Self force in black-hole spacetimes

calculation methods and status

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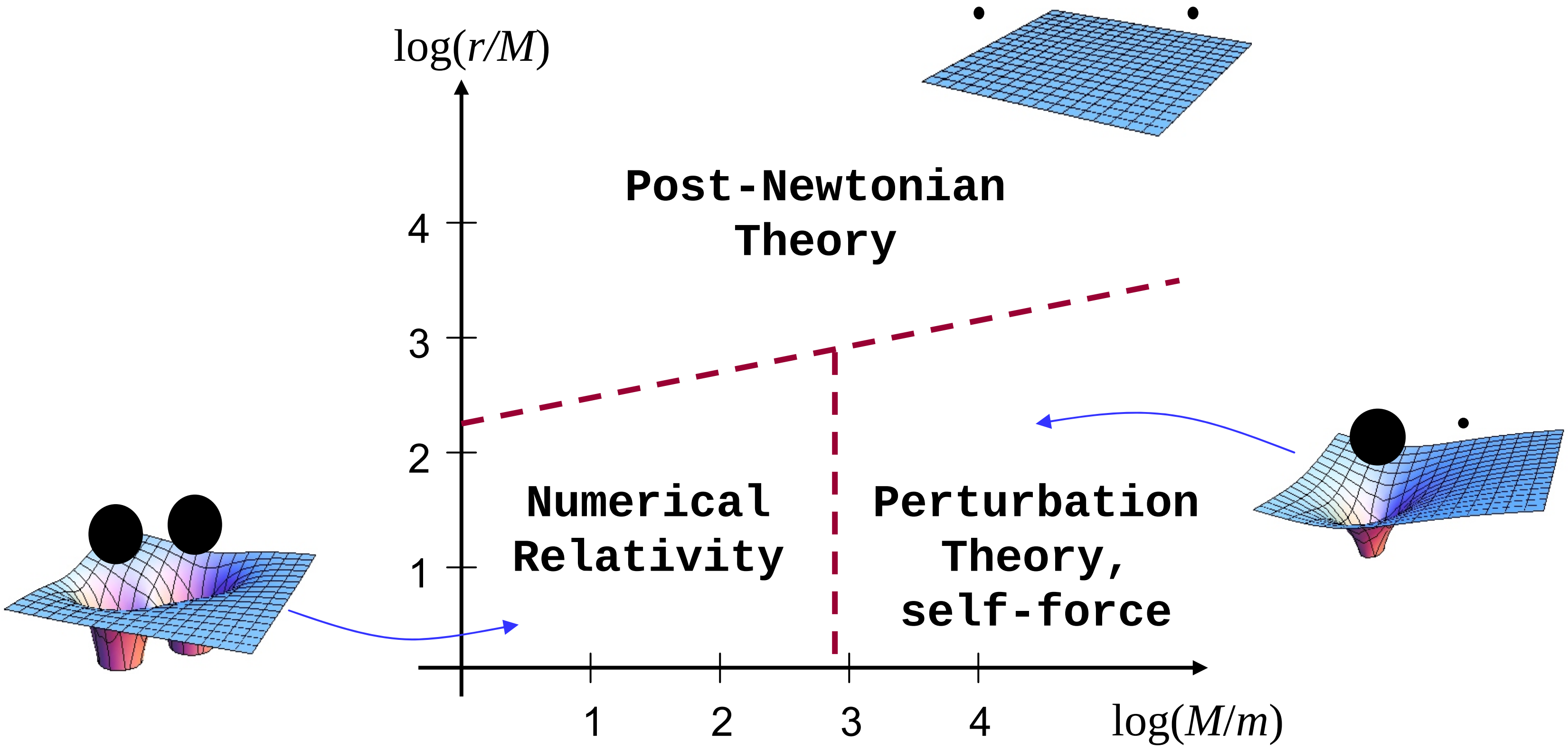
Plan

- **Context and motivation** ▶ 5 min
- **Recap of essential theory** ▶ 5 min
- **Computation strategies** ▶ 30 min
- **Numerical implementation methods** ▶ 10 min
- **Physical effects of the gravitational self-force** ▶ 15 min

Physical context: The 2-body problem in GR

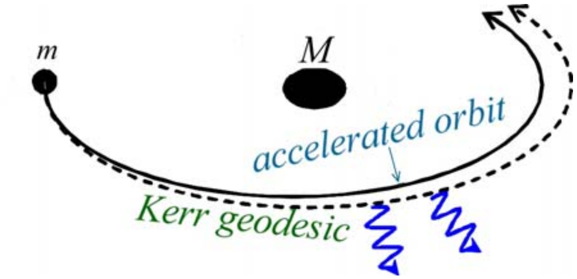


The gravitational self force
Summary of theory (I)

- The MiSaTaQuWa+GrWa equation of motion:

$$mu^\beta \nabla_\beta u^\alpha = F^\alpha_{\rm self} \; (\propto m^2)$$

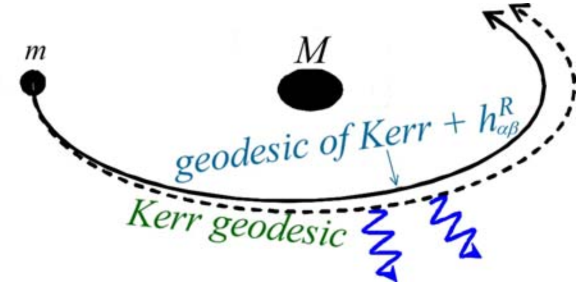
$$\underline{\text{w.r.t. } g^{\rm Kerr}_{\alpha\beta}}$$



- Alternative interpretation (Detweiler & Whiting):

$$m\hat{u}^\beta \hat{\nabla}_\beta \hat{u}^\alpha = 0$$

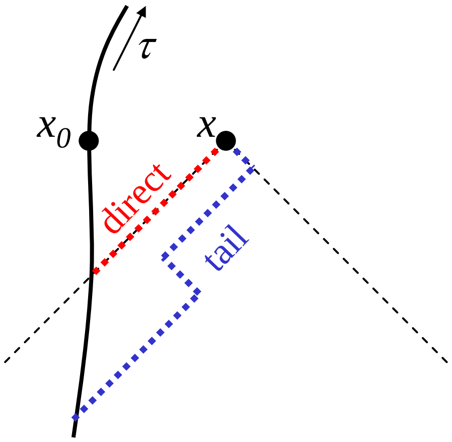
$$\underline{\text{w.r.t. } g^{\rm Kerr}_{\alpha\beta} + h^R_{\alpha\beta}}$$



The gravitational self force

Summary of theory (II)

$$
\begin{aligned}
F^{\alpha}_{\rm self} &= \lim_{x\to x_0} \nabla^{\alpha\mu\nu} h^{\rm tail}_{\mu\nu} = \lim_{x\to x_0} \nabla^{\alpha\mu\nu} \left(h^{\rm ret}_{\mu\nu} - h^{\rm dir}_{\mu\nu}\right) \\
&= \lim_{x\to x_0} \nabla^{\alpha\mu\nu} h^{\rm R}_{\mu\nu} = \lim_{x\to x_0} \nabla^{\alpha\mu\nu} \left(h^{\rm ret}_{\mu\nu} - h^{\rm S}_{\mu\nu}\right)
\end{aligned}
$$



