Evolving Metric Perturbations and Self-Force Calculations in the Lorenz Gauge: progress report

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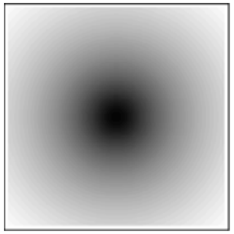
Approaches for SF numerical calculations

- Frequency domain
  - ✓ Easy in Schwarzschild
  - o Hard to generalize to generic Kerr orbits

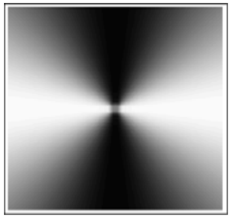
- Time domain
  - ✓ Natural extension to generic worldlines
  - ? Regularization method

Our approach

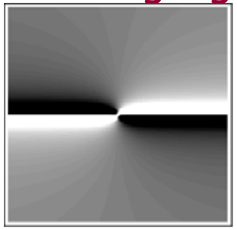
- **Metric perturbations**
  - no need to do metric reconstruction from Weyl scalars
- **Lorenz Gauge**
  - avoid need for gauge transformations
  - particle singularity is isotropic and isolated
  - manifest hyperbolicity
- **Time domain in 2+1D**
  - generic orbits are natural
  - much experience gained from the Teukolsky Equation
- **Regularization**
  - m mode regularization / puncture method



Lorenz gauge



RW gauge



Radiation gauge

Credits: L. Barack

Field equations

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

$$T_{\alpha\beta} = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^4[x^\mu - x_{\mathrm{p}}^\mu(\tau)] u_\alpha u_\beta \, d\tau$$

$$\begin{aligned}
\Box \bar{h}_{\alpha\beta} &= \underbrace{g^{\gamma\delta}\,\partial_\gamma\,\partial_\delta \bar{h}_{\alpha\beta} - g^{\gamma\delta}(\bar{h}_{\alpha\beta,\sigma}\Gamma^{\sigma}_{\delta\gamma}}_{\text{scalar operator } D^2} + 2\bar{h}_{\lambda\beta,\delta}\Gamma^{\lambda}_{\alpha\gamma} + 2\bar{h}_{\lambda\alpha,\delta}\Gamma^{\lambda}_{\beta\gamma}) + \bar{h}_{\lambda\alpha}\mathcal{G}^{\lambda}_{\beta} + \bar{h}_{\lambda\beta}\mathcal{G}^{\lambda}_{\alpha} + \bar{h}_{\lambda\sigma}\mathcal{F}^{\lambda\sigma}_{\alpha\beta} \\
&= D^2\bar{h}_{\alpha\beta} + 4g^{\gamma\delta}\Gamma^{\lambda}_{\gamma(\alpha}\bar{h}_{\beta)\lambda,\delta} + 2\bar{h}_{\lambda(\alpha}\mathcal{G}^{\lambda}_{\beta)} + \bar{h}_{\lambda\sigma}\mathcal{F}^{\lambda\sigma}_{(\alpha\beta)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{\lambda}_{\alpha} &:= \left(\Gamma^{\lambda}_{\sigma\gamma}\Gamma^{\sigma}_{\alpha\delta} + \Gamma^{\lambda}_{\sigma\alpha}\Gamma^{\sigma}_{\gamma\delta} - \Gamma^{\lambda}_{\alpha\gamma,\delta}\right) g^{\gamma\delta} = g^{\gamma\delta}\left[2\Gamma^{\lambda}_{\sigma(\gamma}\Gamma^{\sigma}_{\alpha)\delta} - \Gamma^{\lambda}_{\alpha\gamma,\delta}\right] \\
\mathcal{F}^{\lambda\sigma}_{\alpha\beta} &:= \left(\Gamma^{\lambda}_{\beta\gamma}\Gamma^{\sigma}_{\alpha\delta} + \Gamma^{\lambda}_{\gamma\alpha}\Gamma^{\sigma}_{\beta\delta}\right) g^{\gamma\delta} = 2g^{\gamma\delta}\Gamma^{\lambda}_{\gamma(\alpha}\Gamma^{\sigma}_{\beta)\delta}
\end{aligned}$$

Field equations

$$
\begin{aligned}
\bar{h}_{\alpha\beta} &:= \frac{K_{\alpha\beta}}{r} \\
K_{\alpha\beta} &= \sum_{m=-\infty}^{\infty} \kappa_{\alpha\beta}^{(m)} e^{im\varphi} \\
T_{\alpha\beta} &= \sum_{m=-\infty}^{\infty} \tau_{\alpha\beta}^{(m)} e^{im\varphi}
\end{aligned}
$$

