Post-Newtonian methods

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Orléans School on Mass, Orléans, 24 June 2008

Outline

- Newtonian Dynamics
- Lorentz-Covariant Approach and PN Exps.
- ADM Approach and PN Exps.
- Higher-Order-PN Dynamical Systems

Newtonian Dynamics

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) = 0$$

$$\varrho \partial_t \mathbf{v} + \frac{\varrho}{2} \text{ grad } \mathbf{v}^2 - \varrho \ \mathbf{v} \times \text{curl } \mathbf{v} = -\text{grad } p - \varrho \text{ grad } U$$

$$\Delta U = -4\pi G \varrho$$

$$p = p(\varrho, s)$$

$$\partial_t s + \mathbf{v} \cdot \text{grad } s = 0$$

Lorentz-Covariant Approach and PN Exps.

Papers by L. Blanchet et al.

Einstein field equations (EFE):

$$G^{\mu\nu}(g, \partial g, \partial^2 g; c^{-1}) = \frac{8\pi G}{c^2} \frac{T^{\mu\nu}(g; c)}{c^2}$$

Contracted Bianchi identities:

$$\nabla_\nu G^{\mu\nu} \equiv 0 \quad \rightarrow \quad \nabla_\nu T^{\mu\nu} = 0 \quad \text{EOM}$$

Landau-Lifshitz form:

$$\partial_\lambda H^{\mu\nu\lambda}(g, \partial g) = \frac{16\pi G}{c^4} \tau^{\mu\nu}_{\mathrm{LL}}(g, \partial g)$$

$$\partial_\nu \partial_\lambda H^{\mu\nu\lambda} \equiv 0 \quad \rightarrow \quad \partial_\nu \tau^{\mu\nu}_{\mathrm{LL}} = 0 \quad \text{EOM}$$

$$\tau^{\mu\nu}_{\mathrm{LL}} = -g T^{\mu\nu} + \frac{c^4}{16\pi G} t^{\mu\nu}_{\mathrm{LL}}(g, \partial g)$$

Harmonic Coordinates (HC):

$$\partial_\nu H^{\mu\nu} = 0, \qquad H^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$$

EFE in HC (relaxed EFE):

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta H^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

$$\tau^{\mu\nu} = -gT^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}$$

$$\Lambda^{\mu\nu} = -H^{\alpha\beta}\partial_\alpha\partial_\beta H^{\mu\nu} + \partial_\alpha H^{\mu\beta}\partial_\beta H^{\nu\alpha} + \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\partial_\lambda H^{\alpha\tau}\partial_\tau H^{\beta\lambda}$$
$$-g^{\mu\alpha}g_{\beta\tau}\partial_\lambda H^{\nu\tau}\partial_\alpha H^{\beta\lambda} - g^{\nu\alpha}g_{\beta\tau}\partial_\lambda H^{\mu\tau}\partial_\alpha H^{\beta\lambda} + g_{\alpha\beta}g^{\lambda\tau}\partial_\lambda H^{\mu\alpha}\partial_\tau H^{\beta\nu}$$
$$+\frac{1}{8}(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})(2g_{\lambda\tau}g_{\rho\sigma} - g_{\lambda\sigma}g_{\tau\rho})\partial_\alpha H^{\lambda\sigma}\partial_\beta H^{\tau\rho}$$

Post-Minkowskian expansion: $H^{\mu\nu}(\mathbf{x},t) = \sum_{n=1}^{\infty}\left(\frac{GM}{Rc^2}\right)^n H^{\mu\nu}_{[n]}(\mathbf{x},t)$

$$H^{\mu\nu}(\mathbf{x},t) = -\frac{4G}{c^4}\int d^3x' \tau^{\mu\nu}(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})|\mathbf{x}-\mathbf{x}'|^{-1}$$

Near-zone expansion (post-Newtonian expansion):

$$H^{\mu\nu}_{\rm NZ}(\mathbf{x},t) = -\frac{4G}{c^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^3x' \frac{\partial^n}{c^n\partial t^n}\tau^{\mu\nu}(\mathbf{x}',t)|\mathbf{x}-\mathbf{x}'|^{n-1}$$

Far-zone expansion (post-Newtonian expansion):

$$H^{\mu\nu}_{\rm FZ}(\mathbf{x},t) = -\frac{4G}{c^4 r}\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3x' \frac{\partial^n}{c^n\partial t^n}\tau^{\mu\nu}(\mathbf{x}', t - \frac{r}{c})(\mathbf{x}'\cdot\mathbf{n})^n$$

2PN metric obtained by solving the relaxed EFE iteratively:

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_i V_i + \frac{V^3}{6}\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right] + \frac{4}{c^4}\hat{W}_{ij} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