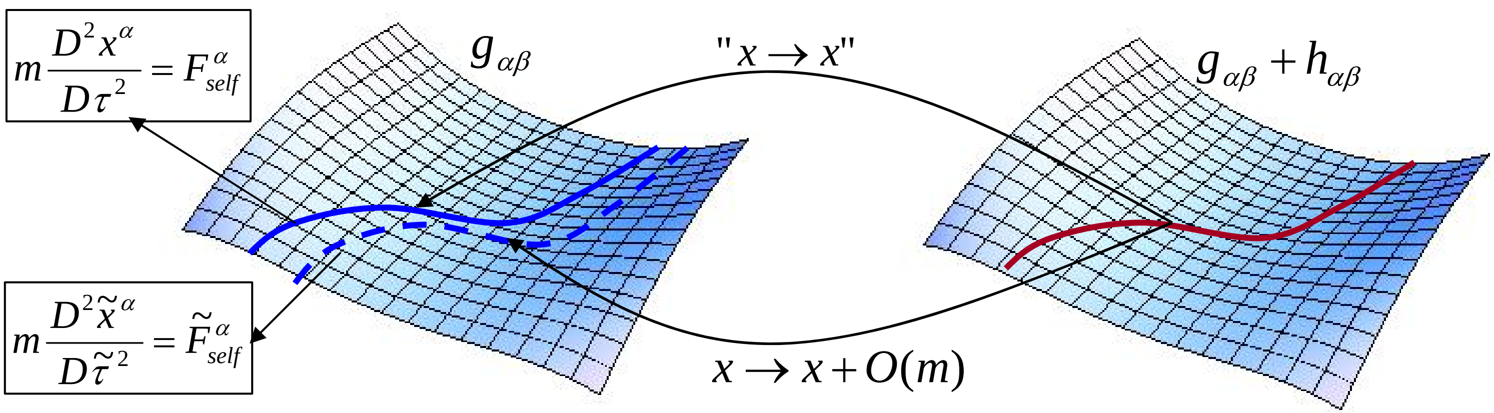
$\nabla^{\alpha\mu\nu} = \nabla^{\alpha\mu\nu}(g_{\beta\gamma}, u^{\beta})$ is the "force" operator (same as for the force exerted by an external perturbation)

Imporatnt notes:

- h^R is only an effective field; the physical field is $h^{\rm ret}$
- MiSaTaQuWa formula applies for a geodesic source
- MiSaTaQuWa formulated in the Lorenz gauge $[\nabla^{\nu}(h_{\mu\nu} - g_{\mu\nu}h/2) = 0]$
- The gravitational self-force is gauge dependent:

The gravitational self force
Summary of theory (III)

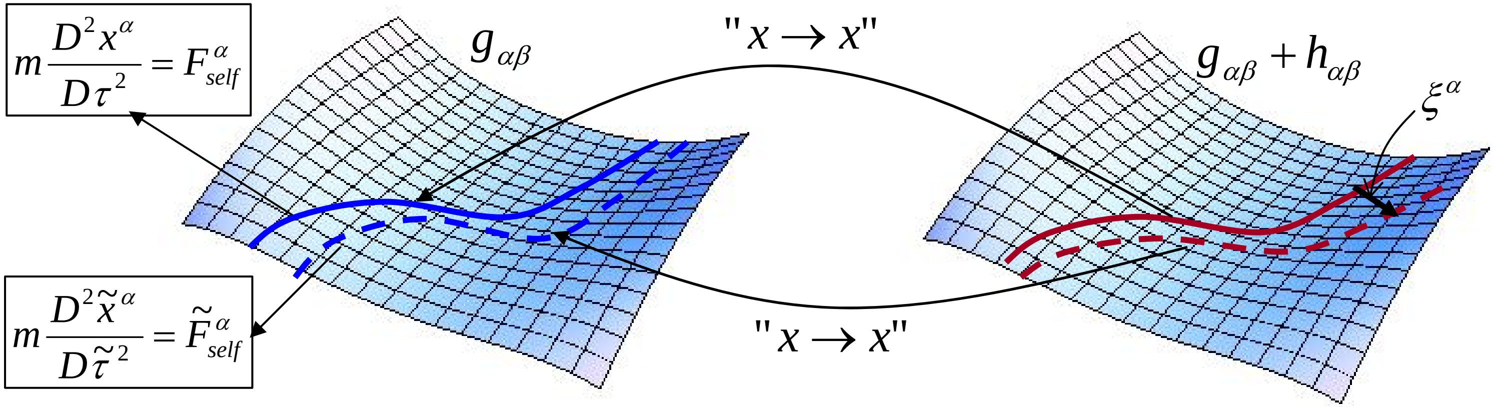
- Gravitational SF defined through mapping:



- Ambiguity in identifying physical points in the two spacetimes brings about ambiguity in the gravitational self force ⊂ gauge freedom

The gravitational self force
Summary of theory (IV)

- To obtain the gauge transformation law of the self force:



$$x^\alpha \;\to\; x^\alpha - \xi^\alpha(\propto m)$$

$$h_{\mu\nu} \;\to\; h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

$$\boxed{\delta_\xi F^\alpha_{\rm self} \equiv \tilde{F}^\alpha_{\rm self} - F^\alpha_{\rm self} = -m\left[(g^{\alpha\lambda}+u^\alpha u^\lambda)\ddot{\xi}_\lambda + R^\alpha{}_{\mu\lambda\nu}u^\mu\xi^\lambda u^\nu\right]}$$

Practical computation schemes

Survey of strategies

- Analytic approximations
  approximate evaluations of the MiSaTaQuWa formula
  - Weak-field, post-Newtonian [**Le Tiec**, Mon]
  - Quasi-local/Matched expansions [**Wardell, Dolan, Casals**, Tue]

- Mode-sum method ("l-*mode regularization*")
  scheme for performing the subtraction 'R-S' mode by mode
  [**Sago**, **Warburton**, Thu]

- Radiation-gauge regularization
  perform the subtraction 'R-S' at the level of the curvature scalar, then reconstruct the R field in the radiation gauge [**Friedman**, Wed]

"regularize the solutions"

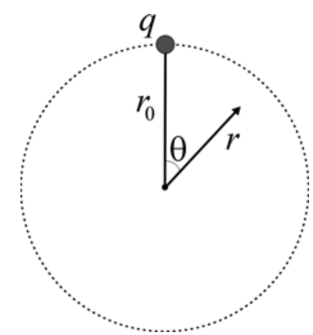
- Direct subtraction in 3+1D or 2+1D ("m-*mode regularization*")
  solve the field eqs. numerically for the regularized variable 'R-S'
  [**Burko**, Thu; **Detweiler**, Wed; **Vega**, Thu; **Diener**, Thu]

"regularize the equations"