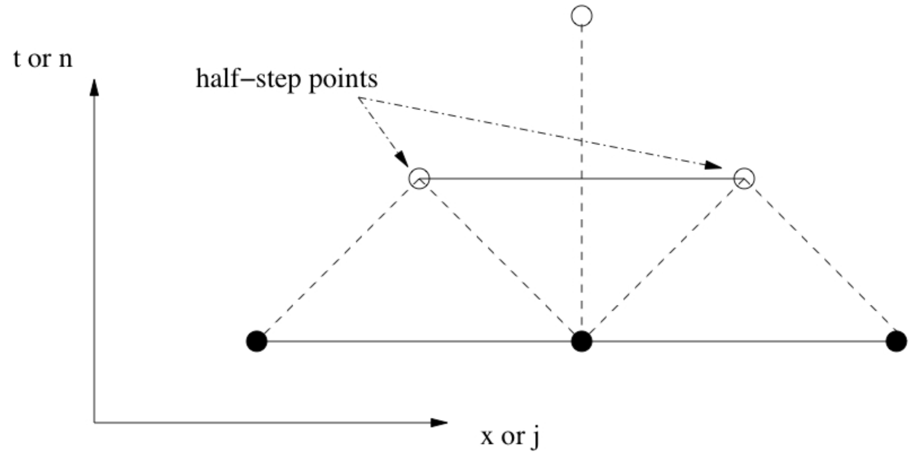
Numerical setup

Krivan, Laguna, Papadopoulos and Andersson (1997):

$$\begin{aligned} \psi &:= e^{im\tilde{\phi}} r^3 \Phi \\ \Pi &:= \partial_t \Phi + b\,\partial_{r^*} \Phi \qquad\qquad b := \frac{r^2 + a^2}{\Sigma} \\ \mathbf{u} &:= (\Phi_R, \Phi_I, \Pi_R, \Pi_I) \end{aligned}$$

$$\partial_t \mathbf{u} \quad + \quad \mathbf{D}\,\partial_{r^*}\mathbf{u} = \mathbf{S}$$

Solution is then obtained by the Lax-Wendroff 2-step scheme



Numerical setup II

E.g., in Schwarzschild:

$$\Pi_{ij} = \partial_0 \kappa_{ij} + \partial_{r_*} \kappa_{ij} \qquad (-\partial_0^2 + \partial_{r*}^2)\kappa_{ij} = (\partial_{r*} - \partial_0)\Pi_{ij}$$

$$\begin{aligned}
\mathbf{u}^{\mathrm{T}} \quad &:= \quad \{\overbrace{\kappa_{00} \;\; \kappa_{10} \;\; \kappa_{20} \;\; \cdots \;\; \kappa_{33}}^{10 \;\; \text{terms}} \; \overbrace{\Pi_{00} \;\; \Pi_{10} \;\; \Pi_{20} \;\; \cdots \;\; \Pi_{33}}^{10 \;\; \text{terms}}\} \\
\mathbf{T}\,\partial_0\mathbf{u} \quad &+ \quad \tilde{\mathbf{D}}\,\partial_{r^*}\mathbf{u} + \tilde{\mathbf{A}}\,\mathbf{u} + \tilde{\mathbf{L}}\,\mathbf{u} = \tilde{\mathbf{S}} \\
\mathbf{T}^{-1}\mathbf{T}\,\partial_0\mathbf{u} \quad &+ \quad \underbrace{\mathbf{T}^{-1}\tilde{\mathbf{D}}}_{\mathbf{D}}\,\partial_{r^*}\mathbf{u} + \underbrace{\mathbf{T}^{-1}\tilde{\mathbf{A}}}_{\mathbf{A}}\,\mathbf{u} + \underbrace{\mathbf{T}^{-1}\tilde{\mathbf{L}}}_{\mathbf{L}}\,\mathbf{u} = \underbrace{\mathbf{T}^{-1}\tilde{\mathbf{S}}}_{\mathbf{S}} \\
\partial_0\mathbf{u} \quad &+ \quad \mathbf{D}\,\partial_{r^*}\mathbf{u} + \mathbf{A}\,\mathbf{u} + \mathbf{L}\,\mathbf{u} = \mathbf{S}
\end{aligned}$$

Field equations

20 equations like:

$$\partial_0 \kappa_{00}^{(m)} = -\,\partial_{r^*}\kappa_{00}^{(m)} + \Pi_{00}^{(m)}$$

$$\begin{aligned}
\partial_0 \Pi_{00}^{(m)} \quad &+ \quad \left[\frac{4M}{r^2}\,\partial_{r^*}\kappa_{00}^{(m)} + \frac{4M}{r^3}(r-2M)\,\partial_{r^*}\kappa_{10}^{(m)} - \partial_{r^*}\Pi_{00}^{(m)}\right] \\
&+ \quad \left[-\frac{2M}{r^4}(r-M) + \frac{m^2(r-2M)}{r^3\,\sin^2\theta}\right]\kappa_{00}^{(m)} \\
&+ \quad \frac{2M}{r^6}(r-2M)^2(2r-3M)\,\kappa_{11}^{(m)} - \frac{2M}{r^7}(r-2M)^2\,\kappa_{22}^{(m)} \\
&- \quad \frac{2M}{\sin^2\theta\, r^7}(r-2M)^2\,\kappa_{33}^{(m)} - \frac{4M}{r^3}(r-2M)\,\Pi_{10}^{(m)} \\
&- \quad \frac{r-2M}{r^3}\left(\partial_{22} + \cot\theta\,\partial_2\right)\kappa_{00}^{(m)} = 16\pi\,(r-2M)\,\tau_{00}^{(m)}
\end{aligned}$$

4 gauge conditions like:

