Post-Newtonian calculation of the gravitational self-force for black hole binaries

Alexandre Le Tiec

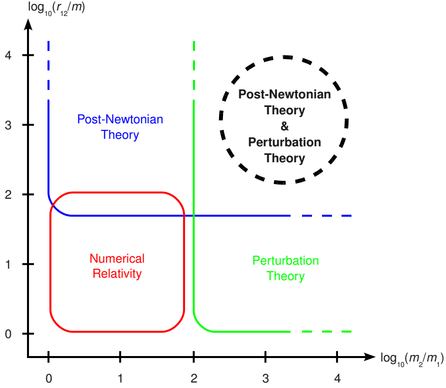
Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

Capra 12 – June 15, 2009

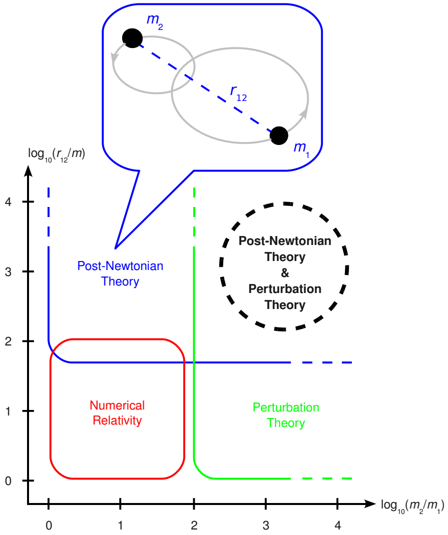
Based on a collaboration with L. Blanchet,
S. Detweiler and B. F. Whiting

Outline

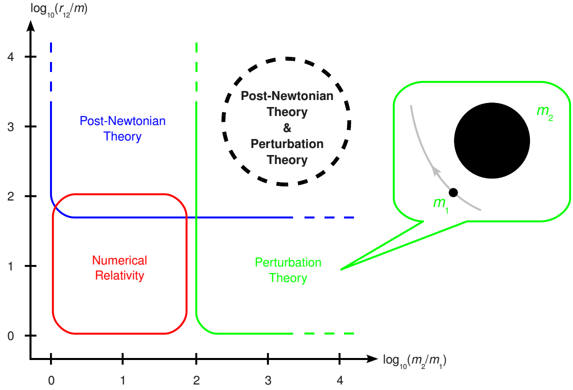
Motivation of this work



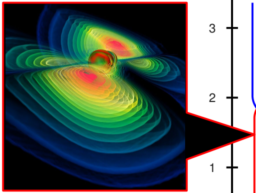
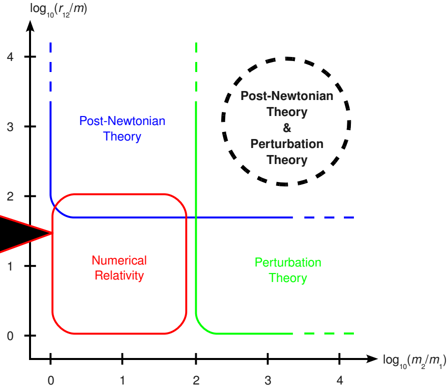
Motivation of this work



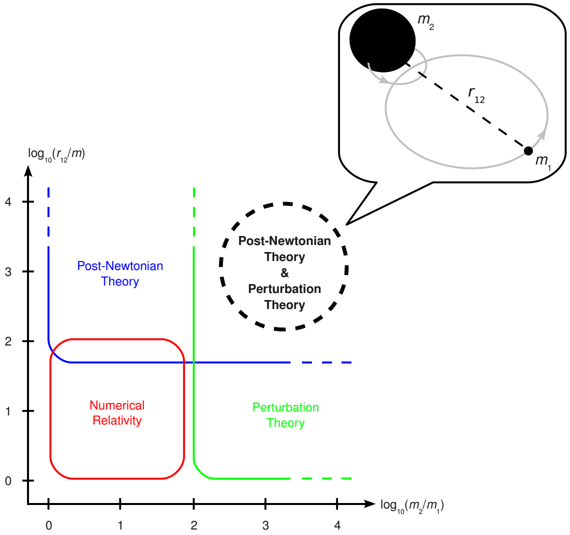
Motivation of this work



Motivation of this work

Motivation of this work



Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)

Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)
  - 1.5PN correction to $\mathcal{F}_{\mathsf{N}}$ [Poisson 93]

Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)
  - 1.5PN correction to $\mathcal{F}_{\mathrm{N}}$ [Poisson 93]
  - 4PN correction to $\mathcal{F}_{\mathrm{N}}$ [Tagoshi & Sasaki 94]

Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)
  - 1.5PN correction to $\mathcal{F}_{\mathrm{N}}$ [Poisson 93]
  - 4PN correction to $\mathcal{F}_{\mathrm{N}}$ [Tagoshi & Sasaki 94]
  - 5.5PN correction to $\mathcal{F}_{\mathrm{N}}$ [Tanaka, Tagoshi & Sasaki 96]

Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)
  - 1.5PN correction to $\mathcal{F}_{\mathrm{N}}$ [Poisson 93]
  - 4PN correction to $\mathcal{F}_{\mathrm{N}}$ [Tagoshi & Sasaki 94]
  - 5.5PN correction to $\mathcal{F}_{\mathrm{N}}$ [Tanaka, Tagoshi & Sasaki 96]
- Similar work with point particle orbiting Kerr BH

Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)
  - 1.5PN correction to $\mathcal{F}_{\rm N}$ [Poisson 93]
  - 4PN correction to $\mathcal{F}_{\rm N}$ [Tagoshi & Sasaki 94]
  - 5.5PN correction to $\mathcal{F}_{\rm N}$ [Tanaka, Tagoshi & Sasaki 96]
- Similar work with point particle orbiting Kerr BH
- Correction from finite mass of the particle on its motion (self-force effect) compared to 2PN prediction [Detweiler 08]

Motivation of this work

Previous work at the interface BH pert. theory/PN theory

- Point particle orbiting Schwarzschild BH (geodesic motion)
  - 1.5PN correction to $\mathcal{F}_{\rm N}$ [Poisson 93]
  - 4PN correction to $\mathcal{F}_{\rm N}$ [Tagoshi & Sasaki 94]
  - 5.5PN correction to $\mathcal{F}_{\rm N}$ [Tanaka, Tagoshi & Sasaki 96]
- Similar work with point particle orbiting Kerr BH
- Correction from finite mass of the particle on its motion (self-force effect) compared to 2PN prediction [Detweiler 08]