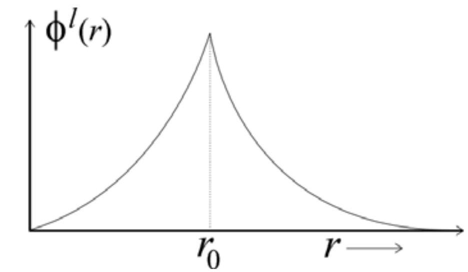
Mode-sum method elementary example

- l-mode contribution to field of scalar charge in flat space:

$$\phi^l(r,\theta) = \frac{q}{r_0} P_l(\cos\theta) \times \begin{cases} (r/r_0)^{-l-1}, & r \geq r_0 \\ (r/r_0)^l, & r \leq r_0 \end{cases}$$



$$F_r^{l,\mathrm{full}}(r_0^\pm) \equiv q\, \nabla_{r^\pm}\phi^l\big|_{x_0} = \mp(l+1/2)\frac{q^2}{r_0^2} - \frac{q^2}{2r_0^2}$$



▶ each $F_r^{l,\mathrm{full}}$ is **finite** ▶ $F_r^{l,\mathrm{full}} \propto l$ as $l \to \infty$

- These features are universal:

$$\boxed{F_\alpha^{l,\mathrm{full}}(r_0^\pm) = \pm L A_\alpha + B_\alpha + L^{-1} C_\alpha + O(L^{-2}) \quad \text{where} \quad L \equiv l + 1/2}$$

"Regularization Parameters", depend on x^α and u^α at x_0

Mode-sum method formulated

$$F_{\text{self}} = \lim_{x\to x_0}\left[F_{\text{full}}(x) - F_{\text{dir}}(x)\right] \qquad \left(F^{\alpha}_{\text{full/dir}} = m\nabla^{\alpha\beta\gamma}h^{\text{ret/dir}}_{\beta\gamma}\right)$$

$$= \lim_{x\to x_0}\sum_l \left[F^l_{\text{full}}(x) - F^l_{\text{dir}}(x)\right]$$

$$= \lim_{x\to x_0}\left[\sum_l \left(F^l_{\text{full}}(x) - LA - B - L^{-1}C\right) - \underline{\sum_l \left(F^l_{\text{dir}}(x) - LA - B - L^{-1}C\right)}\right]$$

$$\boxed{F_{\text{self}} = \sum_l \left(F^l_{\text{full}}(x_0) - LA - B - L^{-1}C\right) + D}$$

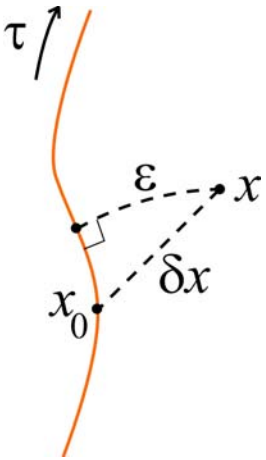
- Regularization parameters A, B, D derived analytically by analyzing the direct force for $x \to x_0$ and $l \to \infty$

Mode-sum method
derivation of the reg. parameters (scalar field)

$$\Phi^{\rm dir}(x) = \frac{q}{\epsilon}\left[1 + O(\delta x)^2\right]$$

$$F_\alpha^{\rm dir}(x) = q\nabla_\alpha \Phi^{\rm dir} = \frac{P_\alpha^{(1)}(\delta x)}{\epsilon_0^3} + \frac{P_\alpha^{(3)}(\delta x)}{\epsilon_0^5} + \frac{P_\alpha^{(7)}(\delta x)}{\epsilon_0^7} + O(\delta x)$$

where ϵ_0 is the $O(\delta x)$ term of ϵ and $P_\alpha^{(n)}$ is a polynomial of order n in δx.



$$F_\alpha^{l,\rm dir} = \lim_{x\to x_0} \sum_m Y^{lm}(\Omega) \oint d\Omega' \, Y^{lm*}(\Omega')\, F_\alpha^{\rm dir}(x)$$

$$= LA_\alpha + B_\alpha \qquad (\Rightarrow C_\alpha = D_\alpha = 0)$$

- parameter values known for generic orbits in Kerr

Mode-sum method summarized

① For a given geodesic source, solve the perturbation equations (mode by mode), and obtain the retarded Metric Perturbation (MP);

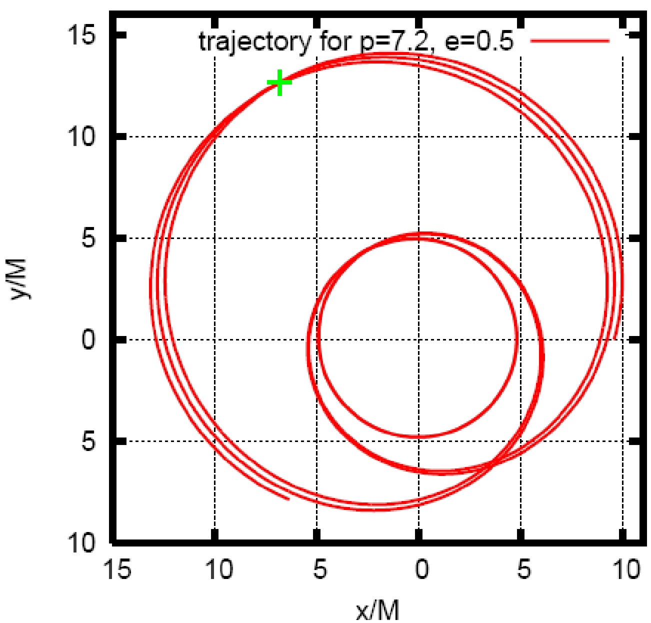
② Construct the “full force” modes (“grad MP”) at the particle;

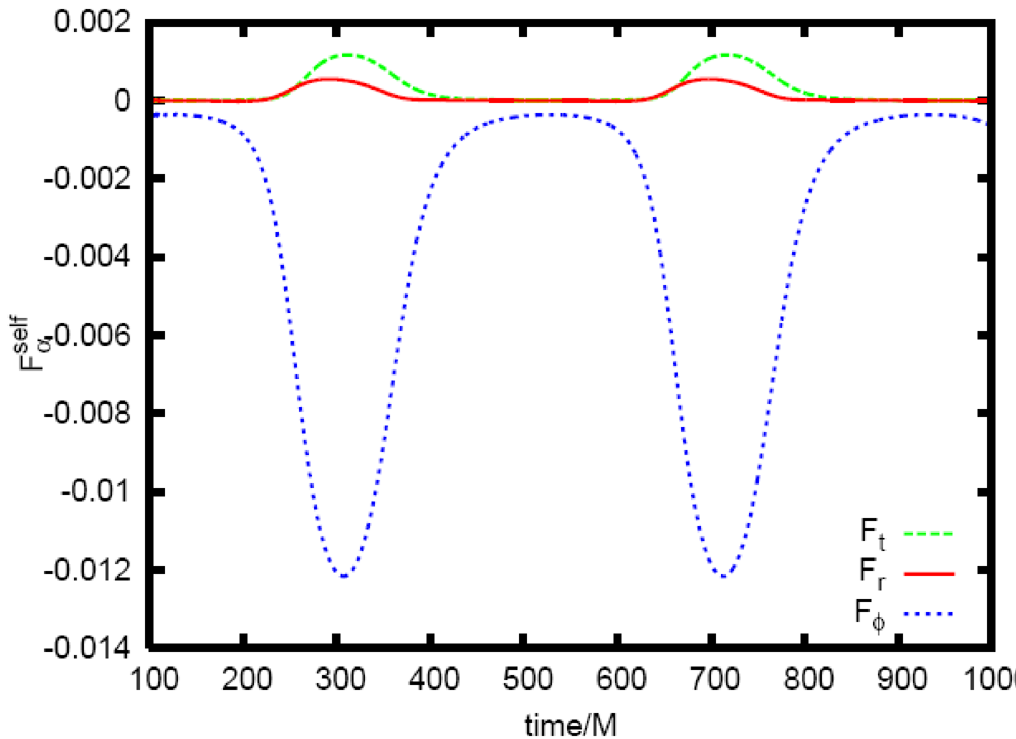
③ Apply the mode-sum formula:

$$\boxed{F_{\text{self}} = \sum_l \left(F^l_{\text{full}}(x_0) - LA - B\right)}$$