Matter description: mass, mass-current, and stress densities:

$$\sigma = \frac{T^{00} + T^{ii}}{c^2}, \qquad \sigma_i = \frac{T^{0i}}{c}, \qquad \sigma_{ij} = T^{ij}$$

Crucial for PN is the virial theorem:

$$\frac{GM}{Rc^2}[\mathrm{PM}] \sim \frac{V^2}{c^2}[\mathrm{PN}] \quad << 1$$

2PN potentials:

$$V(\mathbf{x},t) = \Box_R^{-1}\{-4\pi G\sigma\} \equiv G\int \frac{d^3\mathbf{z}}{|\mathbf{x}-\mathbf{z}|}\sigma(\mathbf{z}, t-|\mathbf{x}-\mathbf{z}|/c)$$

$$V_i = \Box_R^{-1}\{-4\pi G\sigma_i\}$$

$$\hat{W}_{ij} = \Box_R^{-1}\{-4\pi G(\sigma_{ij}-\delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V\}$$

$$\hat{R}_i = \Box_R^{-1}\left\{-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right\}$$

$$\begin{aligned}\hat{X} &= \Box_R^{-1}\Big\{-4\pi G V\sigma_{ii} + 2V_i\partial_t\partial_i V + V\partial_t^2 V \\ &+ \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j\partial_j V_i + \hat{W}_{ij}\partial_{ij}^2 V\Big\}\end{aligned}$$

Burke-Thorne-like potentials:

$$
\begin{aligned}
U^{\mathrm{reac}}(\mathbf{x},t) &= -\frac{G}{5c^5}x^{ij}\mathrm{M}_{ij}^{[5]}(t)+\frac{G}{c^7}\left[\frac{1}{189}x^{ijk}\mathrm{M}_{ijk}^{[7]}(t)\right. \\
&\quad \left.-\frac{1}{70}x^{kk}x^{ij}\mathrm{M}_{ij}^{[7]}(t)\right] \\
U_i^{\mathrm{reac}}(\mathbf{x},t) &= \frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{ijk}\mathrm{M}_{jk}^{[6]}(t)-\frac{4}{45}\epsilon_{ijk}x^{jm}\mathrm{S}_{km}^{[5]}(t)\right]
\end{aligned}
$$

Multipole moments:

$$\mathrm{M}_{ij} = \int d^3\mathbf{y} \left(\hat{y}^{ij}\sigma + \frac{1}{14c^2} y^{kk}\hat{y}^{ij}\partial_t^2\sigma - \frac{20}{21c^2}\hat{y}^{ijk}\partial_t\sigma_k \right)$$

$$\mathrm{M}_{ijk} = \int d^3\mathbf{y}\ \hat{y}^{ijk}\sigma$$

$$\mathrm{S}_{ij} = \int d^3\mathbf{y}\ \epsilon_{km<i}\hat{y}^{j>k}\sigma_m$$

Definitions: $y^{ij} \equiv y^i y^i$, $y^{<ij>} \equiv \hat{y}^{ij} = \mathrm{STF}(y^{ij})$

1PN metric including the gravitational radiation reaction:

$$
\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}(U + U^{\mathrm{reac}}) + \frac{1}{c^4}\left[\partial_t^2\chi - 2U^2 - 4UU^{\mathrm{reac}}\right] + (g_{00})_{(6+8)} \\
g_{0i} &= -\frac{4}{c^3}(U_i + U_i^{\mathrm{reac}}) + (g_{0i})_{(5+7)} \\
g_{ij} &= \delta_{ij}\left[1 + \frac{2}{c^2}(U + U^{\mathrm{reac}})\right] + (g_{ij})_{(4+6)}
\end{aligned}
$$

$$
\begin{aligned}
U(\mathbf{x},t) &= G\int \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\;\sigma(\mathbf{y},t)\\
U_i(\mathbf{x},t) &= G\int \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\;\sigma_i(\mathbf{y},t)\\
\chi(\mathbf{x},t) &= G\int d^3\mathbf{y}\;|\mathbf{x}-\mathbf{y}|\sigma(\mathbf{y},t)
\end{aligned}
$$