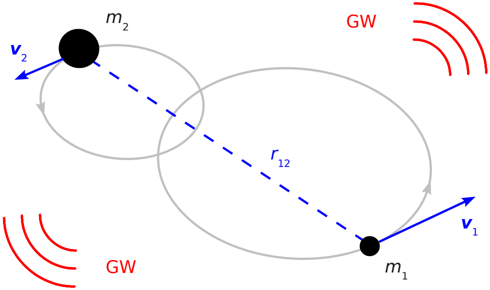
Compare a similar self-force calculation to the **third** post-Newtonian prediction

Post-Newtonian (PN) theory

Perturbation parameter

$$\varepsilon_{\mathrm{pn}} \sim \frac{v_{12}^2}{c^2} \sim \frac{Gm}{r_{12}c^2} \ll 1$$

- Small velocity
- Weak field
- Weak stresses



Templates for inspiraling compact binaries

Conservative part of the equations of motion yields

$$E = \text{binary's center of mass energy}$$

Wave generation formalism yields

$$\mathcal{F} = \text{binary's flux of gravitational radiation}$$

Gravitational wave phase ϕ follows from energy balance

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \quad \Longrightarrow \quad \phi = -\int \frac{E'(\omega)}{\mathcal{F}(\omega)} \omega \, \mathrm{d}\omega$$

Templates for inspiraling compact binaries

Conservative part of the equations of motion yields

$$E = \text{binary's center of mass energy}$$

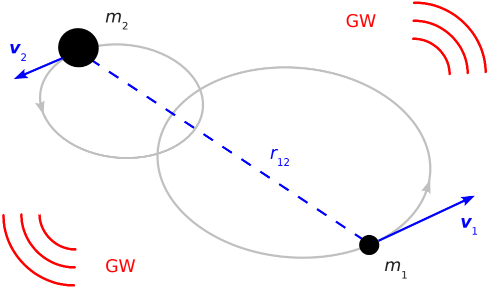
Wave generation formalism yields

$$\mathcal{F} = \text{binary's flux of gravitational radiation}$$

Gravitational wave phase ϕ follows from energy balance

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \quad \Longrightarrow \quad \phi = -\int \frac{E'(\omega)}{\mathcal{F}(\omega)} \omega \, \mathrm{d}\omega$$

Equations of motion of compact binaries



$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = \underbrace{-\frac{Gm_2}{r^2}\mathbf{n} + \frac{\mathbf{A}_{1PN}}{c^2} + \frac{\mathbf{A}_{2PN}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\mathbf{A}_{2.5PN}}{c^5}}_{\text{rad. reac.}} + \underbrace{\frac{\mathbf{A}_{3PN}}{c^6}}_{\text{cons. term}} + \underbrace{\frac{\mathbf{A}_{3.5PN}}{c^7}}_{\text{rad. reac.}} + \cdots$$

3PN equations of motion of compact binaries

- Hamiltonian approach in ADM coordinates + point particles + Hadamard regularization [Jaranowski & Schäfer 98]
- Iteration Einstein equations in harmonic coordinates + point particles + Hadamard regularization [Blanchet & Faye 00]
- Surface integral approach *à la* EIH [Futamase & Itoh 03]

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r^2}\mathbf{n} + \frac{\mathbf{A}_{1\mathrm{PN}}}{c^2} + \frac{\mathbf{A}_{2\mathrm{PN}}}{c^4} + \frac{\mathbf{A}_{2.5\mathrm{PN}}}{c^5} + \frac{\mathbf{A}_{3\mathrm{PN}}}{c^6} + \frac{\mathbf{A}_{3.5\mathrm{PN}}}{c^7} + \cdots$$

3PN equations of motion of compact binaries

- Hamiltonian approach in ADM coordinates + point particles + Hadamard regularization [Jaranowski & Schäfer 98]
- Iteration Einstein equations in harmonic coordinates + point particles + Hadamard regularization [Blanchet & Faye 00]
- Surface integral approach *à la* EIH [Futamase & Itoh 03]

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r^2}\mathbf{n} + \frac{\mathbf{A}_{1PN}}{c^2} + \frac{\mathbf{A}_{2PN}}{c^4} + \frac{\mathbf{A}_{2.5PN}}{c^5} + \frac{\mathbf{A}_{3PN}(\lambda)}{c^6} + \frac{\mathbf{A}_{3.5PN}}{c^7} + \cdots$$

- One remaining regularization ambiguity parameter λ

3PN equations of motion of compact binaries

- Hamiltonian approach in ADM coordinates + point particles + Hadamard regularization [Jaranowski & Schäfer 98]
- Iteration Einstein equations in harmonic coordinates + point particles + Hadamard regularization [Blanchet & Faye 00]
- Surface integral approach *à la* EIH [Futamase & Itoh 03]

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r^2}\mathbf{n} + \frac{\mathbf{A}_{1\text{PN}}}{c^2} + \frac{\mathbf{A}_{2\text{PN}}}{c^4} + \frac{\mathbf{A}_{2.5\text{PN}}}{c^5} + \frac{\mathbf{A}_{3\text{PN}}(\lambda)}{c^6} + \frac{\mathbf{A}_{3.5\text{PN}}}{c^7} + \cdots$$

- One remaining regularization ambiguity parameter λ
- Later determined using dimensional regularization $\left(\lambda = -\frac{1987}{3080}\right)$ [Blanchet, Damour & Esposito-Farèse 04]

3PN equations of motion of compact binaries

- Hamiltonian approach in ADM coordinates + point particles + Hadamard regularization [Jaranowski & Schäfer 98]
- Iteration Einstein equations in harmonic coordinates + point particles + Hadamard regularization [Blanchet & Faye 00]
- Surface integral approach *à la* EIH [Futamase & Itoh 03]

$$\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r^2}\mathbf{n} + \frac{\mathbf{A}_{1PN}}{c^2} + \frac{\mathbf{A}_{2PN}}{c^4} + \frac{\mathbf{A}_{2.5PN}}{c^5} + \frac{\mathbf{A}_{3PN}(\lambda)}{c^6} + \frac{\mathbf{A}_{3.5PN}}{c^7} + \cdots$$

- One remaining regularization ambiguity parameter λ
- Later determined using dimensional regularization $\left(\lambda = -\frac{1987}{3080}\right)$ [Blanchet, Damour & Esposito-Farèse 04]
- As a by product of our PN calculations we shall give an independant confirmation of λ

How to make a meaningful comparison?