- expected $O(L^{-2})$ behaviour provides important check of numerics
- large-L tail can be extrapolated analytically to improve accuracy
- higher-order terms in $1/L$ expansion can be calculated to accelerate convergence

Results for Schwarzschild: scalar-field SF (Haas 2007)





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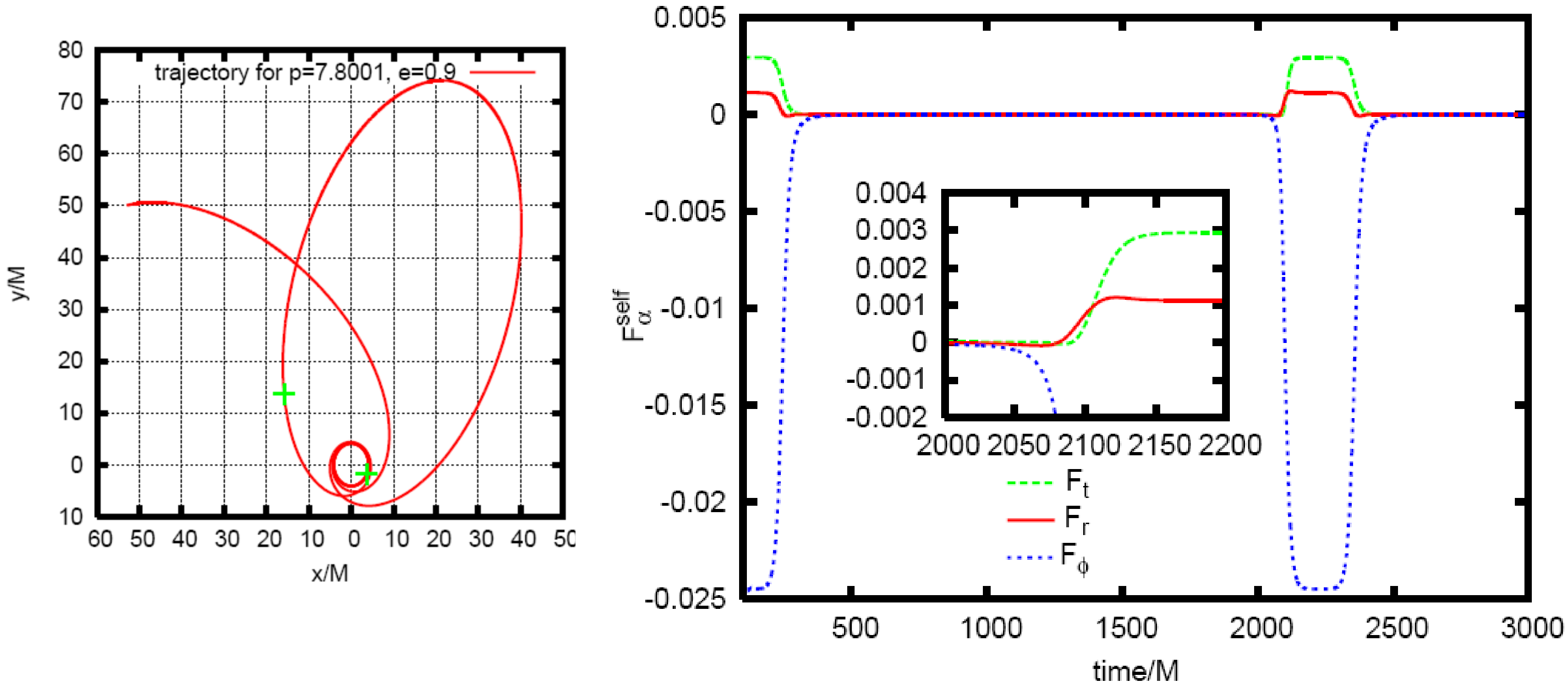
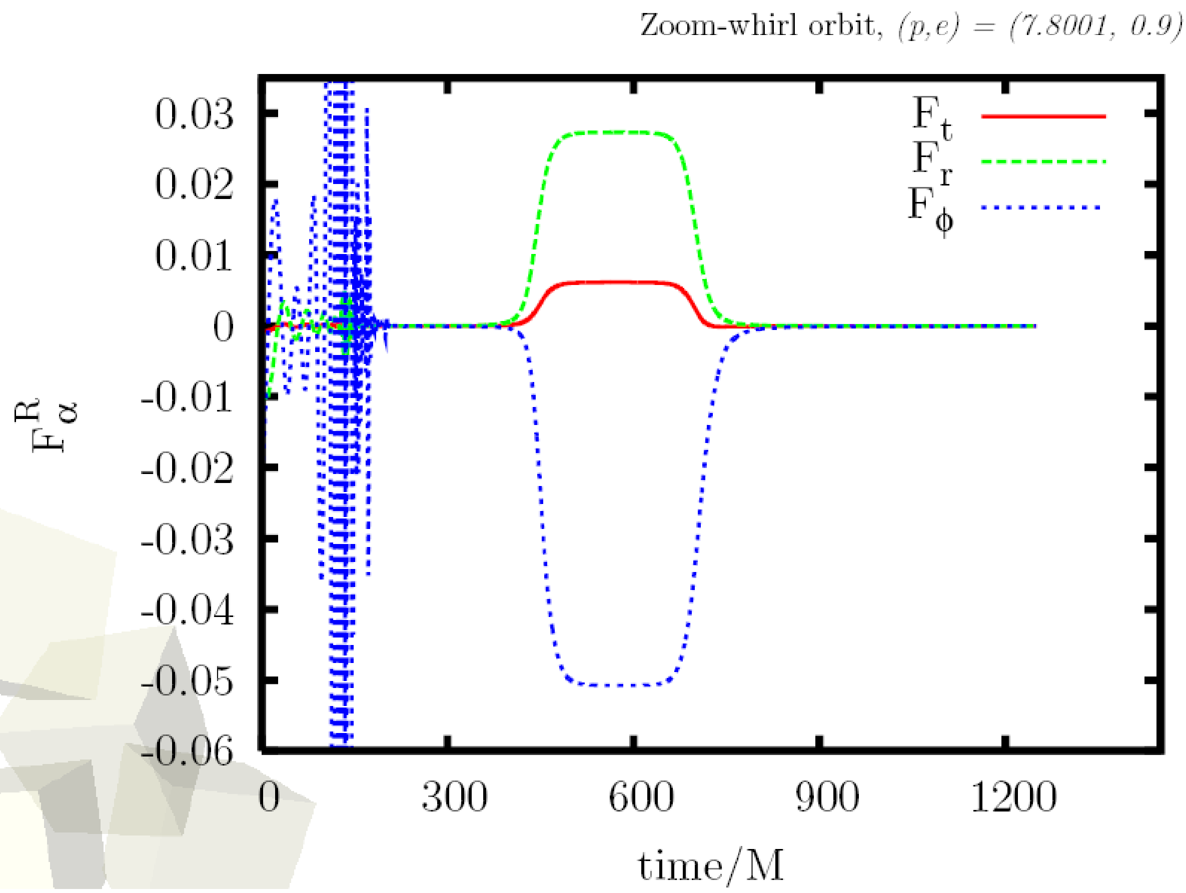
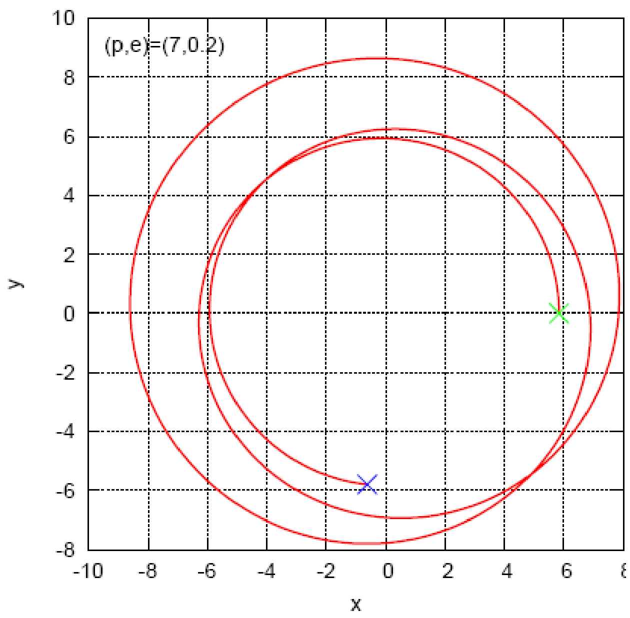
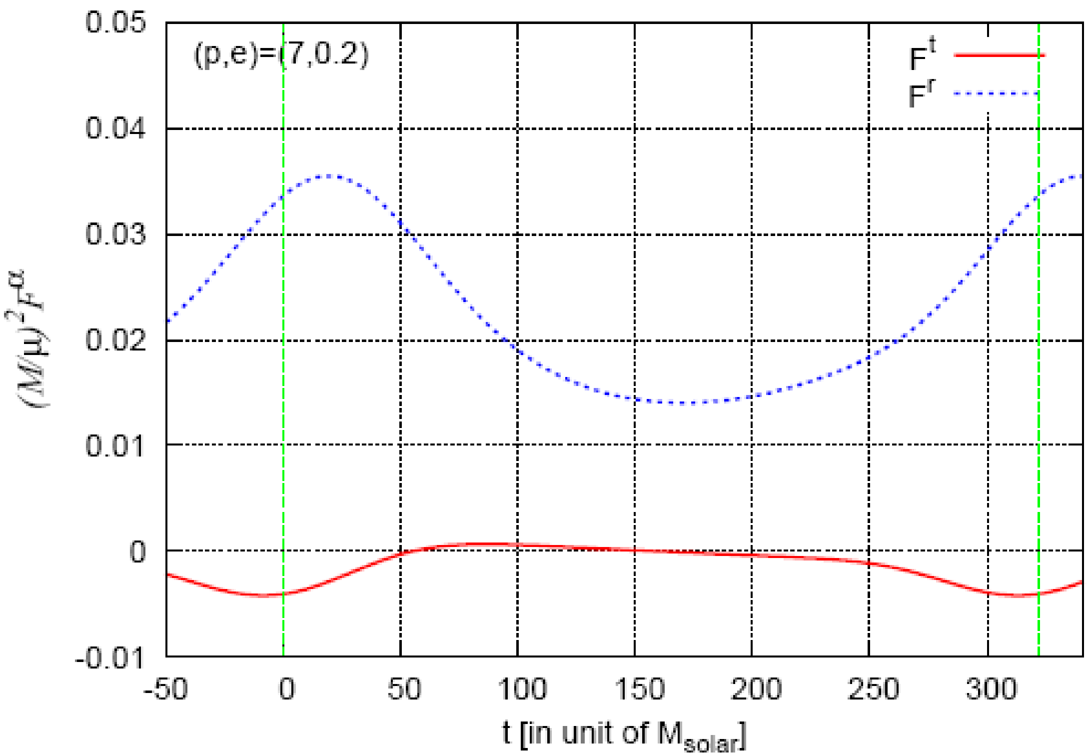