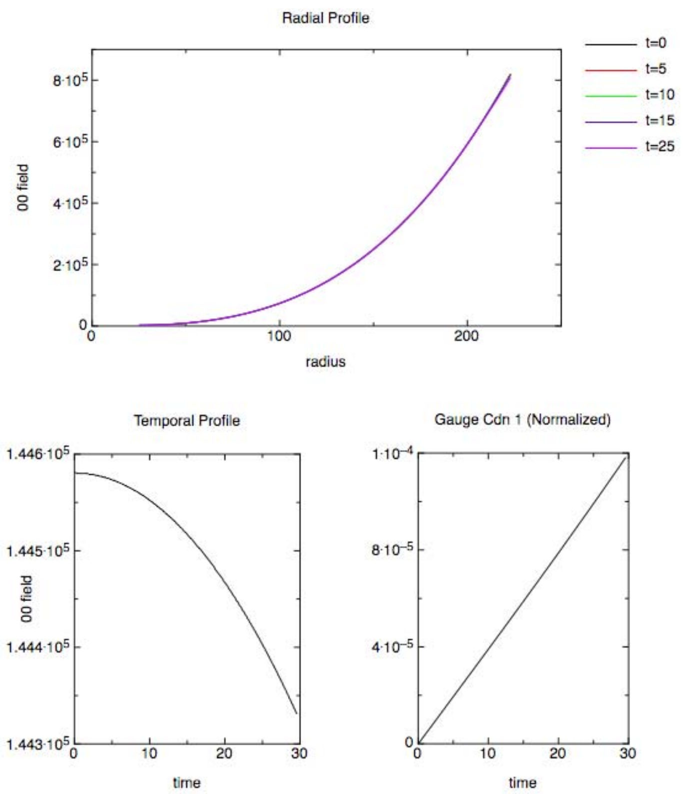
$$\Pi_{00} - \partial_{r^*}\kappa_{00}^{(m)} - \frac{r-2M}{r}\,\partial_{r^*}\kappa_{10}^{(m)} - \frac{r-2M}{r^2}\kappa_{10}^{(m)} - \frac{r-2M}{r^3}\,\cot\theta\kappa_{20}^{(m)}$$
$$-im\frac{r-2M}{r^2\,\sin^2\theta}\kappa_{30}^{(m)} - \frac{r-2M}{r^3}\,\partial_2\kappa_{20}^{(m)} = 0$$

First runs

Vacuum, Minkowski

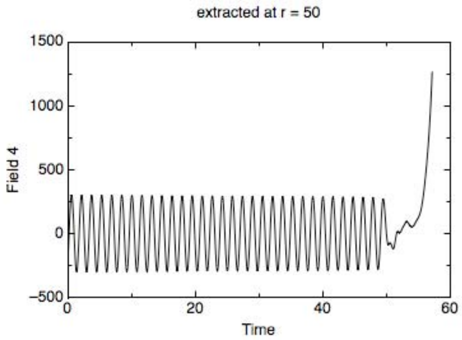
Static ID that satisfy the Lorenz gauge

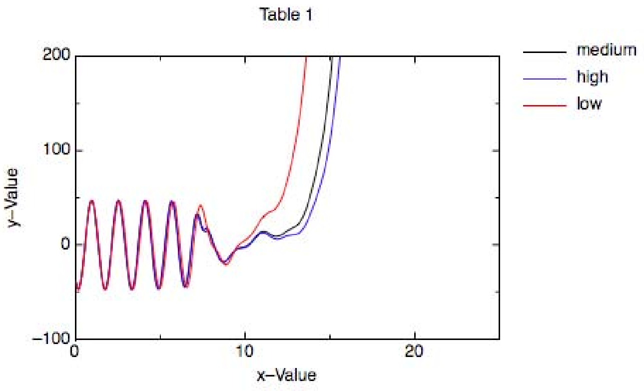
κ_{00}

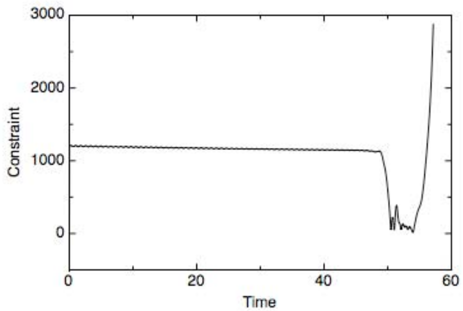


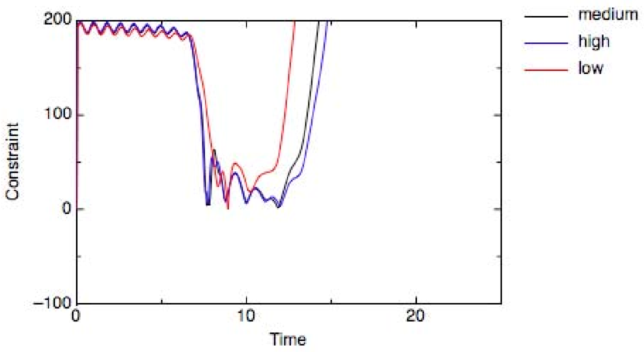
A time dependent solution (exact solution known and oscillatory)