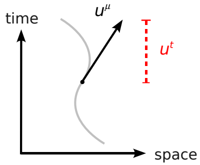
How to make a meaningful comparison?

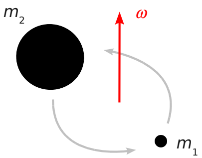
Compute an (almost) gauge-invariant relation

How to make a meaningful comparison?

Compute an (almost) gauge-invariant relation

Time component u^t of the 4-velocity u^μ of the small body, for circular orbits, in the SF limit, as function of the relative angular frequency ω

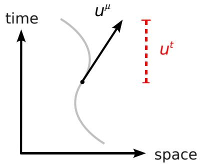




How to make a meaningful comparison?

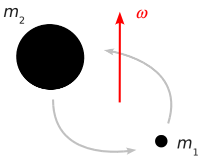
Compute an (almost) gauge-invariant relation

Time component u^t of the 4-velocity u^μ of the small body, for circular orbits, in the SF limit, as function of the relative angular frequency ω



Self-force calculation

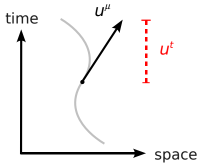
$$u^t = \underbrace{{}_0u^t}_{\text{Schw.}}\left(1 + \frac{1}{2}\bar{u}^\mu \bar{u}^\nu \underbrace{h^{\text{reg}}_{\mu\nu}}_{\text{reg. pert.}}\right)$$



How to make a meaningful comparison?

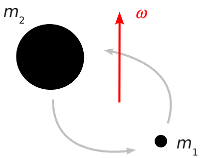
Compute an (almost) gauge-invariant relation

Time component u^t of the 4-velocity u^μ of the small body, for circular orbits, in the SF limit, as function of the relative angular frequency ω



Self-force calculation

$$u^t = \underbrace{{}_0u^t}_{\text{Schw.}}\left(1 + \frac{1}{2}\bar{u}^\mu \bar{u}^\nu \underbrace{h^{\text{reg}}_{\mu\nu}}_{\text{reg. pert.}}\right)$$



Third post-Newtonian calculation

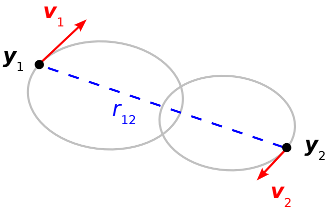
$$u^t = 1 + \sum_{n=1}^{4} a_n(q)\, y^n \quad \text{where} \quad y \equiv \left(\frac{G m_2 \omega}{c^3}\right)^{2/3} \sim \varepsilon_{\text{pn}}$$

Post-Newtonian calculation of u^t

$$u^t = \frac{1}{\sqrt{-\mathrm{Reg}_1\left[g_{\mu\nu}\right] v_1^\mu v_1^\nu}}$$

↑

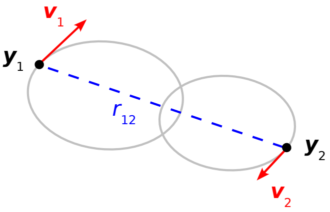
regularized metric at $\mathbf{y}_1$



Post-Newtonian calculation of u^t

$$u^t = \frac{1}{\sqrt{-\mathrm{Reg}_1\left[g_{\mu\nu}\right] v_1^\mu v_1^\nu}}$$

↑
regularized metric at $\mathbf{y}_1$

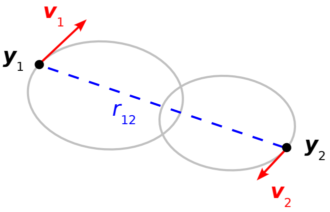


Self-field regularization methods

Post-Newtonian calculation of u^t

$$u^t = \frac{1}{\sqrt{-\mathrm{Reg}_1\left[g_{\mu\nu}\right] v_1^\mu v_1^\nu}}$$

↑
regularized metric at $\mathbf{y}_1$



Self-field regularization methods

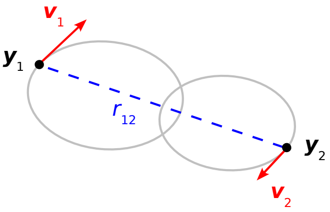
- Had. reg. will yield an ambiguity at 3PN order

Post-Newtonian calculation of u^t

$$u^t = \frac{1}{\sqrt{-\mathrm{Reg}_1\left[g_{\mu\nu}\right] v_1^\mu v_1^\nu}}$$

↑

regularized metric at $\mathbf{y}_1$



Self-field regularization methods

- Had. reg. will yield an ambiguity at 3PN order
- Dim. reg. will be free of any ambiguity at 3PN order

Near-zone post-Newtonian metric

- Iteration Einstein equations in harmonic coordinates yields near-zone metric as 3PN expansion:

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \cdots$$

$$g_{0i} = -\frac{4V^i}{c^3} + \cdots$$

$$g_{ij} = \delta^{ij}\left(1 + \frac{2V}{c^2} + \cdots\right) + \cdots$$

Near-zone post-Newtonian metric

- Iteration Einstein equations in harmonic coordinates yields near-zone metric as 3PN expansion:

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \cdots$$

$$g_{0i} = -\frac{4V^i}{c^3} + \cdots$$

$$g_{ij} = \delta^{ij}\left(1 + \frac{2V}{c^2} + \cdots\right) + \cdots$$

- Elementary potentials $F = \{V, V^i, \cdots\}$ satisfy d'Alembert equations with sources:

$$\Box V = -4\pi G\,\sigma$$

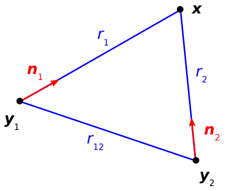
$$\Box V^i = -4\pi G\,\sigma^i$$

$$\vdots$$

Hadamard regularization

- Any potential F is expanded around $r_1 \longrightarrow 0$ as

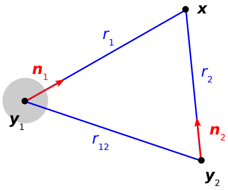
$$F(\mathbf{x}) = \sum_{p \geqslant -p_0} r_1^p f_p(\mathbf{n}_1)$$