Results for Schwarzschild: electromagnetic SF (Haas 2008, Capra 11)



Results for Schwarzschild: gravitational SF in Lorenz gauge (LB & Sago, 2009)





Details of numerical method: Sago, Thursday

The Kerr case:
m-mode regularization

- Metric perturbation equations in **Kerr** separable only in 2+1D:

$$h_{\alpha\beta} = \sum_{m=-\infty}^{\infty} e^{im\varphi} h^m_{\alpha\beta}(t, r, \theta)$$

- In principle, can use 2+1D numerical evolution in conjunction with standard mode-sum regularization — but this is awkward!

- Desired: formulation of the subtraction in 2+1D — **"m-mode scheme"**

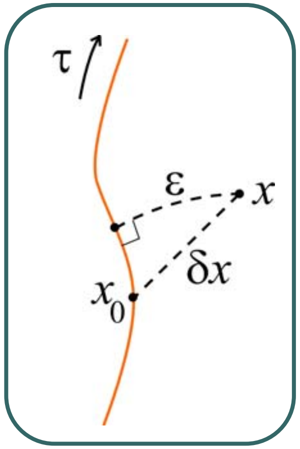
- **Difficulty:** each mode diverges (logarithmically) at worldline

m-mode regularization
logarithmic divergence (scalar field example)

- Decompose $\Phi = \sum_{m=-\infty}^{\infty} e^{im\varphi}\Phi^m(t,r,\theta)$ $\qquad \Phi^m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi e^{-im\varphi} d\varphi$

- Near particle $\Phi(x) \simeq \frac{q}{\epsilon}$ $\qquad \epsilon^2 \simeq P_{\alpha\beta}\delta x^\alpha \delta x^\beta$

$$P_{\alpha\beta} = g_{\alpha\beta}(x_0) + u_\alpha(x_0)u_\beta(x_0)$$



- Pick a worldline point x_0. Look at field near x_0 on surface $t = t_0$:

$$\Phi^m(\rho) \simeq \frac{q}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-im\varphi}}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}} d\varphi$$

$\rho^2 = P_{rr}\delta r^2 + P_{\theta\theta}\delta\theta^2$ (distance in r - θ plane)

m-mode regularization
logarithmic divergence (scalar field example) - cont'd

- Split integral as

$$\int_{-\pi}^{\pi} \frac{e^{-im\varphi}}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi = \int_{-\pi}^{\pi} \frac{e^{-im\varphi} - 1}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi + \int_{-\pi}^{\pi} \frac{1}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi$$