κ_{30}



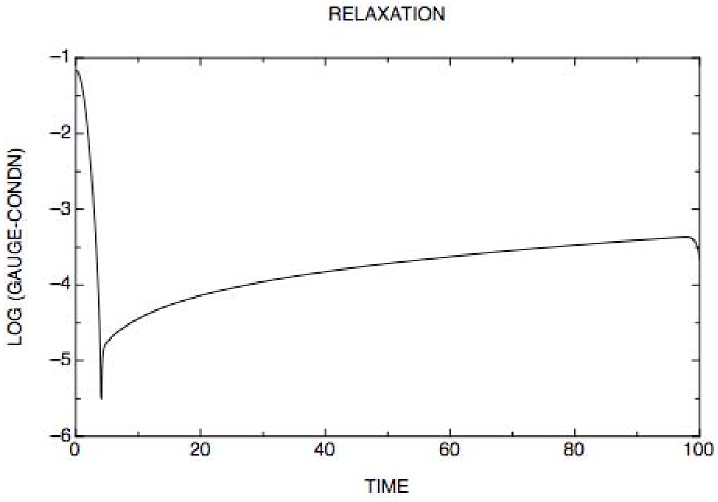


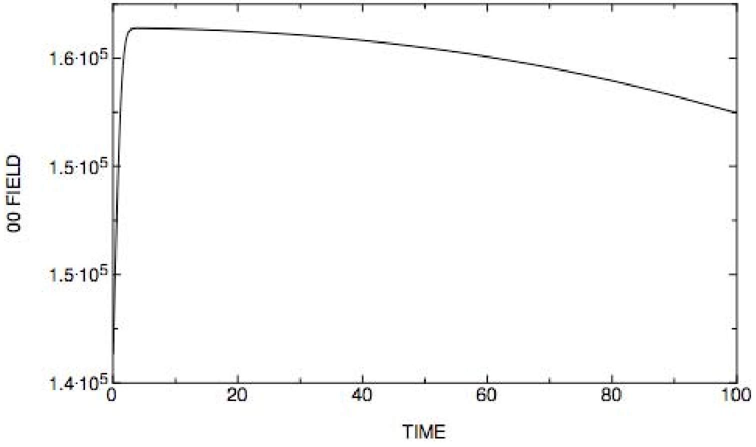




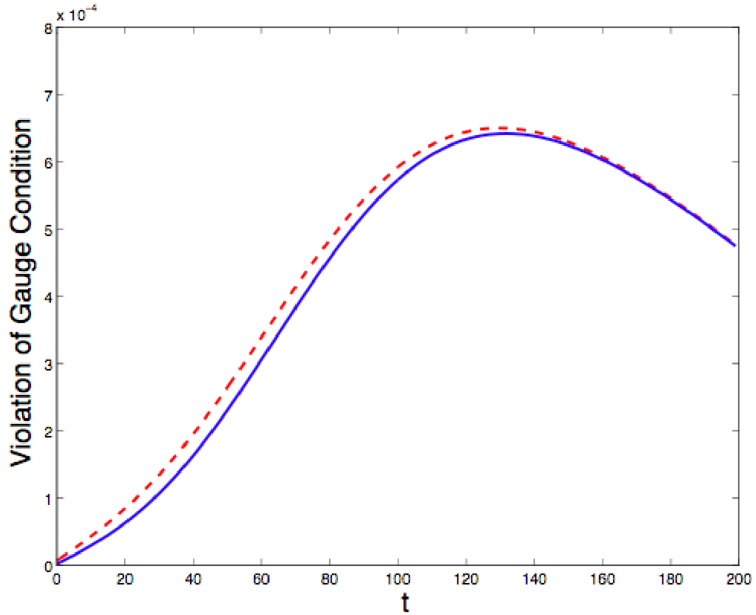
Lorenz violating initial data

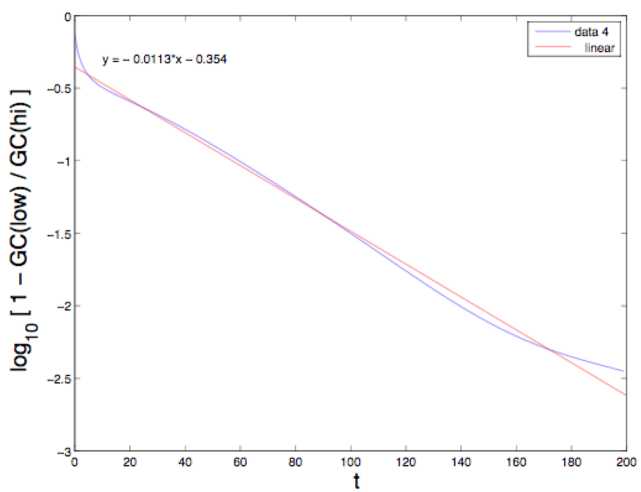
κ_{00}

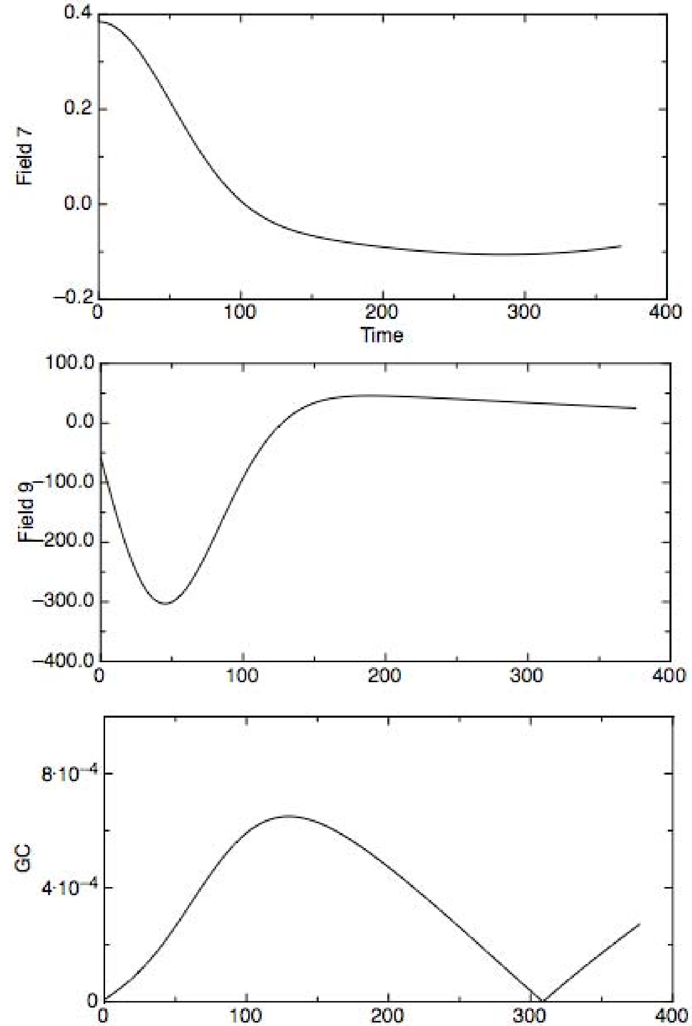




Coupled modes







κ_{13}

κ_{23}

Gauge violating modes

Numerical errors violate the Lorenz gauge.

1) Gauge violating damping terms
2) Evolve the fields with the full Einstein equations

