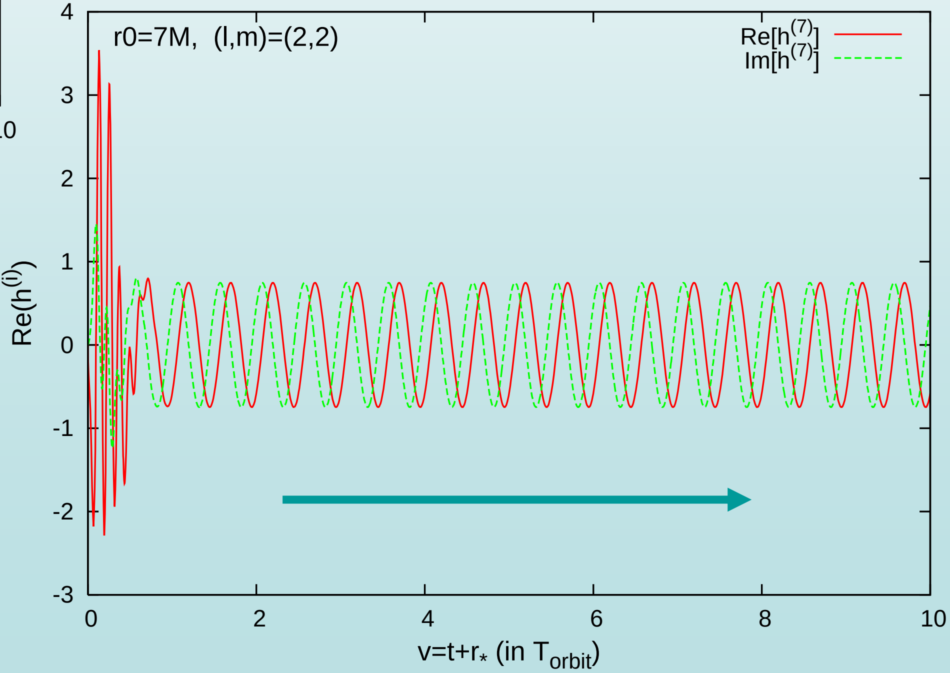
Ingoing boundary condition



$$\psi \xrightarrow{u\to\infty} \psi(t+r_*)$$

Depend only on v at the horizon.





Summation over multipole modes

To obtain the SF, take summation over *l* after regularization.

$$F^{\alpha} = \sum_{l=0}^{\infty} F_{\text{reg}}^{\alpha l} = \sum_{l=0}^{l_{\max}} F_{\text{reg}}^{\alpha l} + \sum_{l=l_{\max}+1}^{\infty} F_{\text{reg}}^{\alpha l}$$

Calculate finite number of modes, $0 \le l \le l_{\max} \quad (l_{\max} = 15)$

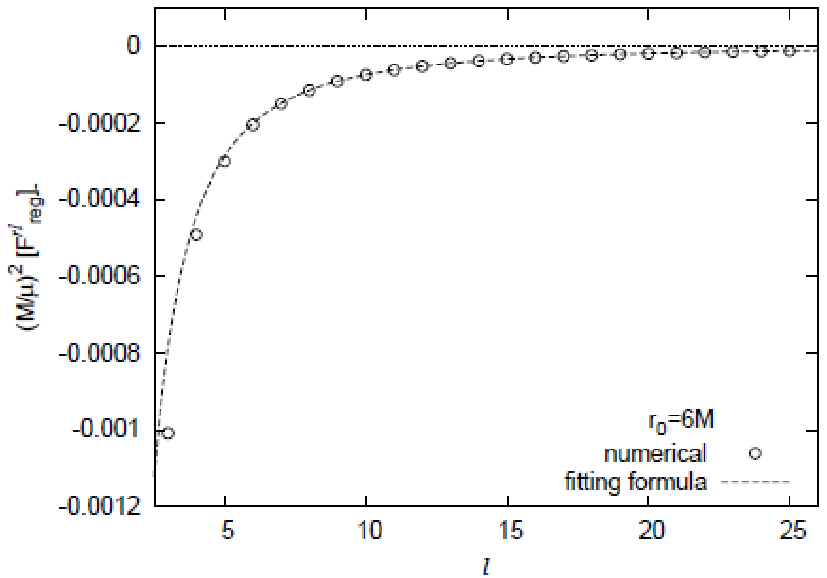
Derive a fitting formula for truncated part.