- 1st integral bounded by a constant:

$$\left| \int_{-\pi}^{\pi} \frac{e^{-im\varphi} - 1}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi \right| \leq \int_{-\pi}^{\pi} \frac{m|\varphi|}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}}\, d\varphi \leq \int_{-\pi}^{\pi} \frac{m}{P_{\varphi\varphi}^{1/2}}\, d\varphi = \frac{2\pi m}{P_{\varphi\varphi}^{1/2}}$$

- 2nd integral is m-independent and integrates explicitly:

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{d\varphi}{(\rho^2 + P_{\varphi\varphi}\varphi^2)^{1/2}} &= \frac{\pi}{\rho_0} \ln \left[\frac{(\rho_0^2 + \rho^2)^{1/2} + \rho_0}{(\rho_0^2 + \rho^2)^{1/2} - \rho_0} \right] \\ &= -\frac{2\pi}{\rho_0} \ln \left(\frac{\rho}{2\rho_0} \right) + O(\rho^2), \end{aligned}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} &\Phi^m(\rho \to 0) = -(q/\rho_0) \ln \left(\frac{\rho}{2\rho_0} \right) \\ &\rho_0 \equiv \pi P_{\varphi\varphi}^{1/2} \end{aligned}}$$

m-mode regularization formulated

- Don't solve for $\Phi_{\rm ret}$; solve for

$$\delta\Phi \equiv \Phi_{\rm ret} - \Phi_{\rm punc}, \qquad \Phi_{\rm punc} = \Phi_{\rm S} + C^0$$

The "punctured" variable $\delta\Phi$ is C^0 and its m-modes $\delta\Phi^m$ are C^1.

- The field equation for $\delta\Phi$ (schematically):

$$\Box\delta\Phi = S_\delta - \Box\Phi_{\rm punc} \equiv \delta S \ (\text{no } \delta \text{ function})$$

- Can show (somewhat surprisingly):

$$F^\alpha_{\rm self,scalar} = q\sum_m \nabla^\alpha\delta\Phi^m, \qquad F^\alpha_{\rm self,grav} = \mu\sum_m \nabla^{\alpha\beta\gamma}\delta h^m_{\beta\gamma}$$

- Note: The above puncture does not guarantee $F^\alpha_{\rm self} = q\nabla^\alpha\delta\Phi$.

[To achieve this requires a higher-order puncture, with $\nabla^\alpha\Phi_{\rm punc}(x_0) = \nabla^\alpha\Phi_{\rm S}(x_0)$]

The m-decomposition spares us the need to go to higher order

m-mode regularization explained

$$\Phi_{\rm S}: \quad O(\epsilon^{-1}) + O(\delta x/\epsilon) + \text{const} + O(\delta x^2/\epsilon) + O(\delta x^2)$$

$$\Phi_{\rm punc} - \Phi_{\rm S}: \quad \text{const} + O(\delta x^2/\epsilon) + O(\delta x^2)$$

$$F_{\rm punc} - F_{\rm S}: \quad O(\delta x/\epsilon) + O(\delta x)$$ [not C^0 but piecewise continuous]

$$F_{\rm self} = \lim_{x\to x_0} \left[F_{\rm full} - F_{\rm S}\right] = \lim_{x\to x_0} \left[\delta F + (F_{\rm punc} - F_{\rm S})\right]$$

$$= \lim_{x\to x_0} \sum_m \left[\delta F^m + \left(F^m_{\rm punc} - F^m_{\rm S}\right)\right]$$

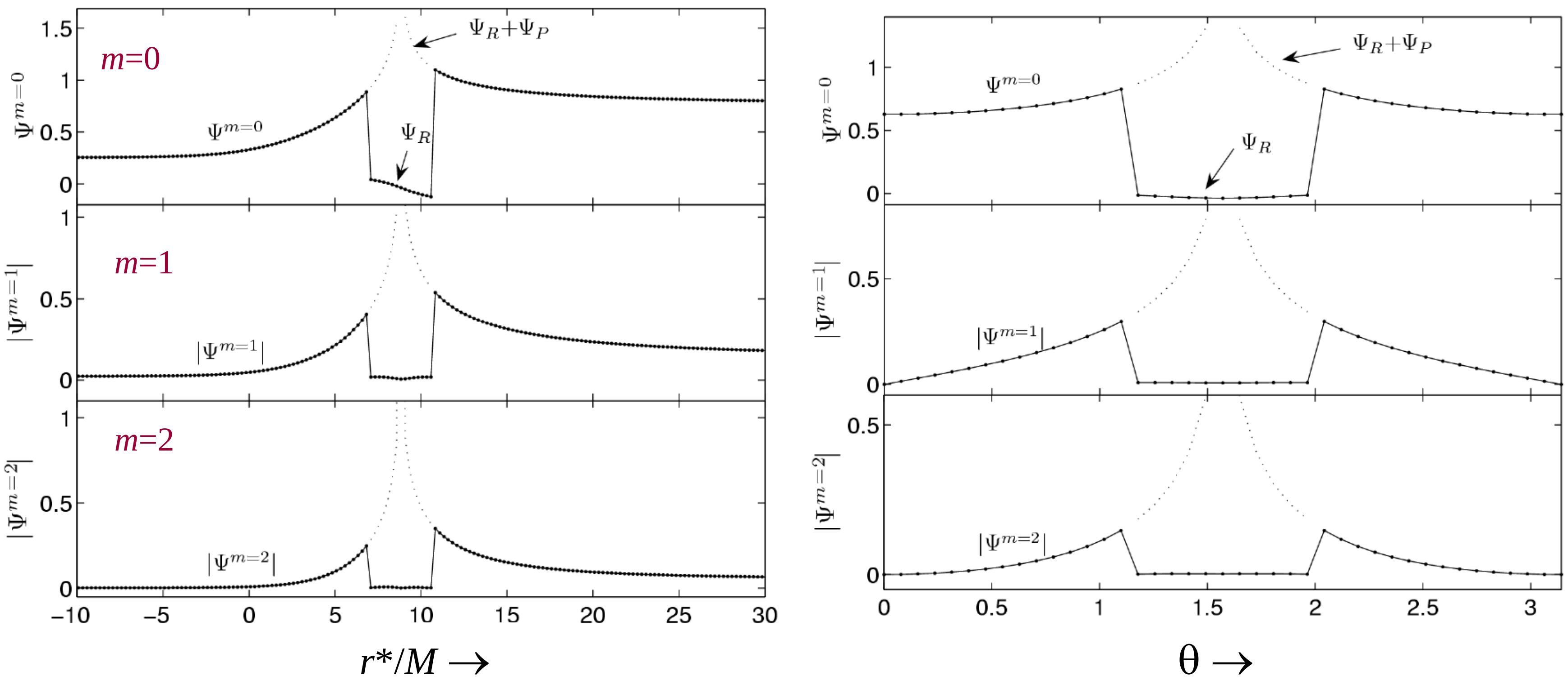
[Fourier series converges uniformly everywhere since $F_{\rm R}$ is smooth]

$$= \sum_m \lim_{x\to x_0} \left[\delta F^m + \left(F^m_{\rm punc} - F^m_{\rm S}\right)\right]$$

$$= \sum_m \delta F^m(x_0)$$

Fourier integral automatically averages over discontinuity (LB, Golbourn and Sago, PRD 2007)