The full field
(physical fields AND numerical error) does not satisfy

$$\Box\bar{h}_{bd} + 2R^{a}{}_{d}{}^{c}{}_{b}\bar{h}_{ac} = -16\pi T_{bd}$$

It satisfies

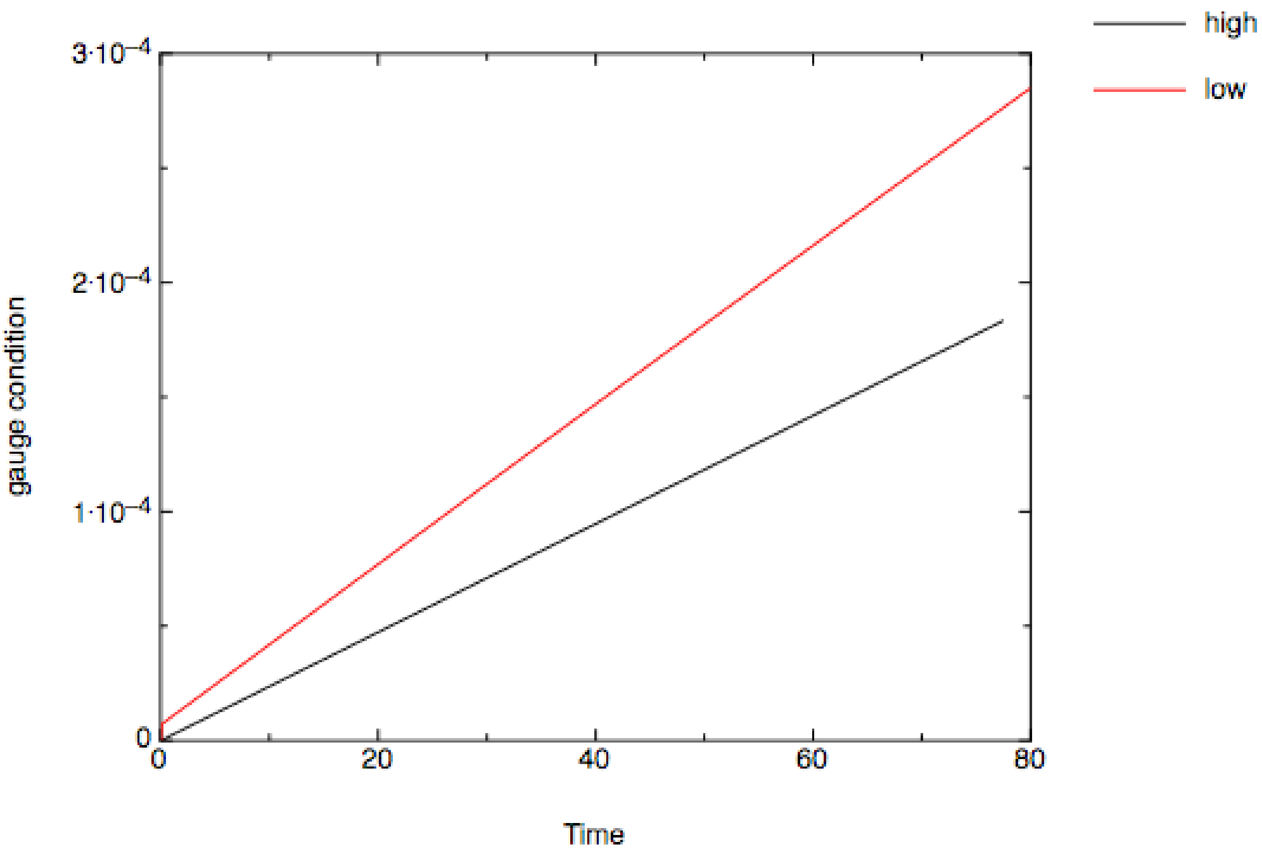
$$\Box\bar{h}_{bd} + 2R^{a}{}_{d}{}^{c}{}_{b}\bar{h}_{ac} \underbrace{+g_{bd}g^{ae}g^{cf}\bar{h}_{ef;ca} - g^{ac}\bar{h}_{cd;ab} - g^{ac}\bar{h}_{cb;ad}}_{\text{Lorenz gauge violating terms}} = -16\pi T_{bd}$$