Hadamard regularization

- Any potential F is expanded around $r_1 \longrightarrow 0$ as

$$F(\mathbf{x}) = \sum_{p \geqslant -p_0} r_1^p f_p(\mathbf{n}_1)$$



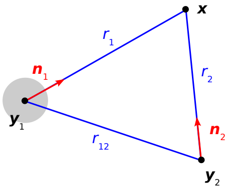
- The Hadamard regularization (HR) of F is given by

$$(F)_1 = \int \frac{\mathrm{d}\Omega}{4\pi} f_0(\mathbf{n}_1)$$

Hadamard regularization

- Any potential F is expanded around $r_1 \to 0$ as

$$F(\mathbf{x}) = \sum_{p \geqslant -p_0} r_1^p f_p(\mathbf{n}_1)$$



- The Hadamard regularization (HR) of F is given by

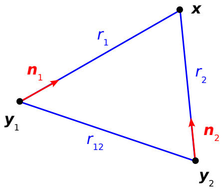
$$(F)_1 = \int \frac{\mathrm{d}\Omega}{4\pi} f_0(\mathbf{n}_1)$$

- The ambiguity parameter at 3PN is related to the non-distributivity of the HR:

$$(FG)_1 \neq (F)_1 (G)_1$$

Dimensional regularization

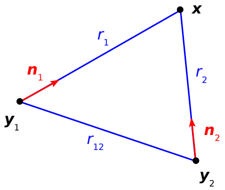
- Work in $d = 3 + \varepsilon$ spatial dimensions



Dimensional regularization

- Work in $d = 3 + \varepsilon$ spatial dimensions
- Any potential $F^{(d)}$ is expanded around $r_1 \longrightarrow 0$ as

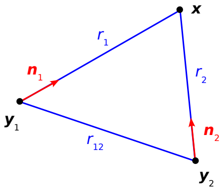
$$F^{(d)}(\mathbf{x}) = \sum_{p \geqslant -p_0, q} r_1^{p+\varepsilon q} f_{p,q}^{(\varepsilon)}(\mathbf{n}_1)$$



Dimensional regularization

- Work in $d = 3 + \varepsilon$ spatial dimensions
- Any potential $F^{(d)}$ is expanded around $r_1 \to 0$ as

$$F^{(d)}(\mathbf{x}) = \sum_{p \geqslant -p_0, q} r_1^{p+\varepsilon q} f_{p,q}^{(\varepsilon)}(\mathbf{n}_1)$$



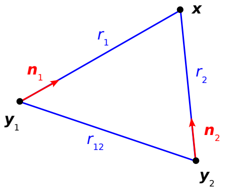
- The dimensional regularisation (DR) of $F^{(d)}$ is given by

$$F^{(d)}(\mathbf{y}_1) = f_{0,0}^{(\varepsilon)}$$

Dimensional regularization

- Work in $d = 3 + \varepsilon$ spatial dimensions
- Any potential $F^{(d)}$ is expanded around $r_1 \rightarrow 0$ as

$$F^{(d)}(\mathbf{x}) = \sum_{p \geqslant -p_0, q} r_1^{p+\varepsilon q} f_{p,q}^{(\varepsilon)}(\mathbf{n}_1)$$



- The dimensional regularisation (DR) of $F^{(d)}$ is given by

$$F^{(d)}(\mathbf{y}_1) = f_{0,0}^{(\varepsilon)}$$

- We observe that $f_{0,0}^{(\varepsilon)}$ does not depend on $\mathbf{n}_1$ at 3PN, which indicates that the DR calculation will be free from ambiguities

Dimensional regularization : a simple example

- Time component of the metric in $d = 3$ spatial dimensions

$$g_{00}(\mathbf{x}) = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} + \cdots$$

Dimensional regularization : a simple example

- Time component of the metric in $d = 3$ spatial dimensions

$$g_{00}(\mathbf{x}) = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} + \cdots$$

- Not defined at the location $\mathbf{y}_1$ in the limit $r_1 \to 0$

Dimensional regularization : a simple example

- Time component of the metric in $d = 3$ spatial dimensions

$$g_{00}(\mathbf{x}) = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} + \cdots$$

- Not defined at the location $\mathbf{y}_1$ in the limit $r_1 \to 0$
- Work in $d = 3 + \varepsilon$ spatial dimensions

$$g_{00}^{(d)}(\mathbf{x}) = -1 + \frac{2KG^{(d)}m_1}{c^2 r_1^{d-2}} + \frac{2KG^{(d)}m_2}{c^2 r_2^{d-2}} + \cdots$$

Dimensional regularization : a simple example

- Time component of the metric in $d = 3$ spatial dimensions

$$g_{00}(\mathbf{x}) = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} + \cdots$$

- Not defined at the location $\mathbf{y}_1$ in the limit $r_1 \to 0$
- Work in $d = 3 + \varepsilon$ spatial dimensions

$$g_{00}^{(d)}(\mathbf{x}) = -1 + \frac{2KG^{(d)}m_1}{c^2 r_1^{d-2}} + \frac{2KG^{(d)}m_2}{c^2 r_2^{d-2}} + \cdots$$

- Choose $d < 2$ such that $g_{00}^{(d)}$ is defined in the limit $r_1 \to 0$

Dimensional regularization : a simple example

- Time component of the metric in $d = 3$ spatial dimensions

$$g_{00}(\mathbf{x}) = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} + \cdots$$

- Not defined at the location $\mathbf{y}_1$ in the limit $r_1 \to 0$
- Work in $d = 3 + \varepsilon$ spatial dimensions

$$g^{(d)}_{00}(\mathbf{x}) = -1 + \frac{2KG^{(d)} m_1}{c^2 r_1^{d-2}} + \frac{2KG^{(d)} m_2}{c^2 r_2^{d-2}} + \cdots$$

- Choose $d < 2$ such that $g^{(d)}_{00}$ is defined in the limit $r_1 \to 0$
- From the principle of analytic continuation (AC), the result is

$$g^{(d)}_{00}(\mathbf{y}_1) = \mathrm{AC}\Big[\lim_{\mathbf{x}\to\mathbf{y}_1} g^{(d)}_{00}(\mathbf{x})\Big] = -1 + \frac{2Gm_2}{c^2 r_{12}} + \mathcal{O}(\varepsilon) + \cdots$$

