A Rigorous Derivation of Gravitational Self-Force

Samuel Gralla and Robert M. Wald

Some Difficulties with Usual Derivations of Gravitational Self-Force

It is of considerable interest to determine the motion of a body in general relativity in the "extreme mass ratio" limit, taking into account the deviations from geodesic motion arising from gravitational self-force ("radiation reaction") effects. There is a general consensus that the gravitational self-force is given by the "MiSaTaQuWa equations": In the absence of "incoming radiation", the deviation from geodesic motion is given by

$$u^\nu \nabla_\nu u^\mu = -u^\rho u^\lambda \left(\nabla_\rho h^{\text{tail}\,\mu}_{\lambda} - \frac{1}{2} \nabla^\mu h^{\text{tail}}_{\rho\lambda} \right)$$

where

$$h^{\rm tail}_{\mu\nu}(\tau) \equiv \int_{-\infty}^{\tau-} \left(G_{+\mu\nu\mu'\nu'} - \frac{1}{2} g_{\mu\nu} G_{+\ \ \alpha\mu'\nu'}^{\ \alpha} \right) u^{\mu'} u^{\nu'} d\tau' \ .$$

where G_+ is the retarded Green's function for the wave operator $\nabla^\alpha \nabla_\alpha \tilde{h}_{\mu\nu} - 2R^\alpha{}_{\mu\nu}{}^\beta \tilde{h}_{\alpha\beta}$. (Note that only the part of G_+ interior to the light cone contributes to $h^{\rm tail}_{\mu\nu}$.)

However, all derivations contain some unsatisfactory features. This is not surprising in view of the fact that "point particles" do not make sense in nonlinear theories like general relativity!

- Derivations that treat the body as a point particle require unjustified "regularizations".

- Derivations using matched asymptotic expansions involve make a number of ad hoc and/or unjustified assumptions.
- The axioms of the Quinn-Wald axiomatic approach have not been shown to follow from Einstein's equation.
- All of the above derivations employ at some stage a "phoney" version of the linearized Einstein equation with a point particle source, wherein the Lorenz gauge version of the linearized Einstein equation is written down, but the Lorenz gauge condition is not imposed.

How Should Gravitational Self-Force Be Derived?

A precise formula for gravitational self-force can hold only in a limit where the size, R, of the body goes to zero. Since "point-particles" (of non-zero mass) do not make sense in general relativity—collapse to a black hole would occur before a point-particle limit could be taken—the mass, M, of the body must also go to zero as $R \to 0$. In the limit as $R, M \to 0$, the worldtube of the body should approach a curve, γ, which should be a geodesic of the "background metric". The self-force should arise as the lowest order in M correction to γ.

This suggests that we consider a one-parameter family of solutions to Einstein's equation, $(g_{ab}(\lambda), T_{ab}(\lambda))$, with

$R(\lambda) \to 0$ and $M(\lambda) \to 0$ as $\lambda \to 0$.

But, what conditions should be imposed on $(g_{ab}(\lambda), T_{ab}(\lambda))$ to ensure that it corresponds to a body that is shrinking down to zero size, but is not undergoing wild oscillations, drastically changing its shape, or doing other crazy things as it does so?

Limits of Spacetimes

As a very simple, explicit example, consider a one-parameter family of Schwarzschild-deSitter metrics with $M = \lambda$

$$ds^2(\lambda) = - (1 - \frac{2\lambda}{r} - Cr^2)dt^2 + (1 - \frac{2\lambda}{r} - Cr^2)^{-1}dr^2 + r^2 d\Omega^2$$

If we take the limit as $\lambda \to 0$ at fixed coordinates (t, r, θ, ϕ) with $r > 0$, it is easily seen that we obtain the deSitter metric—with the deSitter spacetime worldline γ defined by $r = 0$ corresponding to the location of the black hole "before it disappeared".