$$\bar{h}_{\alpha\beta} := \bar{h}^{\text{exact}}_{\alpha\beta} + \Delta_{\alpha\beta}$$

$\bar{h}^{\text{exact}}_{\alpha\beta}$ satisfies the Einstein equations in the Lorenz gauge

$\Delta_{\alpha\beta}$ violates the Einstein equations in the Lorenz gauge; When evolved with the Lorenz-gauge equations, it’s not the Einstein equations we’re solving!

Does this equation remove the problem with gauge violating modes?



Problem: The new field equations mix $\dot{\Pi}_{ij}$ terms

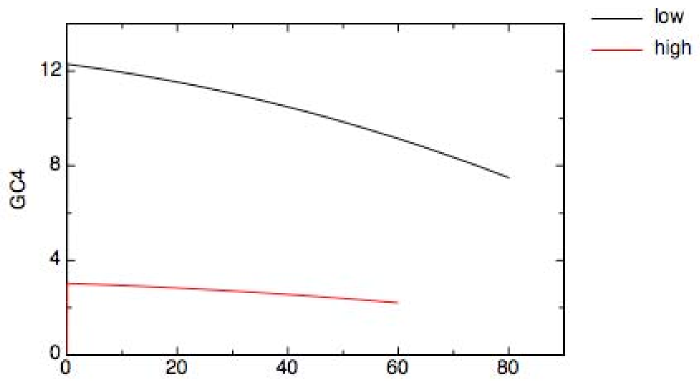
Solution: Define new fields

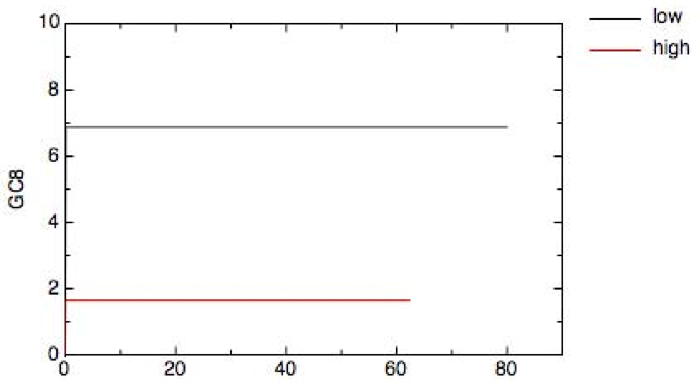
$$
\begin{aligned}
\kappa_{11}^{(m)} \to \underline{\kappa}_{11}^{(m)} &:= \kappa_{11}^{(m)} - \kappa_{00}^{(m)} \\
\kappa_{22}^{(m)} \to \underline{\kappa}_{22}^{(m)} &:= \kappa_{22}^{(m)} - r^2 \kappa_{00}^{(m)} \\
\kappa_{33}^{(m)} \to \underline{\kappa}_{33}^{(m)} &:= \kappa_{33}^{(m)} - r^2 \sin^2\theta \kappa_{00}^{(m)}
\end{aligned}
$$