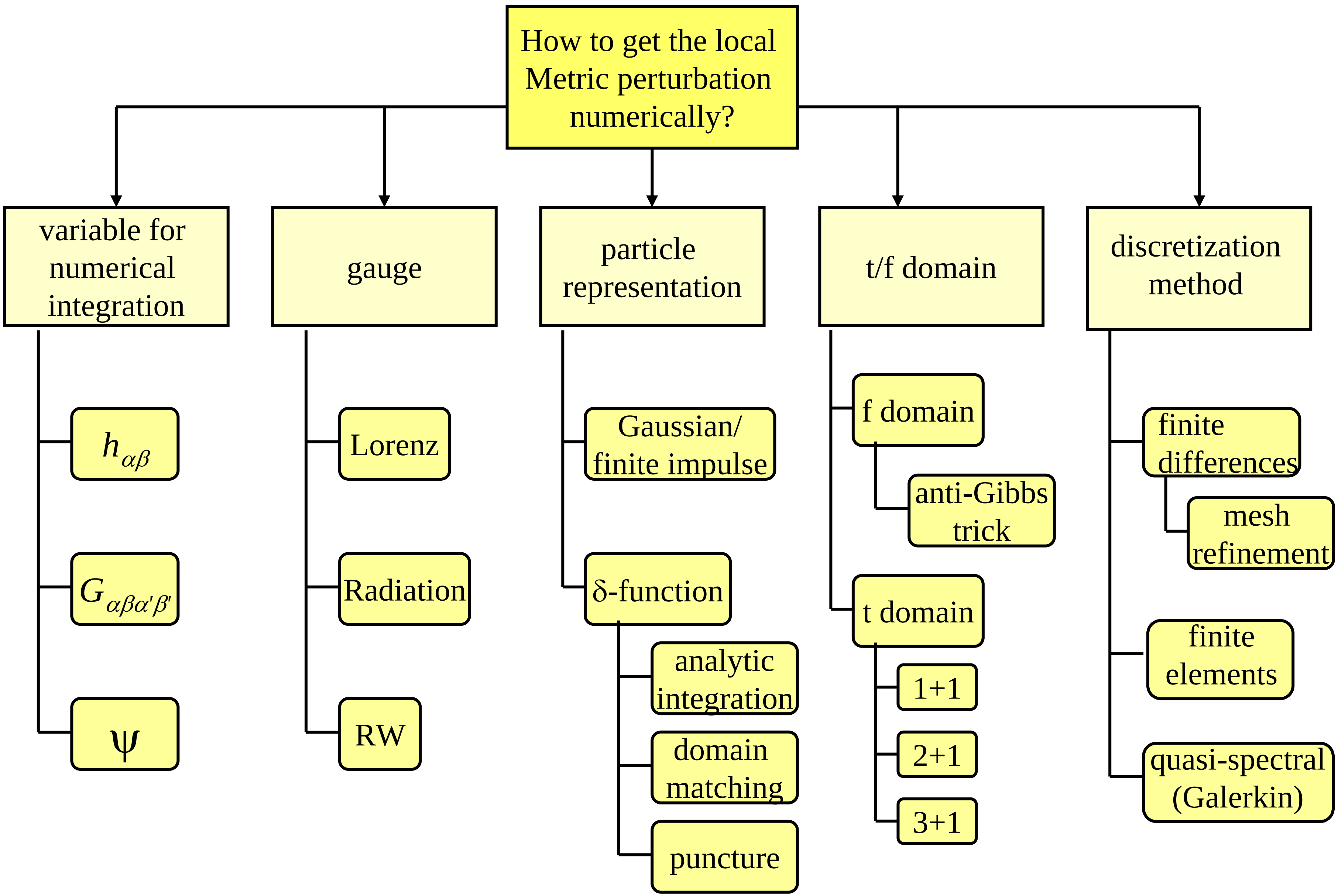
sample results from puncture evolution in 2+1 (scalar field)



LB & Golbourn, PRD 2007

Numerical implementation strategies

How to deal with a point-particle singularity in a numerical treatment of the perturbation equations?



How to get the local Metric perturbation numerically?
variable for numerical integration
gauge
particle representation
t/f domain
discretization method
$h_{\alpha\beta}$
$G_{\alpha\beta\alpha'\beta'}$
ψ
Lorenz
Radiation
RW
Gaussian/ finite impulse
δ-function
analytic integration
domain matching
puncture
f domain
anti-Gibbs trick
t domain
1+1
2+1
3+1
finite differences
mesh refinement
finite elements
quasi-spectral (Galerkin)

Physical effects of the gravitational SF

some results for Schwarzschild

Dissipative & Conservative pieces

$$F_{\rm self} = F_{\rm cons} + F_{\rm diss} \ :$$

$$F_{\rm cons} = \frac{1}{2}\left(F_{\rm ret} + F_{\rm adv}\right)$$

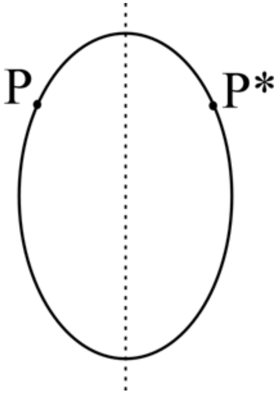
$$F_{\rm diss} = \frac{1}{2}\left(F_{\rm ret} - F_{\rm adv}\right)$$

How to split the SF into "cons" and "diss" in practice?

- In f-domain: get $F_{\rm adv}$ by flipping boundary conditions.
- In t-domain: get $F_{\rm adv}$ by evolving "backward" in time.

Better so: exploit symmetry of Kerr geodesics (*a la* Mino):

$$F^\alpha_{\rm adv} = \pm F^\alpha_{\rm ret}(u^r \to -u^r, u^\theta \to -u^\theta) \quad \left[\begin{array}{ll} (+) & \text{for } \alpha = r,\theta \\ (-) & \text{for } \alpha = t,\varphi \end{array}\right]$$



So (e.g.) for Schwarzschild :

$$\boxed{F^\alpha_{\rm cons} = \frac{1}{2}\left[F^\alpha_{\rm ret}(P) \pm F^\alpha_{\rm ret}(P^*)\right], \qquad F^\alpha_{\rm diss} = \frac{1}{2}\left[F^\alpha_{\rm ret}(P) \mp F^\alpha_{\rm ret}(P^*)\right]}$$