Multipole expansion in FZ (TT-part only):

$$H^{ij}_{\rm FZ}(\mathbf{x},t) = -\frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\Bigg\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{(l)}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\;N_{i_3\ldots i_l}$$

$$+\left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\,\epsilon_{pq(k}\,\mathrm{S}^{(l)}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\;n_q\;N_{i_3\ldots i_l}\Bigg\}$$

$$\mathrm{M}_{ij}(t-\frac{r_*}{c}) = \widehat{\mathrm{M}}_{ij}\left(t-\frac{r_*}{c}\right)$$

$$+\frac{2Gm}{c^3}\int_0^{\infty}dv\,\ln\left(\frac{v}{2b}\right)\widehat{\mathrm{M}}^{(2)}_{ij}(t-\frac{r_*}{c}-v)+O(1/c^4),$$

$$r_* = r + \frac{2Gm}{c^2}\ln\left(\frac{r}{cb}\right)+O(1/c^3)$$

Luminosity and energy loss:

$$\mathcal{L}(t) = \frac{c^3}{32\pi G} \oint_{\rm FZ} (\partial_t H_{ij}^{\rm TT})^2 r^2 d\Omega$$

$$\begin{aligned}
\mathcal{L} &= \frac{G}{5c^5} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n \hat{\mathcal{L}}_n \\
&= \frac{G}{5c^5} \left\{ \mathrm{M}_{ij}^{(3)} \mathrm{M}_{ij}^{(3)} + \frac{1}{c^2} \left[\frac{5}{189} \mathrm{M}_{ijk}^{(4)} \mathrm{M}_{ijk}^{(4)} + \frac{16}{9} \mathrm{S}_{ij}^{(3)} \mathrm{S}_{ij}^{(3)} \right] \right. \\
&+ \left. \frac{1}{c^4} \left[\frac{5}{9072} \mathrm{M}_{ijkm}^{(5)} \mathrm{M}_{ijkm}^{(5)} + \frac{5}{84} \mathrm{S}_{ijk}^{(4)} \mathrm{S}_{ijk}^{(4)} \right] \right\}
\end{aligned}$$

$$- < \frac{d\mathcal{E}(t)}{dt} > \; = \; < \mathcal{L}(t) >$$

The ADM Approach and PN Exps.

Papers by G. Schäfer et al.

Ideal fluid dynamics in canonical form

Stress-energy tensor:

$$T^{\mu\nu} = \varrho(c^2 + h)u_\mu u_\nu + p g_{\mu\nu}, \qquad g_{\mu\nu}u^\mu u^\nu = -1$$

$$dp = \varrho dh - \varrho T ds$$

$$\varrho_* = \sqrt{-g}u^0\varrho, \qquad s, \qquad P_i = \frac{1}{c}\sqrt{-g}T^0_i$$

Kinematics:

$$\{P_i(x,t), \varrho_*(x',t)\} = \frac{\partial}{\partial x'^i}[\varrho_*(x',t)\delta(x-x')]$$

$$\{P_i(x,t), s(x',t)\} = \frac{\partial s(x',t)}{\partial x'^i}\delta(x-x')$$

$$\{P_i(x,t), P_j(x',t)\} = P_i(x',t)\frac{\partial}{\partial x'^j}\delta(x-x') - P_j(x,t)\frac{\partial}{\partial x^i}\delta(x-x')$$

Equations of motion:

$$\frac{\partial \varrho_*}{\partial t} = -\partial_i \left(\frac{\delta H}{\delta P_i} \varrho_* \right), \qquad \partial_\mu (\sqrt{-g}\varrho u^\mu) = 0$$

$$\frac{\partial s}{\partial t} = -\frac{\delta H}{\delta P_i} \partial_i s, \qquad u^\mu \partial_\mu s = 0$$

$$\frac{\partial P_i}{\partial t} = -\partial_j \left(\frac{\delta H}{\delta P_j} P_i \right) - \partial_i \left(\frac{\delta H}{\delta P_j} P_j \right) - \partial_i \left(\frac{\delta H}{\delta \varrho_*} \right) \varrho_* + \frac{\delta H}{\delta s} \partial_i s$$

$$v^i = \frac{\delta H}{\delta P_i}, \qquad v^i = c \frac{u^i}{u^0}$$

Momentum and angular momentum:

$$\int d^3x \; P_i, \qquad \int d^3x \; \epsilon_{ijk} x^j P_k$$

Point-mass dynamics in canonical form

$$p = s = h = 0$$

$$P_i = \sum_a p_{ai}\delta(x - x_a^i), \qquad r_* = \sum_a m_{ai}\delta(x - x_a^i), \qquad v_a^i = \frac{dx_a^i}{dt}$$

Kinematics:

$$\{x_a^i, p_{aj}\} = \delta_{ij}, \qquad \text{zero otherwise}$$

Equations of motion:

$$\frac{dp_{ai}}{dt} = -\frac{\partial H}{\partial x_a^i}$$

$$\frac{dx_a^i}{dt} = \frac{\partial H}{\partial p_{ai}}$$

Independent field variables

3 CC: $$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

1 CC: $\pi^{ii} = 0, \qquad \pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij} K), \quad \pi^i_i = \pi^{ij} h_{ij}^{\mathrm{TT}}$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi^{ij}_{\mathrm{TT}} c^3/16\pi G$: canonical conjugate to h_{ij}^{TT}

$$g^{1/2}\mathrm{R} = \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + \gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$\sqrt{-g}G^{00} = \frac{8\pi G}{c^4}\sqrt{-g}T^{00}$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