which restore the equations to a single $\dot{\Pi}_{ij}$ term

But ID need to be satisfied in the Lorenz gauge

Notice: 4 equations become elliptic: 8 constraints
6 are hyperbolic

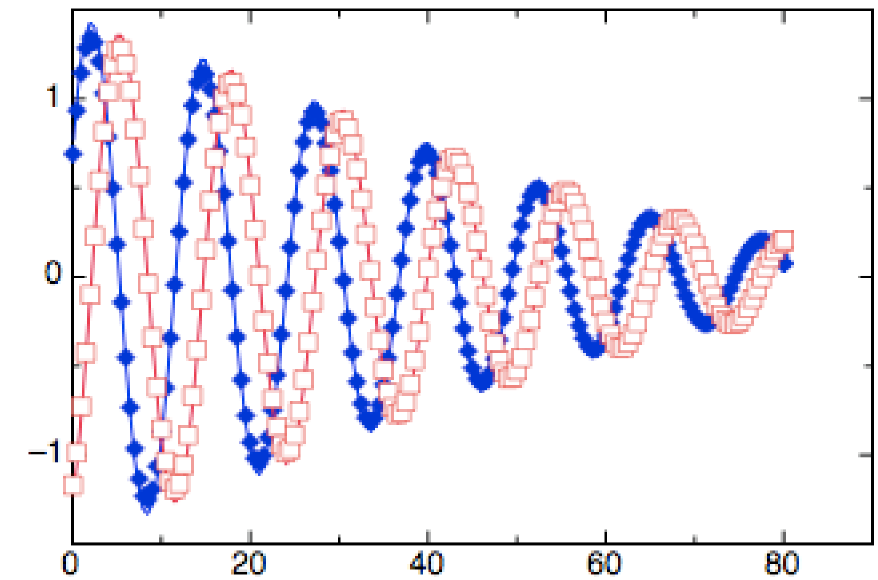


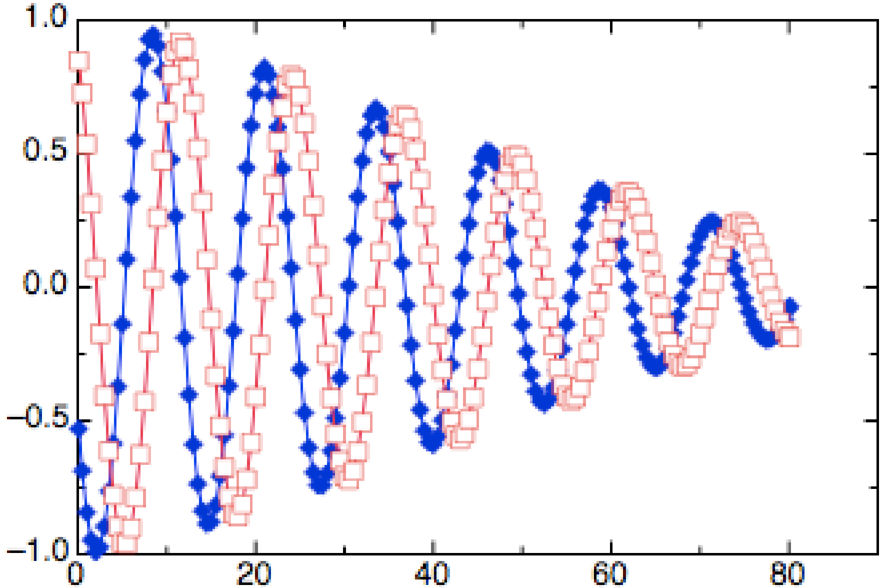


ID for a static κ_{30} field,
Excites also

κ_{13}

κ_{23}

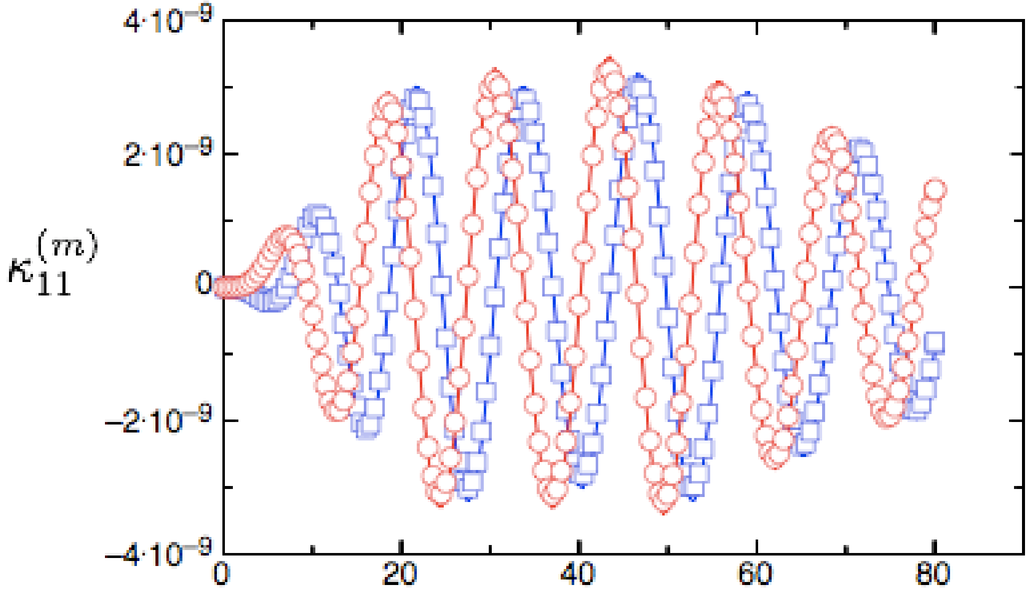




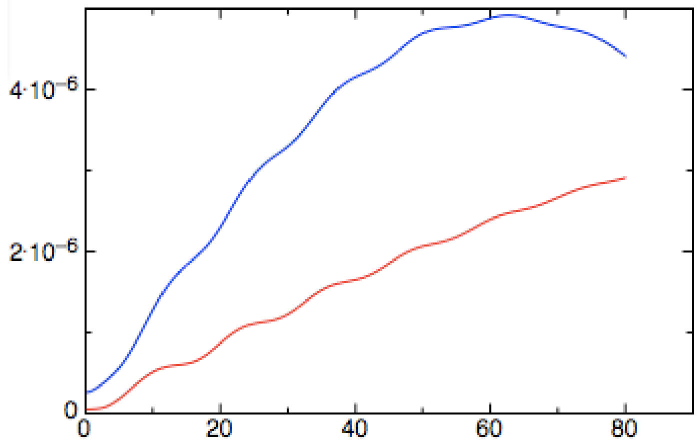
ID for oscillating $\kappa_{22}^{(m)}$ $\kappa_{33}^{(m)}$