Difference between Hadamard and dimensional regularizations

- We do not compute the full 3PN metric $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$ within DR

Difference between Hadamard and dimensional regularizations

- We do not compute the full 3PN metric $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$ within DR
- Instead we compute for each potential F the difference

$$\mathcal{D}F(\mathbf{y}_1) = \underbrace{F^{(d)}(\mathbf{y}_1)}_{\text{dim. reg.}} - \underbrace{(F)_1}_{\text{Had. reg.}}$$

Difference between Hadamard and dimensional regularizations

- We do not compute the full 3PN metric $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$ within DR
- Instead we compute for each potential F the difference

$$\mathcal{D}F(\mathbf{y}_1) = \underbrace{F^{(d)}(\mathbf{y}_1)}_{\text{dim. reg.}} - \underbrace{(F)_1}_{\text{Had. reg.}}$$

- When $\varepsilon \to 0$ that difference is typically of the form

$$\mathcal{D}F(\mathbf{y}_1) = \underbrace{\frac{a_{-1}}{\varepsilon}}_{\text{pole part}} + \underbrace{a_0}_{\text{finite part}} + \mathcal{O}(\varepsilon)$$

Difference between Hadamard and dimensional regularizations

- We do not compute the full 3PN metric $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$ within DR
- Instead we compute for each potential F the difference

$$\mathcal{D}F(\mathbf{y}_1) = \underbrace{F^{(d)}(\mathbf{y}_1)}_{\text{dim. reg.}} - \underbrace{(F)_1}_{\text{Had. reg.}}$$

- When $\varepsilon \to 0$ that difference is typically of the form

$$\mathcal{D}F(\mathbf{y}_1) = \underbrace{\frac{a_{-1}}{\varepsilon}}_{\text{pole part}} + \underbrace{a_0}_{\text{finite part}} + \mathcal{O}(\varepsilon)$$

- Ambiguities within Hadamard regularization are associated with the occurence of poles $\propto \varepsilon^{-1}$ at 3PN order

Comparison of 3PN regularized metrics

- A calculation using Hadamard regularization yields the ambiguous result $(g_{\mu\nu})_1$

Comparison of 3PN regularized metrics

- A calculation using Hadamard regularization yields the ambiguous result $(g_{\mu\nu})_1$
- A calculation using dimensional regularization yields the non-ambiguous result $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$

Comparison of 3PN regularized metrics

- A calculation using Hadamard regularization yields the ambiguous result $(g_{\mu\nu})_1$
- A calculation using dimensional regularization yields the non-ambiguous result $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$
- The two regularized metrics are physically equivalent:

$$g^{(d)}_{\mu\nu}(\mathbf{y}_1) = (g_{\mu\nu})_1 + \big(\text{3PN gauge transformation } \epsilon^\mu\big) + \big(\text{additional word-line shifts } \boldsymbol{\kappa}_A\big)$$

iff the Hadamard ambiguity has a certain value $\lambda' = \frac{129}{440}$

Comparison of 3PN regularized metrics

- A calculation using Hadamard regularization yields the ambiguous result $(g_{\mu\nu})_1$
- A calculation using dimensional regularization yields the non-ambiguous result $g^{(d)}_{\mu\nu}(\mathbf{y}_1)$
- The two regularized metrics are physically equivalent:

$$g^{(d)}_{\mu\nu}(\mathbf{y}_1) = (g_{\mu\nu})_1 + \big(\text{3PN gauge transformation } \epsilon^\mu\big) + \big(\text{additional word-line shifts } \boldsymbol{\kappa}_A\big)$$

  iff the Hadamard ambiguity has a certain value $\lambda' = \frac{129}{440}$
- The fact that $\lambda' \neq \lambda$ confirms that Hadamard regularization is not entirely satisfactory at 3PN order

The dimensionally regularized 3PN metric

$$g_{00}^{(d)}(\mathbf{y}_1) = -1 + \frac{2Gm_2}{c^2 r_{12}}$$

The dimensionally regularized 3PN metric

$$g_{00}^{(d)}(\mathbf{y}_1) = -1 + \frac{2Gm_2}{c^2 r_{12}} + \frac{Gm_2}{c^4 r_{12}}\left[4v_2^2 - (n_{12}v_2)^2 - 3\frac{Gm_1}{r_{12}} - 2\frac{Gm_2}{r_{12}}\right]$$

The dimensionally regularized 3PN metric

$$
\begin{aligned}
g_{00}^{(d)}(\mathbf{y}_1) = & -1 + \frac{2Gm_2}{c^2 r_{12}} + \frac{Gm_2}{c^4 r_{12}}\left[4v_2^2 - (n_{12}v_2)^2 - 3\frac{Gm_1}{r_{12}} - 2\frac{Gm_2}{r_{12}}\right] \\
& + \frac{Gm_2}{c^6 r_{12}}\left[\frac{3}{4}(n_{12}v_2)^4 - 3(n_{12}v_2)^2 v_2^2 + 4v_2^4 + \frac{Gm_2}{r_{12}}\left(3(n_{12}v_2)^2\right.\right. \\
& \left. - v_2^2 + 2\frac{Gm_2}{r_{12}}\right) + \frac{Gm_1}{r_{12}}\left(-\frac{87}{4}(n_{12}v_1)^2 + \frac{47}{2}(n_{12}v_1)(n_{12}v_2)\right. \\
& \left.\left. - \frac{55}{4}(n_{12}v_2)^2 - \frac{39}{2}(v_1v_2) + \frac{23}{4}v_1^2 + \frac{47}{4}v_2^2 - \frac{Gm_1}{r_{12}} + \frac{17}{2}\frac{Gm_2}{r_{12}}\right)\right]
\end{aligned}
$$