$\sqrt{-g}G^0_i = \frac{8\pi G}{c^4}\sqrt{-g}T^0_i$

ADM Hamiltonian:

$$H\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right] = -\frac{c^4}{16\pi G}\int d^3x\ \Delta\phi\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right]$$

Routh functional:

$$R\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \partial_t h^{\mathrm{TT}}_{ij}\right] = H - \frac{c^3}{16\pi G}\int d^3x\ \pi^{ij}_{\mathrm{TT}}\partial_t h^{\mathrm{TT}}_{ij}$$

$$\frac{\delta \int R(t')dt'}{\delta h_{ij}^{\mathrm{TT}}(x^k, t)} = 0, \quad \dot{p}_{ai} = -\frac{\partial R}{\partial x_a^i}, \quad \dot{x}_a^i = \frac{\partial R}{\partial p_{ai}}$$

on-field-shell Routh functional:

$$R_{\mathrm{on}}(t) = R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}[x_a^k, p_{ak}], \partial_t h_{ij}^{\mathrm{TT}}[x_a^k, p_{ak}]\right]$$

$$\dot{p}_{ai}(t) = -\frac{\delta \int R_{\mathrm{on}}(t')dt'}{\delta x_a^i(t)}, \qquad \dot{x}_a^i(t) = \frac{\delta \int R_{\mathrm{on}}(t')dt'}{\delta p_{ai}(t)}$$

$$\frac{\delta \int R_{\mathrm{on}}(t')dt'}{\delta z(t)} = \frac{\partial R_{\mathrm{on}}}{\partial z(t)} - \frac{d}{dt}\frac{\partial R_{\mathrm{on}}}{\partial \dot{z}(t)} + \ldots, \quad z = (x_a^i, p_{ai})$$

Brill-Lindquist BHs

$$-\left(1+\frac{1}{8}\,\phi\right)\Delta\phi=\frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}}=0=p_{ai})$$

$$\phi=\frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$

$$\alpha_a=m_a-\frac{m_a+m_b}{2}+\frac{c^2r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2r_{ab}/G}\right)^2}-1\right)$$

$$H_{\mathrm{BL}}=(\alpha_1+\alpha_2)\;c^2=(m_1+m_2)\;c^2-G\;\frac{\alpha_1\alpha_2}{r_{12}}$$

Metric in d-dimensional conformally flat space:

$$g_{ij} = \left(1 + \frac{1}{4}\frac{d-2}{d-1}\phi\right)^{\frac{4}{d-2}} \delta_{ij}$$

$$\phi = \frac{4G}{c^2}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)$$

$$\Psi = 1 + \frac{1}{4}\frac{d-2}{d-1}\phi$$

Skeleton dynamics (Faye/Jaranowski/GS 2004)

PN expansion possible to all orders in powers of $1/c^2$

$$h_{ij}^{\rm TT} = 0, \quad g_{ij} = \Psi^{\frac{4}{d-2}} \delta_{ij}$$

$$\Delta\phi = -\,\Psi^{-\frac{3d-2}{d-2}} \pi^{i}_{\ j} \pi^{j}_{\ i} - \frac{16\pi G}{c^2} \sum_a m_a \delta_a \Psi^{-1} \left(1 + \Psi^{-\frac{4}{d-2}} \frac{p_a^2}{m_a^2 c^2}\right)^{\frac{1}{2}}$$

$$\partial_j \pi^{j}_{\ i} = -\,\frac{8\pi G}{c^3} \sum_a p_{ai} \delta_a$$

$$\pi^{i}{}_{j}^{\,(\mathrm{part})} = \mathrm{STF}(2\partial_i V_j) \equiv \partial_i V_j + \partial_j V_i - \frac{2}{d}\delta_{ij}\partial_k V_k$$

$$V_i = \frac{G}{c^3}\frac{\Gamma(\frac{d-2}{2})}{2\pi^{\frac{d-2}{2}}}\sum_a \left(\frac{4p_{ai}}{r_a^{d-2}} - \frac{(d-2)p_{aj}}{(d-1)(4-d)}\partial_{ij} r_a^{4-d}\right)$$

$$\pi^{i}{}_{j}\pi^{j}{}_{i} = 2\pi^{i}{}_{j}\partial_i V_j$$

$$\Psi^{\frac{2-3d}{d-2}}\pi^{i}{}_{j}\pi^{j}{}_{i} \to -2\Psi^{\frac{2-3d}{d-2}} V_j \partial_i \pi^{i}{}_{j} = \frac{16\pi G}{c^3}\Psi^{\frac{2-3d}{d-2}}\sum_a p_{aj} V_j \delta_a$$