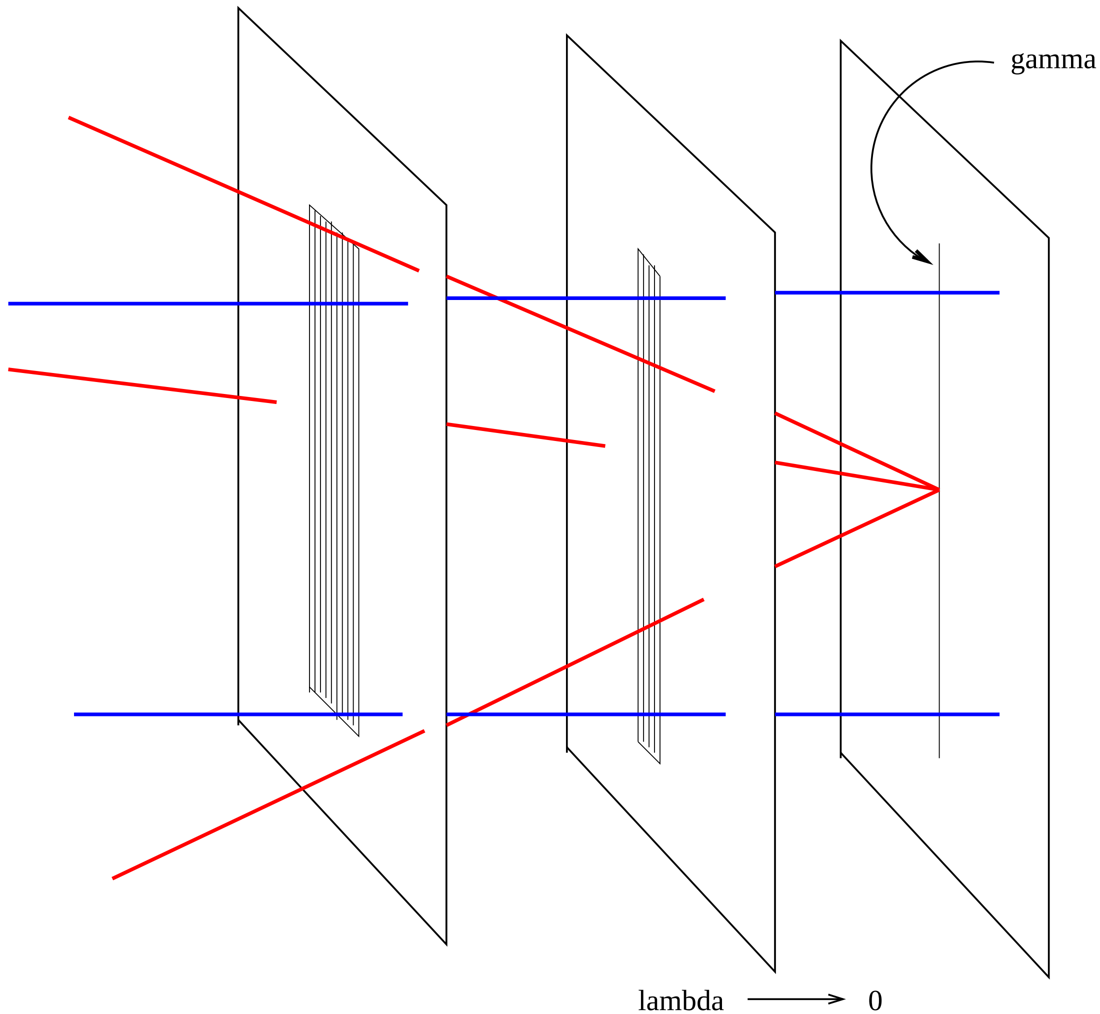
However, there is also another limit that can be taken. At each time t_0, can "blow up" the metric $g_{ab}(\lambda)$ by multiplying it by λ^{-2}, i.e., define $\bar{g}_{ab}(\lambda) = \lambda^{-2} g_{ab}(\lambda)$. Correspondingly rescale the coordinates by defining $\bar{r} = r/\lambda$, $\bar{t} = (t - t_0)/\lambda$. Then

$$\begin{aligned} d\bar{s}^2(\lambda) = \; & - \; (1 - 2/\bar{r} - \lambda^2 C \bar{r}^2) d\bar{t}^2 \\ & + \; (1 - 2/\bar{r} - \lambda^2 C \bar{r}^2)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2 \end{aligned}$$

In the limit as $\lambda \to 0$ (at fixed $(\bar{t}, \bar{r}, \theta, \phi)$) the "deSitter background" becomes irrelevant. The limiting metric is simply the Schwarzschild metric of unit mass. The fact that the limit as $\lambda \to 0$ exists can be attributed to the fact that the Schwarzschild black hole is shrinking to zero in a manner where, in essence, nothing changes except the overall scale.

The simultaneous existence of both types of limits contains a great deal of information about the one-parameter family of spacetimes $g_{\mu\nu}(\lambda)$.

Illustration of the Two Types of Limits



Our Basic Assumptions

We consider a one parameter family of solutions $g_{ab}(\lambda)$ satisfying the following properties:

- (i) Existence of the "ordinary limit": $g_{ab}(\lambda)$ is such that there exists coordinates x^α such that $g_{\mu\nu}(\lambda, x^\alpha)$ is jointly smooth in (λ, x^α), at least for $r > \bar{R}\lambda$ for some constant $\bar{R}$, where $r \equiv \sqrt{\sum (x^i)^2}$ $(i = 1, 2, 3)$. For all λ and for $r > \bar{R}\lambda$, $g_{ab}(\lambda)$ is a vacuum solution of Einstein's equation. Furthermore, $g_{\mu\nu}(\lambda = 0, x^\alpha)$ is smooth in x^α, including at $r = 0$, and, for $\lambda = 0$, the curve γ defined by $r = 0$ is timelike.
- (ii) Existence of the "scaled limit": For each t_0, we

define $\bar{t} \equiv (t - t_0)/\lambda$, $\bar{x}^i \equiv x^i/\lambda$. Then the metric $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha) \equiv \lambda^{-2} g_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha)$ is jointly smooth in $(\lambda, t_0; \bar{x}^\alpha)$ for $\bar{r} \equiv r/\lambda > \bar{R}$.