The dimensionally regularized 3PN metric

$$
\begin{aligned}
g_{00}^{(d)}(\mathbf{y}_1) = & -1 + \frac{2Gm_2}{c^2 r_{12}} + \frac{Gm_2}{c^4 r_{12}}\left[4v_2^2 - (n_{12}v_2)^2 - 3\frac{Gm_1}{r_{12}} - 2\frac{Gm_2}{r_{12}}\right] \\
& + \frac{Gm_2}{c^6 r_{12}}\left[\frac{3}{4}(n_{12}v_2)^4 - 3(n_{12}v_2)^2 v_2^2 + 4v_2^4 + \frac{Gm_2}{r_{12}}\left(3(n_{12}v_2)^2\right.\right. \\
& \left. - v_2^2 + 2\frac{Gm_2}{r_{12}}\right) + \frac{Gm_1}{r_{12}}\left(-\frac{87}{4}(n_{12}v_1)^2 + \frac{47}{2}(n_{12}v_1)(n_{12}v_2)\right. \\
& \left.\left. - \frac{55}{4}(n_{12}v_2)^2 - \frac{39}{2}(v_1v_2) + \frac{23}{4}v_1^2 + \frac{47}{4}v_2^2 - \frac{Gm_1}{r_{12}} + \frac{17}{2}\frac{Gm_2}{r_{12}}\right)\right] \\
& + \frac{Gm_2}{c^8 r_{12}}\left[-\frac{5}{8}(n_{12}v_2)^6 - 5(n_{12}v_2)^2 v_2^4 + 3(n_{12}v_2)^4 v_2^2 + 4v_2^6\right. \\
& \left. + \cdots + \frac{G^2 m_1^2}{r_{12}^2}(n_{12}v_1)^2\left\{\frac{12021}{100} + \frac{182}{5\varepsilon} + \cdots\right\} + \cdots\right] + \cdots
\end{aligned}
$$

Result for circular orbits

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8} q^2 + \cdots\right) y^3 \\
&+ \left(\frac{2835}{128} - \mathcal{C} q + \mathcal{D} q^2 + \cdots\right) y^4 \\
&\qquad \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}
$$

Result for circular orbits

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y \qquad \longleftarrow \text{Newtonian} \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8} q^2 + \cdots\right) y^3 \\
&+ \left(\frac{2835}{128} - \mathcal{C} q + \mathcal{D} q^2 + \cdots\right) y^4 \\
&\quad\;\; \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}
$$

Result for circular orbits

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y && \longleftarrow \text{Newtonian} \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 && \longleftarrow \text{1PN} \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8}q^2 + \cdots\right) y^3 \\
&+ \left(\frac{2835}{128} - \mathcal{C}q + \mathcal{D}q^2 + \cdots\right) y^4 \\
&\quad \vdots \qquad \vdots \qquad \vdots
\end{aligned}
$$

Result for circular orbits

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y && \longleftarrow \text{Newtonian} \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 && \longleftarrow \text{1PN} \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8} q^2 + \cdots\right) y^3 && \longleftarrow \text{2PN} \\
&+ \left(\frac{2835}{128} - \mathcal{C} q + \mathcal{D} q^2 + \cdots\right) y^4 \\
&\quad\ \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}
$$

Result for circular orbits

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y && \longleftarrow \text{Newtonian} \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 && \longleftarrow \text{1PN} \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8} q^2 + \cdots\right) y^3 && \longleftarrow \text{2PN} \\
&+ \left(\frac{2835}{128} - \mathcal{C} q + \mathcal{D} q^2 + \cdots\right) y^4 && \longleftarrow \text{3PN} \\
&\quad \vdots \qquad \vdots \qquad \vdots
\end{aligned}
$$

Result for circular orbits

Schwarzschild ↘

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y && \longleftarrow \text{Newtonian} \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 && \longleftarrow \text{1PN} \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8}q^2 + \cdots\right) y^3 && \longleftarrow \text{2PN} \\
&+ \left(\frac{2835}{128} - \mathcal{C}q + \mathcal{D}q^2 + \cdots\right) y^4 && \longleftarrow \text{3PN} \\
&\quad\ \ \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}
$$

Result for circular orbits

Schwarzschild ↘ ↙ Self-force

$$u^t = 1 + \left(\frac{3}{2} - q + q^2 + \cdots\right) y \qquad \longleftarrow \text{Newtonian}$$

$$+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 \qquad \longleftarrow \text{1PN}$$

$$+ \left(\frac{135}{16} - 5q + \frac{97}{8}q^2 + \cdots\right) y^3 \qquad \longleftarrow \text{2PN}$$

$$+ \left(\frac{2835}{128} - \mathcal{C}q + \mathcal{D}q^2 + \cdots\right) y^4 \qquad \longleftarrow \text{3PN}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Result for circular orbits

Schwarzschild ↘ ↙ Self-force

$$
\begin{aligned}
u^t = 1 &+ \left(\frac{3}{2} - q + q^2 + \cdots\right) y && \longleftarrow \text{Newtonian} \\
&+ \left(\frac{27}{8} - 2q + 3q^2 + \cdots\right) y^2 && \longleftarrow \text{1PN} \\
&+ \left(\frac{135}{16} - 5q + \frac{97}{8} q^2 + \cdots\right) y^3 && \longleftarrow \text{2PN} \\
&+ \left(\frac{2835}{128} - \mathcal{C} q + \mathcal{D} q^2 + \cdots\right) y^4 && \longleftarrow \text{3PN} \\
&\quad \vdots \qquad \vdots \qquad \vdots
\end{aligned}
$$

Objective: compare $\mathcal{C}_{\mathrm{PN}}$ and $\mathcal{C}_{\mathrm{SF}}$

Result for circular orbits

- General expression of u^t for circular orbits

$$u^t = \underbrace{{}_0u^t}_{\text{Schw.}} - q \underbrace{{}_1u^t}_{\text{SF}} + q^2 \, {}_2u^t + \mathcal{O}(q^3)$$

Result for circular orbits

- General expression of u^t for circular orbits

$$u^t = \underbrace{{}_0u^t}_{\text{Schw.}} - q \underbrace{{}_1u^t}_{\text{SF}} + q^2\,{}_2u^t + \mathcal{O}(q^3)$$

- The 3PN calculation with dimensional regularization gives

$${}_1u^t = y + 2y^2 + 5y^3 + \underbrace{\left(\frac{121}{3} - \frac{41}{32}\pi^2\right)}_{\text{3PN coefficient } \mathcal{C}_{\text{PN}}} y^4 + \mathcal{O}(y^5)$$

Result for circular orbits

- General expression of u^t for circular orbits

$$u^t = \underbrace{{}_0u^t}_{\text{Schw.}} - q \underbrace{{}_1u^t}_{\text{SF}} + q^2\,{}_2u^t + \mathcal{O}(q^3)$$

