Elementary introduction to the gravitational self force

ADL

$$\mathcal{F}_{\rm rad} = \frac{2}{3}\frac{q^2}{c^3}\ddot{\mathbf{v}}$$

"Self-force" because the force is proportional to q^2.

Einstein tensor: $G_{ab}(g) = 8\pi\pi T_{ab}$

Expand in powers of h:

$$G(g+h) = G(g) + G^{(1)}(g,h) + G^{(2)}(g,h) + \ldots$$

$$E_{ab}(h) = -G^{(1)}_{ab}(g,h) = -\frac{\delta G_{ab}}{\delta g_{cd}} h_{cd} = -8\pi\pi T_{ab}.$$

$$\begin{aligned} 2E_{ab}(h) \;=\; & \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} \\ & + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h), \end{aligned}$$

If h_{ab} solves

$$E_{ab}(h) = -8\pi T_{ab}.$$

then the Einstein equation is satisfied through first order in h.

The Bianchi identity implies that

$$\nabla^a E_{ab}(k) = 0$$

Integrability condition on T_{ab}:

$$\nabla^a T_{ab} = 0.$$

The second simplest perturbation problem in GR:

Perturbation of Minkowskii space in the neighborhood of a geodesic Γ:

$$E_{ab}(H) = 0$$

$$g_{ab} = \eta_{ab} + H_{ab}$$

Thorne and Hartle (1985) and Zhang (1986):

- Minkowskii-like, locally inertial, harmonic coordinates
- t is the proper time along Γ
- $x = y = z = 0$ on Γ.
- Spatial, symmetric, trace-free *multipole moments* of the external spacetime $\mathcal{E}_{ij}$, $\mathcal{B}_{ij}$, functions only of t

Evaluated on Γ:

$$\mathcal{E}_{ij} \equiv R_{titj},$$

$$\mathcal{B}_{ij} \equiv \epsilon_i{}^{pq} R_{pqjt}/2,$$

$$H_{ab}dx^a dx^b \; = \; -\mathcal{E}_{ij}x^i x^j (\,\mathrm{d}t^2 + f_{kl}\mathrm{d}x^k\,\mathrm{d}x^l) + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i\,\mathrm{d}t\,\mathrm{d}x^k + \mathrm{O}(r^3/\mathcal{R}^3)$$

The easiest perturbation problem in General Relativity: A δ-function point mass m at rest in Minkowskii space.

$$E^{(\eta)}_{ab}(h) = 8\pi T_{ab}$$

$$\text{with} \qquad T_{ab} = m\delta^t_a\delta^t_b\,\delta(r)$$

$$h^{\mathrm{S}}_{ab}dx^a dx^b = \frac{2m}{r}(dt^2 + dr^2)$$

An “easy” perturbation of Schwarzschild problem:

Time independent limit, quadrupole perturbation of Schwarzschild:

$$
\begin{aligned}
(g_{ab}^{\mathrm{Schw}} + h_{ab}^{\mathrm{Schw}})\,\mathrm{d}x^a\mathrm{d}x^b = &-\Big(1-\frac{2m}{r}\Big)\Big[1-\mathcal{E}_{ij}x^ix^j\Big(1-\frac{2m}{r}\Big)\Big]\mathrm{d}t^2 \\
&+\frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_ix^px^i\Big(1-\frac{2m}{r}\Big)\mathrm{d}t\,\mathrm{d}x^k + \Big(\frac{1}{1-2m/r}-\mathcal{E}_{ij}x^ix^j\Big)\mathrm{d}r^2 \\
&+\Big[r^2-\big(r^2-m^2\big)\mathcal{E}_{ij}x^ix^j\Big]\big(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\big).
\end{aligned}
$$

In this expression x^i represents x,y and z which are related to r, θ and ϕ in the usual way in Cartesian space.

Various limits:

$m = 0 \Rightarrow g_{ab}^{\mathrm{Schw}} + h_{ab}^{\mathrm{Schw}} = \eta_{ab} + H_{ab}$

$\mathcal{E}_{ij} = \mathcal{B}_{ij} = 0 \Rightarrow g_{ab}^{\mathrm{Schw}} + h_{ab}^{\mathrm{Schw}} = g_{ab}^{\mathrm{Schw}}$

All terms linear in m:

$$
h_{ab}^{\mathrm{S}} = \ \mathrm{d}x^a\mathrm{d}x^b = \frac{2m}{r}\Big(1+2\mathcal{E}_{ij}x^ix^j\Big)\mathrm{d}t^2 + \frac{2m}{r}\mathrm{d}r^2 - \frac{8m}{3}\epsilon_{kpq}\mathcal{B}^q{}_ix^px^i\mathrm{d}t\,\mathrm{d}x^k
$$

$$
E_{ab}^{(\eta+H)}(h^{\mathrm{S}}) = 8\pi m\delta_a^t\delta_b^t\,\delta(r)
$$

If the “actual” metric is

$$g_{ab}^{\rm act} \approx \eta_{ab} + H_{ab} + h_{ab}^{\rm S}$$

then m stays on the geodesic Γ of

$$g_{ab}^{\rm act} - h_{ab}^{\rm S} = \eta_{ab} + H_{ab}.$$

In a perturbation problem, find $h_{ab}^{\rm ret}$ for source m then

$$g_{ab}^{\rm act} = g_{ab}^{\rm Schw} + h_{ab}^{\rm ret} \approx \eta_{ab} + H_{ab} + h_{ab}^{\rm S}$$

Then m stays on a geodesic of

$$g_{ab}^{\rm act} - h_{ab}^{s} = g_{ab}^{\rm Schw} + h_{ab}^{\rm ret} - h_{ab}^{\rm S} = g_{ab}^{\rm Schw} + h_{ab}^{\rm R}$$

where

$$h_{ab}^{\rm R} = h_{ab}^{\rm ret} - h_{ab}^{\rm S}$$

Conclusion: The gravitational self-force appears as geodesic motion in a vacuum spacetime metric $g_{ab} + h_{ab}^{\rm R}$ where $h_{ab}^{R} = h_{ab}^{\rm ret} - h_{ab}^{\rm S}$

$$h_{ab} = g_{ab}^{\text{pert}} - g_{ab}^{\text{S}}$$

A self-force effect on quasi-circular orbit of Schwarzschild:

$$\Omega^2 = \frac{m}{r^3} - \frac{r-3m}{2r^2} u^a u^b \partial_r h_{ab}$$

for a quasi-circular orbit.

Gauge confusion in Newtonian gravity

$$m_1 r_1 = m_2 r_2$$

$$R = r_1 + r_2 = r_1(1 + m_1/m_2)$$

Newton's law of gravity gives

$$m_1 \Omega^2 r_1 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \ .$$

The self-force decreases Ω:

$$\begin{aligned} \Omega^2 &= \frac{G m_2}{r_1 (r_1 + r_2)^2} \\ &= \frac{G m_2}{r_1^3 (1 + m_1/m_2)^2} \ , \\ &= \frac{G m_2}{r_1^3} (1 - 2 m_1/m_2 + \ldots) \ . \end{aligned}$$

The self-force increases Ω:

$$\Omega^2 = \frac{G m_2 (1 + m_1/m_2)}{R^3}$$

Newtonian potential:

$$\nabla^2 \phi = 4\pi\rho$$

Newtonian potential:

$$\nabla^2 \phi = 4\pi m \delta(r)$$