further details: A. Gopakumar, 11$^{\rm th}$ Capra Meeting on RR

Post-Newtonian expansions

$$R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \partial_t h_{ij}^{\mathrm{TT}}\right] - Mc^2 = \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n R_n\left[x_a^i, p_{ai}, \hat{h}_{ij}^{\mathrm{TT}}, \partial_t \hat{h}_{ij}^{\mathrm{TT}}\right]$$

$$h_{ij}^{\mathrm{TT}} = \frac{G}{c^4}\hat{h}_{ij}^{\mathrm{TT}}$$

$$\left(\Delta - \frac{\partial_t^2}{c^2}\right) h^{\mathrm{TT}} = \frac{G}{c^4} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n D_n^{\mathrm{TT}}[x, x_a(t), p_a(t), \hat{h}^{\mathrm{TT}}(t), \partial_t \hat{h}^{\mathrm{TT}}(t)]$$

Higher-Order-PN Dynamical Systems

3PN binary BH conservative dynamics

$$
\begin{aligned}
H(t) &= m_1 c^2 + m_2 c^2 + H_N + \frac{1}{c^2} H_{[1PN]} \\
&+ \frac{1}{c^4} H_{[2PN]} + \frac{1}{c^6} H_{[3PN]} + \ldots \\
&+ \frac{1}{c^5} H_{[2.5PN]}(t) + \frac{1}{c^7} H_{[3.5PN]}(t) + \ldots
\end{aligned}
$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1 m_2 / M, \qquad M = m_1 + m_2$

$\nu = \mu / M, \qquad 0 \leq \nu \leq 1/4$

test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$

CMF: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} \equiv \mathbf{p}_1 / \mu,$

$p_r = (\mathbf{n} \cdot \mathbf{p}), \qquad \mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
\hat{H}_{[2PN]} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2p_r^2p^4 \\
&+ 3(1-\nu)\nu^2p_r^4p^2-5\nu^3p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

Remark on EOB

Research by Damour et al. since 1998

One-body dynamics in deformed Schwarzschild/Kerr metric related to binary dynamics in the CMF through canonical transformation.

Dynamical invariants

radial action $i_r(E,j)$ with $E = \hat{H}$ and $p_r^2 + j^2/r^2 = p^2$:

$$i_r(E,j) = \frac{1}{2\pi} \oint dr\ p_r$$

phase of revolution Φ:

$$\frac{\Phi}{2\pi} = 1 + k = -\frac{\partial}{\partial j} i_r(E,j)$$

orbital period P:

$$\frac{P}{2\pi GM} = \frac{\partial}{\partial E} i_r(E,j)$$

periastron advance k:

$$k = \frac{1}{c^2}\frac{3}{j^2}\left\{1 + \frac{1}{c^2}\left[\frac{5}{4}(7-2\nu)\frac{1}{j^2} + \frac{1}{2}(5-2\nu)\,E\right]\right.$$

$$\left. + \frac{1}{c^4}\left[a_1(\nu)\frac{1}{j^4} + a_2(\nu)\frac{E}{j^2} + a_3(\nu)\,E^2\right]\right\}$$

$$\frac{P}{2\pi GM} = \frac{1}{(-2E)^{3/2}}\left\{1 - \frac{1}{c^2}\frac{1}{4}(15-\nu)E\right.$$

$$+ \frac{1}{c^4}\left[\frac{3}{2}(5-2\nu)\frac{(-2E)^{3/2}}{j} - \frac{3}{32}(35+30\nu+3\nu^2)\,E^2\right]$$

$$\left. + \frac{1}{c^6}\left[a_2(\nu)\frac{(-2E)^{3/2}}{j^3} - 3a_3(\nu)\frac{(-2E)^{5/2}}{j} + a_4(\nu)\,E^3\right]\right\}$$

$$
\begin{aligned}
a_1(\nu) &= \frac{5}{2}\left(\frac{77}{2} + \left(\frac{41}{64}\pi^2 - \frac{125}{3}\right)\nu + \frac{7}{4}\nu^2\right) \\
a_2(\nu) &= \frac{105}{2} + \left(\frac{41}{64}\pi^2 - \frac{218}{3}\right)\nu + \frac{45}{6}\nu^2 \\
a_3(\nu) &= \frac{1}{4}(5 - 5\nu + 4\nu^2) \\
a_4(\nu) &= \frac{5}{128}(21 - 105\nu + 15\nu^2 + 5\nu^3)
\end{aligned}
$$