- The 3PN calculation with dimensional regularization gives

$${}_1u^t = y + 2y^2 + 5y^3 + \underbrace{\left(\frac{121}{3} - \frac{41}{32}\pi^2\right)}_{\text{3PN coefficient } \mathcal{C}_{\text{PN}}} y^4 + \mathcal{O}(y^5)$$

- A 2nd order BH perturbation calculation can be compared to

$${}_2u^t = y + 3y^2 + \frac{97}{8}y^3 + \left(\frac{725}{12} - \frac{41}{64}\pi^2\right) y^4 + \mathcal{O}(y^5)$$

Comparison of the PN and SF calculations

- The 3PN calculation with dimensional regularization gives

$$\mathcal{C}_{\text{PN}} = \frac{121}{3} - \frac{41}{32}\pi^2 = 27.6879\cdots$$

Comparison of the PN and SF calculations

- The 3PN calculation with dimensional regularization gives

$$\mathcal{C}_{\text{PN}} = \frac{121}{3} - \frac{41}{32}\pi^2 = 27.6879\cdots$$

- The SF approach gives the exact value of ${}_1u^t$ (modulo the numerical error)

Comparison of the PN and SF calculations

- The 3PN calculation with dimensional regularization gives

$$\mathcal{C}_{\mathrm{PN}} = \frac{121}{3} - \frac{41}{32}\pi^2 = 27.6879\cdots$$

- The SF approach gives the exact value of ${}_1u^t$ (modulo the numerical error)
- This numerical result is fitted by a PN series

Comparison of the PN and SF calculations

- The 3PN calculation with dimensional regularization gives

$$\mathcal{C}_{\mathrm{PN}} = \frac{121}{3} - \frac{41}{32}\pi^2 = 27.6879\cdots$$

- The SF approach gives the exact value of ${}_1u^t$ (modulo the numerical error)
- This numerical result is fitted by a PN series
- The fit yields for the 3PN coefficient [Detweiler & Whiting]

$$\mathcal{C}_{\mathrm{SF}} = 27.677 \pm 0.005 + (\text{systematic effects})$$

Comparison of the PN and SF calculations

- The 3PN calculation with dimensional regularization gives

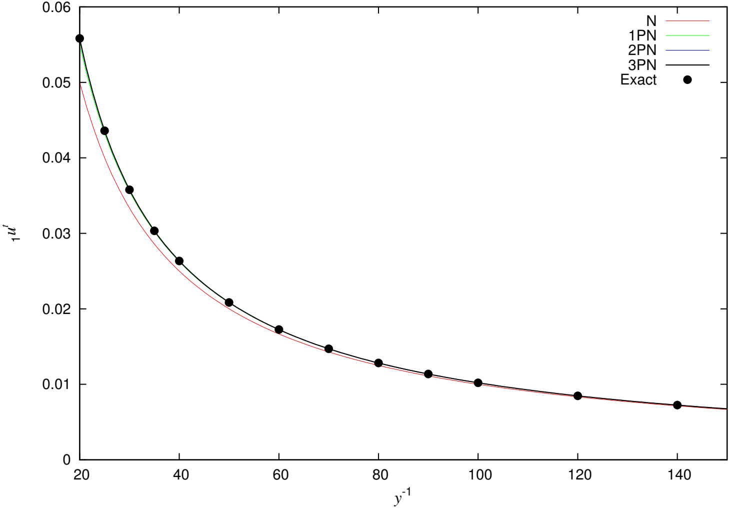
$$\mathcal{C}_{\text{PN}} = \frac{121}{3} - \frac{41}{32}\pi^2 = 27.6879\cdots$$

- The SF approach gives the exact value of ${}_1u^t$ (modulo the numerical error)
- This numerical result is fitted by a PN series
- The fit yields for the 3PN coefficient [Detweiler & Whiting]

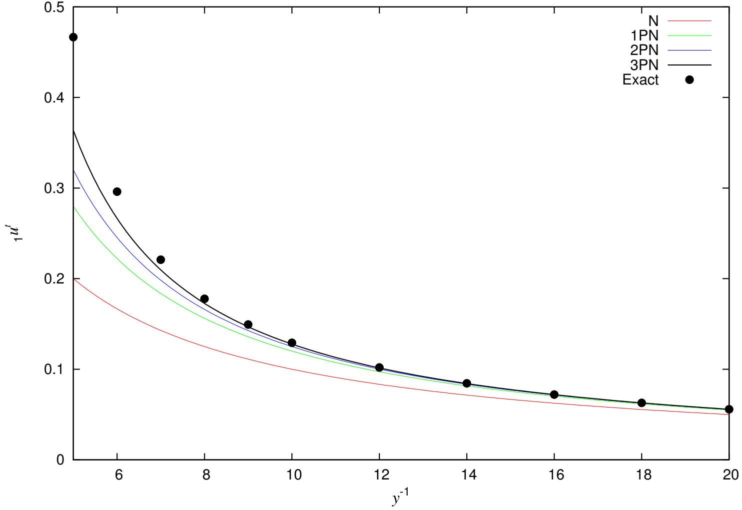
$$\mathcal{C}_{\text{SF}} = 27.677 \pm 0.005 + (\text{systematic effects})$$

- The two calculations are therefore in agreement at the 2σ level

Self-force contribution ${}_1u^t$



Self-force contribution ${}_1u^t$



Link with energy and angular momentum

- For a circular orbit in the SF limit [Detweiler 08]:

$$(e - \omega j)^{-1} = u^t \quad \text{where} \quad \begin{cases} e = \text{ene.}/m \\ j = \text{ang. mom.}/m \end{cases}$$

Link with energy and angular momentum

- For a circular orbit in the SF limit [Detweiler 08]:

$$(e - \omega j)^{-1} = u^t \quad \text{where} \quad \begin{cases} e = \text{ene.}/m \\ j = \text{ang. mom.}/m \end{cases}$$

- Does a similar relation hold for an arbitrary mass ratio q in PN theory?