Dissipation of energy and angular momentum

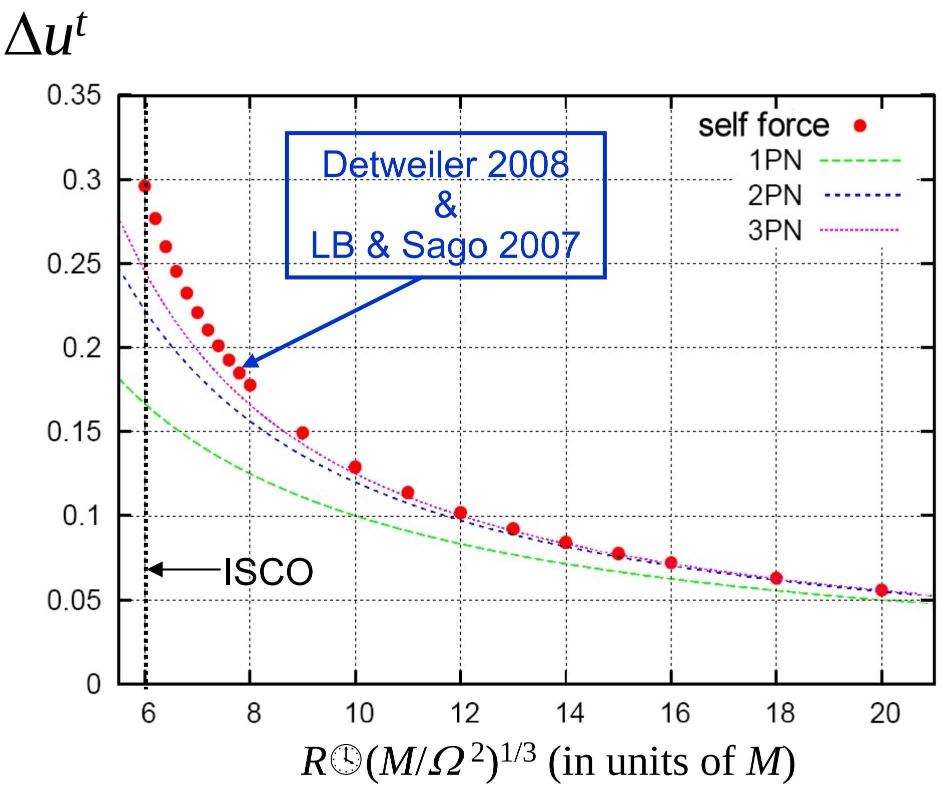
$$\langle F_t \rangle = \langle F_t^{\rm diss} \rangle = -\mu \left(\langle \dot{E}_\infty^{\rm GW} \rangle + \langle \dot{E}_{\rm EH}^{\rm GW} \rangle \right)$$

$$\langle F_\varphi \rangle = \langle F_\varphi^{\rm diss} \rangle = \mu \left(\langle \dot{L}_\infty^{\rm GW} \rangle + \langle \dot{L}_{\rm EH}^{\rm GW} \rangle \right)$$

(average taken over a radial period, for a particle on a bound geodesic)

- Infinity fluxes obtained from the asymptotic perturbation using Isaacson's effective energy-momentum.
- For horizon absorption cannot use Isaacson's high-freq. approximation; instead:
  - f-domain: Press-Teukolsky (1974) based on Hawking-Hartle formalism
  - t-domain: formalism by Poisson (2004)
- Balance equations demonstrated for eccentric orbits (LB & Sago 2009)
- No new physics but good test of self-force code

conservative effects for circular orbits



frequency

$$\Omega \equiv d\varphi/dt = (M/r_0^3)^{1/2}\left[1 - \frac{r_0(r_0-3M)}{2\mu M}\, F_r^{\text{cons}}\right]$$

and "time function"

$$u^t = (1-3M/r_0)^{-1/2}\left(1 - \frac{r_0}{2\mu}\, F_r^{\text{cons}}\right)$$

are both invariant under gauge transformations that respect the helical symmetry of the orbit.

the conservative ISCO shift (I)

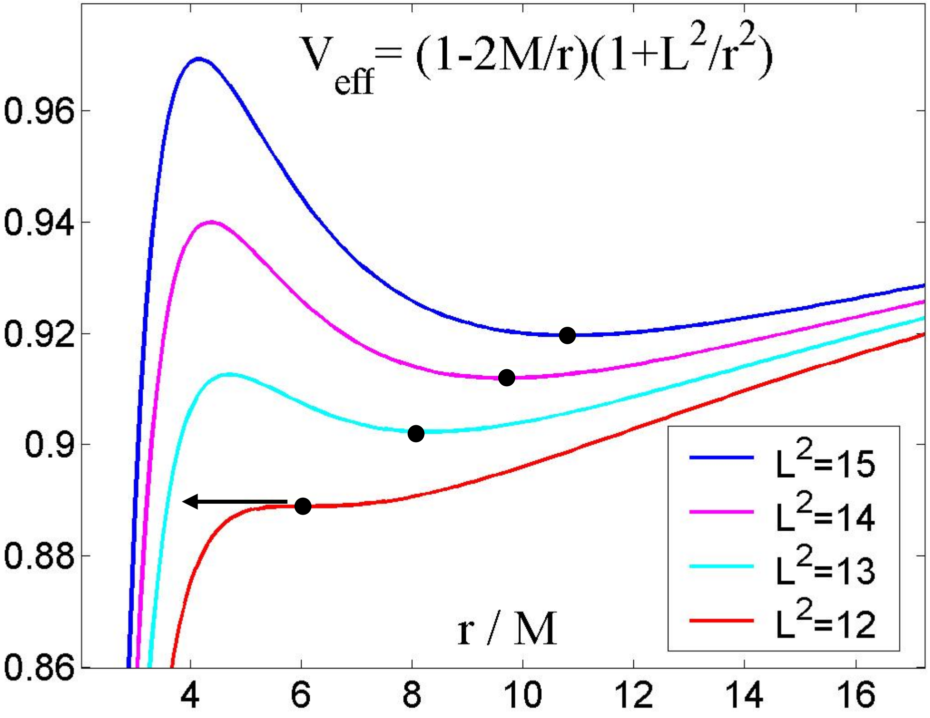
$$\ddot{r} = -\frac{1}{2}\frac{\partial V_{\rm eff}(r, L^2)}{\partial r} + \mu^{-1} F^r_{\rm self}$$

$$\dot{E} = -\mu^{-1} F_t^{\rm self}, \qquad \dot{L} = \mu^{-1} F_\varphi^{\rm self}$$

- Near stable equilibrium radial motion is a linear oscillator:

$$\ddot{r} + \omega_r^2 r = 0$$

- ISCO where ω_r=0
- At $O(\mu^0)$ [no SF]: $r_{\rm isco}$=6M



the conservative ISCO shift (II)

$O(\mu)$ dissipative effect

"smears" the ISCO $\rightarrow$ transition regime of width Δr ✪ $\mu^{2/5}$ (Ori & Thorne 2000)

$O(\mu)$ conservative effect (dissipation ignored)

"shifts" the ISCO by a well-defined amount δr ✪ μ, determined from analysis of slightly eccentric geodesics near $r = 6M$ (details in N. Sago's talk):

$$\boxed{\delta r_{\rm isco} \approx 3.269\,\mu, \qquad \delta\Omega/\Omega \approx 0.4870\,\mu/M}$$

(LB & Sago 2009)

- ISCO shift due to scalar SF calculated by Diaz-Rivera *et al* (2004).
- Conservative ISCO frequency shift can be used for comparison with/ calibration of PN calculations

Prospects

- Schwarzschild:
  - extract all physics (precession effects, importance of conservative self force, gauge invariants and comparison with PN theory)
  - test-bed for advanced numerical methods (mesh refinement; finite element)
- Kerr, scalar self-force:
  - extend existing f-domain code (Warburton) to eccentric & inclined orbits
  - test-bed for *m*-mode regularization in t-domain
  - study forced resonances.
- Kerr, gravitational self-force:
  - implement regularization in 2+1D or 3+1D, radiation-gauge regularization
  - try 1+1D or 1D with coupled spherical harmonics
  - "spheroidal tensor harmonics"?
- Orbital evolution: osculating geodesics, multiple-scale analysis, and all that.