ISCO

$\hat{H} = \hat{H}(\mathbf{p}, \mathbf{r}), \qquad p^2 = p_r^2 + j^2/r^2, \qquad p_r = (\mathbf{p} \cdot \mathbf{r})/r$

circular orbits: $p_r = 0, \qquad p^2 = j^2/r^2, \qquad \hat{H} = \hat{H}(j, r)$

circular motion: $\frac{\partial}{\partial r}\hat{H}(j, r) = 0 \longrightarrow \hat{H}(j)$

orbital frequency: $\omega = \frac{d\hat{H}(j)}{dj} \longrightarrow \hat{H}(\omega)$

ISCO: $\frac{d\hat{H}(\omega)}{d\omega} = 0$, alternatively $\frac{\partial^2}{\partial r^2}\hat{H}(j, r) = 0$

$$\text{SBH:} \quad E(x) = \frac{1 - 2x}{(1 - 3x)^{1/2}} - 1$$

$$= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM\omega}{c^3} \right)^{2/3}$$

circular orbits:

$$\omega_{\rm circ} = \omega_{\rm radial} + \omega_{\rm periastron} = 2\pi \frac{1+k}{P}, \quad x = \left(\frac{GM\omega_{\rm circ}}{c^3} \right)^{2/3}$$

$$c^2 E_{3PN} \equiv \hat{H}_N + \hat{H}_{[1PN]} + \hat{H}_{[2PN]} + \hat{H}_{[3PN]}$$

$$\begin{aligned} E_{3PN}(x) &= -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu \right) x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2 \right) x^3 \\ &+ \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2 \right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3 \right) x^4 \end{aligned}$$

ISCO: $\dfrac{dE_{3PN}}{dx} = 0$

2.5PN binary BH (orbital) dissipative dynamics

$$\frac{1}{c^5} H_{[2.5PN]}(t) = \frac{2G}{5c^5} \frac{d^3 Q_{ij}(t)}{dt^3} \left(\frac{p_{1i}p_{1j}}{m_1} + \frac{p_{2i}p_{2j}}{m_2} - \frac{Gm_1 m_2}{r_{12}} \right)$$

$$Q_{ij}(t) = \sum_{a=1,2} m_a (x_a^i x_a^j - \frac{1}{3}\mathbf{x}_a^2 \delta_{ij})$$

3.5PN Hamiltonian: Königsdörffer/Faye/GS (2003)

Toward binary rotating black holes

Action principle:

$$W = \int dt \Bigg(\sum_a p_{ai} \dot{x}^i_a + \sum_a S^{(i)}_a \Omega^{(i)}_a + \frac{1}{16\pi} \int d^3x \, \pi^{ij}_{\mathrm{TT}} \dot{h}^{\mathrm{TT}}_{ij} - H_{ADM} \left[x^i_a, p_{ai}, S^{(j)}_a, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}} \right] \Bigg)$$

- $\Omega^{(i)}_a = \frac{1}{2} \epsilon_{ijk} \Lambda_{a(l)(j)} \dot{\Lambda}_{a(l)(k)}$,
  $\Lambda_{a(i)(k)} \Lambda_{a(j)(k)} = \Lambda_{a(k)(i)} \Lambda_{a(k)(j)} = \delta_{ij}$
- independent variables: p_{ai}, x^i_a, $S^{(i)}_a$, $\Lambda_{a(i)(j)}$

Equations of motion for matter:

$$\dot{x}_a^i(t) = \frac{\delta \int dt' H_{ADM}}{\delta p_{ai}(t)}, \quad \dot{p}_{ai}(t) = -\frac{\delta \int dt' H_{ADM}}{\delta x_a^i(t)}$$

$$\Omega_a^{(i)}(t) = \frac{\delta \int dt' H_{ADM}}{\delta S_a^{(i)}(t)}, \quad \dot{S}_a^{(i)}(t) = \epsilon_{ijk}\Omega_a^{(j)}(t) S_a^{(k)}(t)$$

$$H = \int d^3\mathbf{x}(N\mathcal{H} - N^i\mathcal{H}_i) + E[\gamma_{ij}]$$

$$\mathcal{H} = \mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}}$$

$$\mathcal{H}_i = \mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}}$$

$$H^{\text{matter}} = \int d^3\mathbf{x}(N\mathcal{H}^{\text{matter}} - N^i\mathcal{H}_i^{\text{matter}})$$