Link with energy and angular momentum

- For a circular orbit in the SF limit [Detweiler 08]:

$$(e - \omega j)^{-1} = u^t \quad \text{where} \quad \begin{cases} e = \text{ene.}/m \\ j = \text{ang. mom.}/m \end{cases}$$

- Does a similar relation hold for an arbitrary mass ratio q in PN theory?
- Poincaré invariance of 3PN Lagrangian yields well-defined notions of energy and angular momentum in PN theory

Link with energy and angular momentum

- For a circular orbit in the SF limit [Detweiler 08]:

$$(e - \omega j)^{-1} = u^t \quad \text{where} \quad \begin{cases} e = \text{ene.}/m \\ j = \text{ang. mom.}/m \end{cases}$$

- Does a similar relation hold for an arbitrary mass ratio q in PN theory?
- Poincaré invariance of 3PN Lagrangian yields well-defined notions of energy and angular momentum in PN theory
- From 3PN calculations using Hadamard regularization [de Andrade, Blanchet & Faye 01]:

$$E_{\text{PN}}(\lambda) = \frac{1}{2} m_1 v_1^2 - \frac{G m_1 m_2}{2 r_{12}} + \cdots - \frac{11}{3} \frac{G^4 m_1^3 m_2^2}{c^6 r_{12}^4} \lambda + 1 \leftrightarrow 2$$

$$\mathbf{J}_{\text{PN}} = m_1 \mathbf{y}_1 \times \mathbf{v}_1 + \cdots + 1 \leftrightarrow 2$$

Link with energy and angular momentum

- A PN calculation with Hadamard regularization gives

$$\left(\frac{E_{\mathrm{PN}}}{m_1} - \omega\frac{J_{\mathrm{PN}}}{m_1}\right)^{-1} = 1 + \frac{3}{2}y + \frac{27}{8}y^2 + \frac{135}{14}y^3 + \frac{2835}{128}y^4 + \cdots$$

Link with energy and angular momentum

- A PN calculation with Hadamard regularization gives

$$\left(\frac{E_{\text{PN}}}{m_1} - \omega \frac{J_{\text{PN}}}{m_1}\right)^{-1} = 1 + \frac{3}{2}y + \frac{27}{8}y^2 + \frac{135}{14}y^3 + \frac{2835}{128}y^4 + \cdots$$
$$- \frac{1}{2}\left[y + 2y^2 + 5y^3 + \left(\frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda\right)y^4\right] q + \cdots$$

- Conjecture: we get the SF contribution up to a factor of 2

Link with energy and angular momentum

- A PN calculation with Hadamard regularization gives

$$\left(\frac{E_{\text{PN}}}{m_1} - \omega \frac{J_{\text{PN}}}{m_1}\right)^{-1} = 1 + \frac{3}{2}y + \frac{27}{8}y^2 + \frac{135}{14}y^3 + \frac{2835}{128}y^4 + \cdots$$
$$- \frac{1}{2}\left[y + 2y^2 + 5y^3 + \left(\frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda\right) y^4\right] q + \cdots$$

- Conjecture: we get the SF contribution up to a factor of 2
- Then comparing the PN and SF calculations we get:

$$\mathcal{C}_{\text{SF}} = \frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda \quad \Longrightarrow \quad \lambda = -0.6436 \pm 0.0007$$
$$+ \text{ (systematic effects)}$$

Link with energy and angular momentum

- A PN calculation with Hadamard regularization gives

$$\left(\frac{E_{\mathrm{PN}}}{m_1} - \omega \frac{J_{\mathrm{PN}}}{m_1}\right)^{-1} = 1 + \frac{3}{2}y + \frac{27}{8}y^2 + \frac{135}{14}y^3 + \frac{2835}{128}y^4 + \cdots$$
$$- \frac{1}{2}\left[y + 2y^2 + 5y^3 + \left(\frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda\right) y^4\right] q + \cdots$$

- Conjecture: we get the SF contribution up to a factor of 2
- Then comparing the PN and SF calculations we get:

$$\mathcal{C}_{\mathrm{SF}} = \frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda \quad \Longrightarrow \quad \lambda = -0.6436 \pm 0.0007$$
$$+ \text{ (systematic effects)}$$

- Recall $\lambda = -\frac{1987}{3080} = -0.6451\cdots$

Link with energy and angular momentum

- A PN calculation with Hadamard regularization gives

$$\left(\frac{E_{\mathrm{PN}}}{m_1} - \omega\frac{J_{\mathrm{PN}}}{m_1}\right)^{-1} = 1 + \frac{3}{2}y + \frac{27}{8}y^2 + \frac{135}{14}y^3 + \frac{2835}{128}y^4 + \cdots$$
$$- \frac{1}{2}\left[y + 2y^2 + 5y^3 + \left(\frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda\right)y^4\right]q + \cdots$$

- Conjecture: we get the SF contribution up to a factor of 2
- Then comparing the PN and SF calculations we get:

$$\mathcal{C}_{\mathrm{SF}} = \frac{14953}{420} - \frac{41}{32}\pi^2 - \frac{22}{3}\lambda \quad \Longrightarrow \quad \lambda = -0.6436 \pm 0.0007$$
$$+ \text{ (systematic effects)}$$

- Recall $\lambda = -\frac{1987}{3080} = -0.6451\cdots$

Beware this is not a proof!

Conclusion

- Successfull (?) comparison of the PN formalism (at the 3PN level) and the SF approach through gauge invariant variables

Conclusion

- Successfull (?) comparison of the PN formalism (at the 3PN level) and the SF approach through gauge invariant variables
- 3PN approx. $\sim 1\%$ accurate up to $r_{12} \sim 10M$ and $\sim 5\%$ accurate up to $r_{12} \sim 7M$ in the extreme mass ratio regime

Conclusion

- Successfull (?) comparison of the PN formalism (at the 3PN level) and the SF approach through gauge invariant variables
- 3PN approx. $\sim 1\%$ accurate up to $r_{12} \sim 10M$ and $\sim 5\%$ accurate up to $r_{12} \sim 7M$ in the extreme mass ratio regime
- Hadamard regularization not entirely satisfactory at 3PN order

Conclusion

- Successfull (?) comparison of the PN formalism (at the 3PN level) and the SF approach through gauge invariant variables
- 3PN approx. $\sim 1\%$ accurate up to $r_{12} \sim 10M$ and $\sim 5\%$ accurate up to $r_{12} \sim 7M$ in the extreme mass ratio regime
- Hadamard regularization not entirely satisfactory at 3PN order
- Independant confirmation of the 3PN equations of motion of compact binaries within dimensional regularization