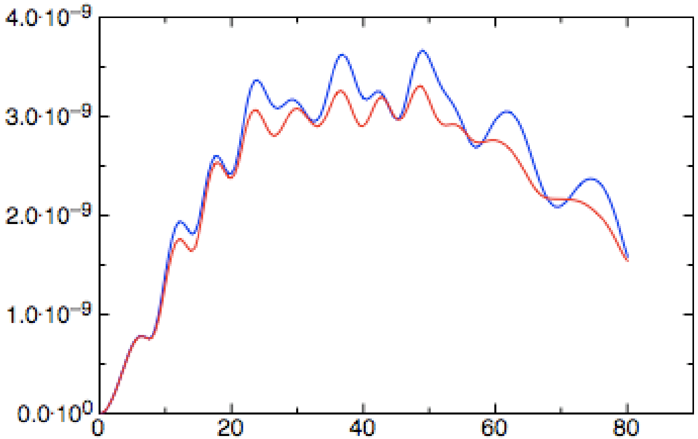
Excites other fields: $\kappa_{11}^{(m)}$ $\kappa_{12}^{(m)}$

Excited (initially zero) fields:

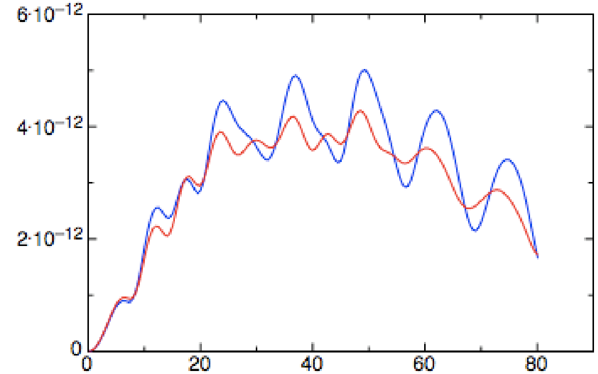


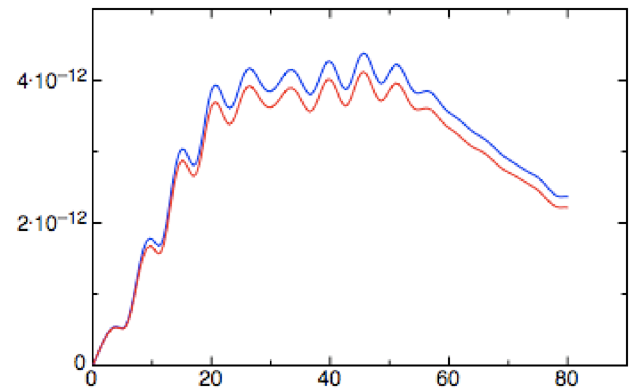
Lorenz gauge





Elliptic Einstein equations





Gauge violating damping terms

$$\Box \bar{h}_{bd} + 2R^a{}_d{}^c{}_b \bar{h}_{ac} - K\left(t_b Z_d + t_d Z_b\right) = -16\pi T_{bd}$$

$$\Box Z_b - K g^{ad} \nabla_a \left(t_b Z_d + t_d Z_b\right) = 0$$

$$Z_b := \nabla^a \bar{h}_{ab}$$

Allows to pose Lorenz-violating ID

Numerical instability in Schwarzschild and Kerr

$$\kappa_{\mu\nu}^{(m)} \rightarrow \tilde{\kappa}_{\mu\nu}^{(m)} := \left(\frac{\Delta}{r^2}\right)^{\delta_{\mu 1}+\delta_{\nu 1}} \kappa_{\mu\nu}^{(m)}$$

$$\partial_0 \kappa_{00}^{(m)} = -\,\partial_{r^*} \kappa_{00}^{(m)} + \Pi_{00}^{(m)}$$

Self force integrated waveforms:

$$u^\alpha(\tau) = u^\alpha|_{\rm bg}(\tau) + \frac{1}{\mu}\int_{-\infty}^{\tau} f(\tau')\, d\tau$$

Quasi-circular Schwarzschild orbits should be doable now! Results by next Capra meeting (not the Florida one!)

Quasi-self-consistency

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = \varepsilon a^{(1)\,\nu} + \varepsilon^2 a^{(2)\,\nu} + O(\varepsilon^3)$$

$$a = \underbrace{a^{(0)}}_{=0} + [\underbrace{a_D^{(1)}}_{\substack{\text{phase to}\\ O(\epsilon^{-1})}} + \underbrace{a_C^{(1)}]\,\epsilon + [a_D^{(2)}}_{\text{phase to } O(\epsilon^0)} + \underbrace{a_C^{(2)}]\,\epsilon^2}_{\text{phase to } O(\epsilon)} + \cdots$$

Comparison with the “radiation reaction without radiation reaction” approach. Conservative effects?

Focus on gauge-independent quantities: the waveform

- Barack and Ori (2001): “The meaningful description of the gravitational self force must include both $f^{\alpha}_{\rm SF}$ and the metric perturbations $h_{\alpha\beta}$.”
- Detweiler (2008): “The value in calculating the self force, in any particular gauge, is to apply it to a question whose answer is related to some physical observable. And a physical observable ought to be independent of the gauge choice.”