Constant-euclidean-length spin:

$$\gamma^{ik}\gamma^{jl}\hat{S}_{ij}\hat{S}_{kl} = \mathrm{const}$$

$$e_{il}e_{lj} = \gamma_{ij}\ , \quad e_{ij} = e_{ji}$$

$$\hat{S}_{kl} = e_{ki}e_{lj}S_{(i)(j)}, \quad S_{(i)(j)}S_{(i)(j)} = \mathrm{const}$$

$$S_{(i)(j)} = \epsilon_{ijk}S_{(k)}$$

$$p_{ai} = \int_a d^3x \mathcal{H}_i^{\mathrm{matter}}$$

$$J_{aij} = \int_a d^3x (x^i \mathcal{H}_j^{\mathrm{matter}} - x^j \mathcal{H}_i^{\mathrm{matter}}) = x_a^i p_{aj} - x_a^j p_{ai} + S_{a(i)(j)}$$

$$\{x_a^i, p_{aj}\} = \delta_{ij}, \qquad \{S_{a(i)}, S_{a(j)}\} = \epsilon_{ijk}S_{a(k)}$$

$$\mathcal{H}^{\rm matter} = -np\ \delta - \frac{1}{2} t^k_{ij} \gamma^{ij}{}_{,k} - \left[\frac{p_l}{m - np} \gamma^{ij} \gamma^{kl} \hat{S}_{jk} \delta \right]_{,i}$$

$$\begin{aligned} \mathcal{H}^{\rm matter}_i &= p_i \delta + \frac{1}{2} \left[\gamma^{mk} \hat{S}_{ik} \delta \right]_{,m} \\ &- \left[\frac{p_l p_k}{np(m - np)} (\gamma^{mk} \delta^p_i + \gamma^{mp} \delta^k_i) \gamma^{ql} \hat{S}_{qp} \delta \right]_{,m} \end{aligned}$$

$$\sqrt{\gamma} T_{ij} = -\tfrac{p_i p_j}{np} \delta + t^k_{ij,k} + \mathcal{O}(G)$$

$$t^k_{ij} \equiv \gamma^{kl} \tfrac{\hat{S}_{l(i} p_{j)}}{np} \delta + \gamma^{kl} \gamma^{mn} \tfrac{\hat{S}_{m(i} p_{j)} p_n p_l}{(np)^2 (m - np)} \delta$$

$$\frac{\delta H^{\rm matter}}{\delta \gamma^{ij}} = \frac{1}{2} N \sqrt{\gamma} T_{ij} + \mathcal{O}(G)$$

Poincaré algebra:

- Global Poincaré group is a consequence of asymptotic flatness
- Generators P^μ and $J^{\mu\nu}$ are conserved

$$\begin{aligned} \{P^\mu, P^\nu\} &= 0 \\ \{P^\mu, J^{\rho\sigma}\} &= -\eta^{\mu\rho} P^\sigma + \eta^{\mu\sigma} P^\rho \\ \{J^{\mu\nu}, P^{\rho\sigma}\} &= -\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\sigma\mu} J^{\rho\nu} - \eta^{\sigma\mu} J^{\rho\mu} \end{aligned}$$

  - Energy: $E \equiv P^0$
  - Momentum: P^i
  - Angular momentum: J^{ij}
  - Boost: $J^{i0} \equiv K^i \equiv G^i - t\, P^i$
  - Center-of-Mass: $X^i \equiv G^i / E$

- E, P^i, J^{ij}, and G^i are surface integrals at spatial infinity
- E and G^i:

$$E = -\frac{1}{16\pi} \int d^3\mathbf{x}\, \Delta\phi$$

$$G^i = -\frac{1}{16\pi} \int d^3\mathbf{x}\, x^i \Delta\phi$$

Leading-order (LO) in spin:

- LO spin-orbit

$$H_{\mathrm{SO}}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a} \mathbf{P}_a - 2\mathbf{P}_b \right]$$

- LO spin_1-spin_2

$$H_{\mathrm{SS}}^{\mathrm{LO}} = \sum_a \sum_{b \neq a} \frac{1}{2r_{ab}^3} \left[3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

- LO center-of-mass vector

$$\mathbf{G}_{\mathrm{SO}}^{\mathrm{LO}} = \sum_a \frac{1}{2m_a} (\mathbf{P}_a \times \mathbf{S}_a) \,, \qquad \mathbf{G}_{\mathrm{SS}}^{\mathrm{LO}} = 0$$

NLO spin-orbit Hamiltonian (Damour/Jaranowski/GS 2008):

$$H_{\text{SO}}^{\text{NLO}} = -\frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^2}\left[\frac{5m_2\mathbf{p}_1^2}{8m_1^3} + \frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^2} - \frac{3\mathbf{p}_2^2}{4m_1m_2}\right.$$
$$\left. + \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{4m_1^2} + \frac{3(\mathbf{p}_2\cdot\mathbf{n}_{12})^2}{2m_1m_2}\right]$$
$$+ \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^2}\left[\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2}\right]$$
$$+ \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{p}_2)}{r_{12}^2}\left[\frac{2(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1m_2} - \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})}{4m_1^2}\right]$$
$$- \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^3}\left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1}\right]$$
$$+ \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{ab}^3}\left[6m_1 + \frac{15m_2}{2}\right] + (1 \leftrightarrow 2)$$