An Additional Uniformity Requirement

We have

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{t}, \bar{x}^i) = g_{\mu\nu}(\lambda; t_0 + \lambda\bar{t}, \lambda\bar{x}^i) \, .$$

Introduce new variables $\alpha \equiv r$ and $\beta \equiv \lambda/r = 1/\bar{r}$. Let f denote a component of $g_{ab}(\lambda)$ in the coordinates x^α. However, instead of viewing f as a function of (λ, x^α), we view f as a function of $(\alpha, \beta, t, \theta, \phi)$, where θ and ϕ are defined in terms of x^i by the usual formula for spherical polar angles, i.e.,

$$f(\alpha, \beta) = g_{\mu\nu}(\alpha\beta, t_0; \alpha, \theta, \phi) = \bar{g}_{\bar{\mu}\bar{\nu}}(\alpha\beta, t_0; \bar{t} = 0, 1/\beta, \theta, \phi) \, .$$

Then, by assumption (ii) we see that for $0 < \beta < 1/\bar{R}$, f

is smooth in (α, β) for all α *including* $\alpha = 0$. By assumption (i), we see that for all $\alpha > 0$, f is smooth in (α, β) for $\beta < 1/\bar{R}$, *including* $\beta = 0$. Furthermore, for $\beta = 0$, f is smooth in α, *including* $\alpha = 0$.

We now impose the additional uniformity requirement on our one-parameter family of spacetimes: f is jointly smooth in (α, β) at $(0, 0)$. We already know from our previous assumptions that $g_{\mu\nu}(\lambda; t_0, r, \theta, \phi)$ and its derivatives with respect to x^α approach a limit if we let $\lambda \to 0$ at fixed r and then let $r \to 0$. The uniformity requirement implies that the same limits are attained whenever λ and r both go to zero in any way such that λ/r goes to zero.

The uniformity requirement implies that in a neighborhood of $(\alpha, \beta) = (0, 0)$ (with $\alpha, \beta \geq 0$), we can uniformly approximate f by a series in α and β. This means that we can approximate $g_{\mu\nu}$ by a series in r and λ/r, i.e., we have

$$g_{\mu\nu}(\lambda; t, r, \theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} r^n \left(\frac{\lambda}{r}\right)^m (a_{\mu\nu})_{nm}(t, \theta, \phi)$$

This yields a far zone expansion. Equivalently, we have

$$\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{t}, \bar{r}, \theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} (\lambda \bar{r})^n \left(\frac{1}{\bar{r}}\right)^m (a_{\mu\nu})_{nm}(t_0 + \lambda \bar{t}, \theta, \phi)$$

Taylor expanding this formula with respect to the time variable yields a near zone expansion.

Since we can express $\bar{g}_{\bar{\mu}\bar{\nu}}$ at $\lambda = 0$ as a series in $1/\bar{r}$ as $\bar{r} \longrightarrow \infty$ and since $\bar{g}_{\bar{\mu}\bar{\nu}}$ at $\lambda = 0$ does not depend on $\bar{t}$, we see that $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda = 0)$ is an asymptotically flat spacetime.

Geodesic Motion

The far zone expansion tells us that

$$g_{\mu\nu}(\lambda; t, r, \theta, \phi) = \sum_{n,m} (a_{\mu\nu})_{nm}(t, \theta, \phi) r^n \left(\frac{\lambda}{r}\right)^m$$

We choose the coordinates x^α so that at $\lambda = 0$ they correspond to Fermi normal coordinates about the worldline γ. It follows that near γ (i.e., for small r) the metric $g_{\mu\nu}$ must take the form

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r) + \lambda \left(\frac{C_{\mu\nu}(t, \theta, \phi)}{r} + O(1) \right) + O(\lambda^2)$$

Now, for $r > 0$, the coefficient of λ, namely

$$\gamma_{\mu\nu} = \frac{C_{\mu\nu}}{r} + O(1)$$

must satisfy the vacuum linearized Einstein equation off of the background spacetime $g_{\mu\nu}(\lambda = 0)$. However, since each component of $\gamma_{\mu\nu}$ is a locally L^1 function, it follows immediately that $\gamma_{\mu\nu}$ is well defined as a distribution. It is not difficult to show that, as a distribution, $\gamma_{\mu\nu}$ satisfies the linearized Einstein equation with source of the form $N_{\mu\nu}(t)\delta^{(3)}(x^i)$, where $N_{\mu\nu}$ is given by a formula involving the limit as $r \to 0$ of the angular average of $C_{\mu\nu}$ and its first derivative. The linearized Bianchi identity then immediately implies that (i) $N_{\mu\nu}$ is of the form $M(t)u_\mu u_\nu$, where u^μ is the unit tangent to γ; (ii) if $M \neq 0$, then γ is a geodesic; and (iii) M is independent of t.