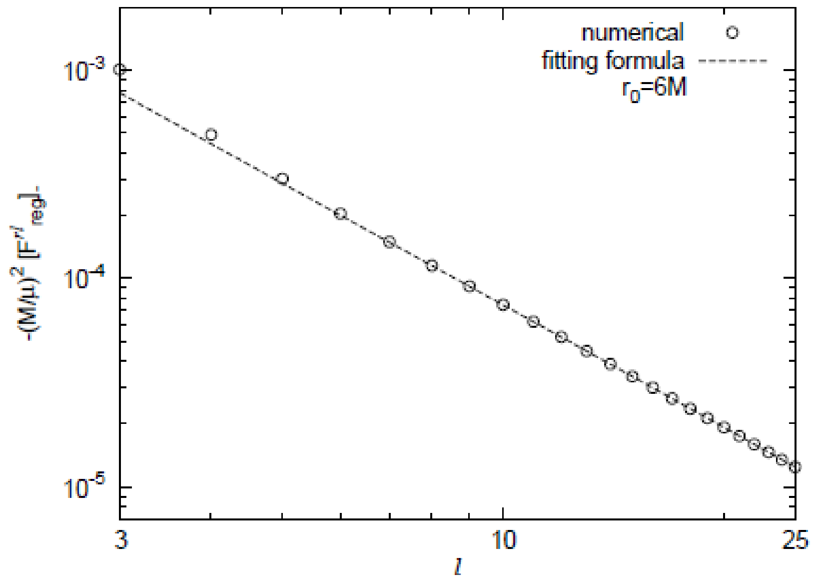
$$F_{\text{reg}}^{\alpha l} = \frac{D_2^{\alpha}}{L^2} + \frac{D_4^{\alpha}}{L^4} + \cdots$$

Contribution from Large *l*-tail (I)

<u>Fitting formula for large *l* modes</u>

$$F_{\text{reg}}^{\alpha l} = \frac{D_2^{\alpha}}{L^2} + \frac{D_4^{\alpha}}{L^4}$$





To derive the fitting parameters, use six data points, $10 \le l \le 15$

Fitting error in large-*l* contribution is $O(10^{-4})$.

Contribution from Large *l*-tail (II)

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Example

N	$[F^r_{l>15}]_-$ (fit using $10 \le l \le 15$)	Relative difference w.r.t $N=2$	$[F^r_{l>15}]_+$ (fit using $10 \le l \le 15$)	Relative difference w.r.t $N=2$
		$r_0 = 6M$		
1	$-5.117158\ [2\times10^{-3}]$	-1.4×10^{-2}	$-5.117176\ [2\times10^{-3}]$	-1.4×10^{-2}
2	$-5.046144\ [3\times10^{-5}]$	0	$-5.046189\ [3\times10^{-5}]$	0
3	$-5.046984\ [6\times10^{-5}]$	-1.7×10^{-4}	$-5.046812\ [2\times10^{-4}]$	-1.2×10^{-4}
4	$-5.046316\ [4\times10^{-4}]$	-3.4×10^{-5}	$-5.046133\ [9\times10^{-4}]$	-1.1×10^{-5}
		$r_0 = 100M$		
1	$-5.904165\ [1\times10^{-3}]$	-7.5×10^{-3}	$-5.904159\ [1\times10^{-3}]$	-7.5×10^{-3}
2	$-5.859951\ [2\times10^{-5}]$	0	$-5.859925\ [2\times10^{-5}]$	0
3	$-5.860397\ [8\times10^{-6}]$	-7.6×10^{-5}	$-5.860450\ [6\times10^{-6}]$	-9.0×10^{-5}
4	$-5.860430\ [7\times10^{-5}]$	-8.2×10^{-5}	$-5.860380\ [5\times10^{-5}]$	-7.8×10^{-5}

Accuracy of $O(10^{-4})$ by using six data points, $10 \le l \le 15$

Contribution from Large *l*-tail (III)