NLO Spin$_1$-Spin$_2$ Hamiltonian (Steinhoff/Hergt/GS 2008):

$$H_{\rm SS}^{\rm NLO} = \frac{1}{2m_1 m_2 r_{12}^3}\Big[\tfrac{3}{2}((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_2)\cdot\mathbf{n}_{12})$$
$$+6((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2)\cdot\mathbf{n}_{12})$$
$$-15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})$$
$$-3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})$$
$$+3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})$$
$$+3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})$$
$$+3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})$$
$$-3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2)$$
$$-\tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1) + \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\Big]$$

$$
\begin{aligned}
&+\frac{3}{2m_1^2 r_{12}^3}[-((\mathbf{p}_1 \times \mathbf{S}_1) \cdot \mathbf{n}_{12})((\mathbf{p}_1 \times \mathbf{S}_2) \cdot \mathbf{n}_{12}) \\
&+(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{p}_1 \cdot \mathbf{n}_{12})^2 - (\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_2 \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{n}_{12})] \\
&+\frac{3}{2m_2^2 r_{12}^3}[-((\mathbf{p}_2 \times \mathbf{S}_2) \cdot \mathbf{n}_{12})((\mathbf{p}_2 \times \mathbf{S}_1) \cdot \mathbf{n}_{12}) \\
&+(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{p}_2 \cdot \mathbf{n}_{12})^2 - (\mathbf{S}_2 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{p}_2)(\mathbf{p}_2 \cdot \mathbf{n}_{12})] \\
&+\frac{6(m_1+m_2)}{r_{12}^4}[(\mathbf{S}_1 \cdot \mathbf{S}_2) - 2(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_2 \cdot \mathbf{n}_{12})]
\end{aligned}
$$

NLO center-of-mass:

$$\mathbf{G}_{\rm SO}^{\rm NLO} = -\sum_a \frac{\mathbf{P}_a^2}{8m_a^3}(\mathbf{P}_a \times \mathbf{S}_a)$$
$$+\sum_a \sum_{b\neq a} \frac{m_b}{4m_a r_{ab}}\left[((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab})\frac{5\mathbf{x}_a + \mathbf{x}_b}{r_{ab}} - 5(\mathbf{P}_a \times \mathbf{S}_a)\right]$$
$$+\sum_a \sum_{b\neq a} \frac{1}{r_{ab}}\left[\frac{3}{2}(\mathbf{P}_b \times \mathbf{S}_a) - \frac{1}{2}(\mathbf{n}_{ab} \times \mathbf{S}_a)(\mathbf{P}_b \cdot \mathbf{n}_{ab}) - ((\mathbf{P}_a \times \mathbf{S}_a)\cdot \mathbf{n}_{ab})\frac{\mathbf{x}_a + \mathbf{x}_b}{r_{ab}}\right]$$

$$\mathbf{G}_{\rm SS}^{\rm NLO} = \frac{1}{2}\sum_a \sum_{b\neq a}\left\{ [3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b)]\,\frac{\mathbf{x}_a}{r_{ab}^3} + (\mathbf{S}_b \cdot \mathbf{n}_{ab})\frac{\mathbf{S}_a}{r_{ab}^2}\right\}$$

Further conservative binary Hamiltonians (Hergt/GS 2008):

$$\begin{aligned} H &= H_N + H_{1PN} + H_{2PN} + H_{3PN} + H_{SO}^{1PN} + H_{S_1S_2}^{1PN} \\ &+ H_{SO}^{2PN} + H_{S_1S_2}^{2PN} + H_{p^2S_1S_2} \\ &+ H_{S_1^2,S_2^2} + H_{p_1S_2^3} + H_{p_2S_1^3} + H_{p_1S_1S_2^2} + H_{p_2S_2S_1^2} \end{aligned}$$

$$\mathcal{H}^{\text{matter}} = m_1 \left(1 - \frac{1}{2} (\mathbf{a}_1 \cdot \partial_1)^2\right) \delta_1 + \frac{1}{2}\mathbf{p}_1 \cdot (\mathbf{a}_1 \times \partial_1)\, \delta_1 + (1 \leftrightarrow 2)$$

$$\mathcal{H}_i^{\text{matter}} = p_{1i}\delta_1 + \frac{m_1}{2} (\mathbf{a}_1 \times \partial_1)_i \left(1 - \frac{1}{6} (\mathbf{a}_1 \cdot \partial_1)^2\right) \delta_1 + (1 \leftrightarrow 2)$$

$$\mathbf{S}_1 = \mathbf{a}_1 m_1, \quad \mathbf{S}_2 = \mathbf{a}_2 m_2$$