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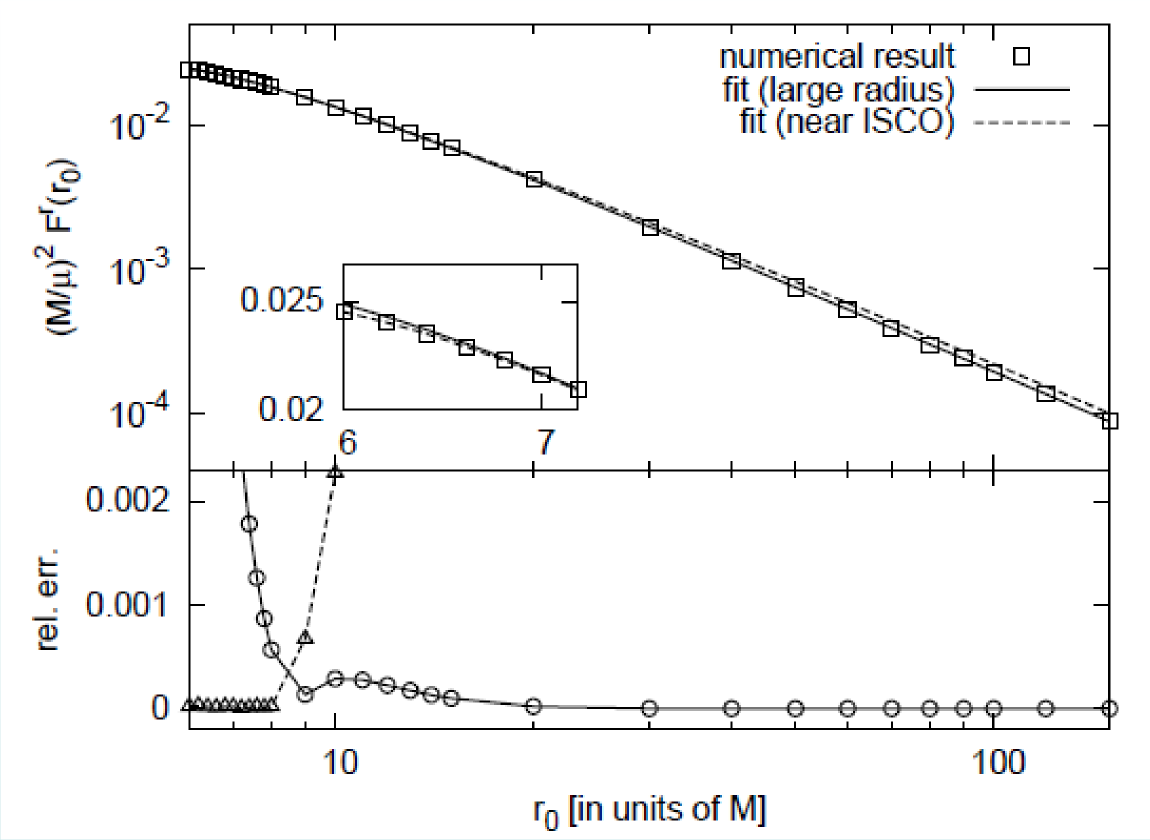
<u>Example</u> Fitting by using sixteen data points, $10 \le l \le 25$

N	$[F^r_{l>15}]_-$ (fit using $10 \le l \le 25$)	Relative difference w.r.t reference case	$[F^r_{l>15}]_+$ (fit using $10 \le l \le 25$)	Relative difference w.r.t reference case
		$r_0 = 6M$		
1	$-5.106648\ [1 \times 10^{-3}]$	-1.2×10^{-2}	$-5.106701\ [1 \times 10^{-3}]$	-1.2×10^{-2}
2	$-5.046340\ [2 \times 10^{-5}]$	-3.9×10^{-5}	$-5.046533\ [3 \times 10^{-5}]$	-6.8×10^{-5}
3	$-5.046634\ [3 \times 10^{-5}]$	-9.7×10^{-5}	$-5.046985\ [4 \times 10^{-5}]$	-1.6×10^{-4}
4	$-5.046500\ [1 \times 10^{-4}]$	-7.1×10^{-5}	$-5.047089\ [1 \times 10^{-4}]$	-1.8×10^{-4}
5	$-5.046769\ [4 \times 10^{-4}]$	-1.2×10^{-4}	$-5.047224\ [5 \times 10^{-4}]$	-2.1×10^{-4}
		$r_0 = 100M$		
1	$-5.897633\ [8 \times 10^{-4}]$	-6.4×10^{-3}	$-5.897625\ [8 \times 10^{-4}]$	-6.4×10^{-3}
2	$-5.860129\ [1 \times 10^{-5}]$	-3.0×10^{-5}	$-5.860106\ [1 \times 10^{-5}]$	-3.1×10^{-5}
3	$-5.860367\ [7 \times 10^{-6}]$	-7.1×10^{-5}	$-5.860349\ [8 \times 10^{-6}]$	-7.2×10^{-5}
4	$-5.860371\ [2 \times 10^{-5}]$	-7.2×10^{-5}	$-5.860304\ [2 \times 10^{-5}]$	-6.5×10^{-5}
5	$-5.860407\ [9 \times 10^{-5}]$	-7.8×10^{-5}	$-5.860308\ [1 \times 10^{-4}]$	-6.5×10^{-5}

Radial Force (circular orbit)



<u>Fitting formula for large radius</u>

$$F^r \cong \frac{\mu^2}{r_0^2}\left[a_0 + a_1\frac{M}{r_0} + a_2\left(\frac{M}{r_0}\right)^2 + a_3\left(\frac{M}{r_0}\right)^3\right]$$

$$a_0 = 1.999991\,,\ a_1 = -6.9969\,,\ a_2 = 6.29,\ a_3 = -24.6$$