Capra XII Bloomington, IN
Second order self-force
or
Numerical relativity meets the self-force

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Motivation:

The self-force problem for quasi-circular inspiral in Schwarzschild, (finding all components of the self-force in the Lorenz gauge) has been solved.

Components of the self-force show:

$$\begin{aligned}\frac{dE}{ds} &= -\frac{1}{2}u^a u^b \frac{\partial h^{\mathrm{R}}_{ab}}{\partial t}, \\ \frac{dJ}{ds} &= \frac{1}{2}u^a u^b \frac{\partial h^{\mathrm{R}}_{ab}}{\partial \phi}, \\ \frac{d}{ds}\Big(\frac{r\dot{R}}{r-2m} + u^a h^{\mathrm{R}}_{ar}\Big) &= \frac{1}{2}u^a u^b \frac{\partial}{\partial r}(g_{ab} + h^{\mathrm{R}}_{ab}).\end{aligned}$$

Analytic work shows that

$$\frac{d\Omega}{dt} = \frac{dE}{dt} \times \frac{d\Omega/dr}{dE/dr} + O(\mu^2)$$

gives the rate of change of Ω for quasi-circular inspiral. Numerical work has been used to evaluate $d\Omega/dt$.

A "success" of self-force analysis has been a proof that this expression for $d\Omega/dt$ is correct.

This result is completely equivalent to analyses that were probably done in the 1970's, and have certainly been done many times since.

Determine $h^{\rm S}_{ab}$ from tidally distorted Schwarzschild

Tidal environment:

$$g_{ab}dx^a dx^b \approx \eta_{ab}dx^a dx^b + H_{ab}dx^a dx^b \;=\; \eta_{ab} - \mathcal{E}_{ij}x^i x^j (\,\mathrm{d}t^2 + f_{kl}\mathrm{d}x^k\,\mathrm{d}x^l) \\ + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i\,\mathrm{d}t\,\mathrm{d}x^k + \mathrm{O}(r^3/\mathcal{R}^3)$$

Tidally distorted Schwarzschild:

$$\begin{aligned}(g^{\rm Schw}_{ab} + h^{\rm Schw}_{ab})\,\mathrm{d}x^a\mathrm{d}x^b = &-\Big(1-\frac{2m}{r}\Big)\Big[1-\mathcal{E}_{ij}x^i x^j\Big(1-\frac{2m}{r}\Big)\Big]\mathrm{d}t^2 \\ &+\frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i\Big(1-\frac{2m}{r}\Big)\mathrm{d}t\,\mathrm{d}x^k + \Big(\frac{1}{1-2m/r}-\mathcal{E}_{ij}x^i x^j\Big)\mathrm{d}r^2 \\ &+\Big[r^2-\big(r^2-m^2\big)\mathcal{E}_{ij}x^i x^j\Big]\big(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\big).\end{aligned}$$

$h^{\rm S1}_{ab}$: terms proportional to m

$h^{\rm S2}_{ab}$: terms proportional to m^2

Convention for expanding the Einstein tensor

$$G(g+h) \;=\; G(g) + G^{(1)}(g,h) + G^{(2)}(g,h) + G^{(3)}(g,h) + O(h^4),$$

where $G^{(n)}(g,h) = O(h^n)$.

$G^{(1)}_{ab}(g,h) = E_{ab}(h)$, the usual linear operator.

$G^{(2)}(g,h)$ is made of terms $h\nabla h + \nabla h \nabla h$

$G^{(3)}(h)$ is unfamiliar.

Assume that g_{ab} is a vacuum solution.

3+1 treatment of first order self-force

We wish to solve $G(g + h^{\rm ret}) = 8\pi T + O(h^2)$.

Assume that $h^s_{ab} = h^{\rm S}_{ab} + O(mr^3/\mathcal{R}^4)$. The $O(mr^3/\mathcal{R}^4)$ is chosen to provide convenient global behavior for h^s_{ab}

$$
\begin{aligned}
G(g + h^{\rm ret}) &= G(g + h^{\rm R} + h^s) \\
&= G(g) + G^{(1)}(g, h^{\rm R} + h^s) = 8\pi T + O(h^2) \\
&= G^{(1)}(g, h^{\rm R}) + G^{(1)}(g, h^s) = 8\pi T + O(h^2)
\end{aligned}
$$

Rewrite this as

$$
\begin{aligned}
G^{(1)}(g, h^{\rm R}) &= 8\pi T - G^{(1)}(g + h^{\rm R}, h^s) + O(h^2) \\
&= 8\pi T - G^{(1)}(g, h^s) + O(h^2)
\end{aligned}
$$

Define

$$8\pi T - G^{(1)}(g, h^s) \equiv S_{ab}$$

S_{ab} is known analytically and $S_{ab} = O(mr\mathcal{R}^4)$ near $r = 0$.

Finally just solve

$$G^{(1)}(g, h^{\rm R}) = S_{ab}$$

while adjusting the worldline of m to be a geodesic in $g_{ab} + h^{\rm R}_{ab}$. $h^{\rm R}$ is C^2 at $r = 0$, and C^∞ elsewhere.

After finding $h^{\rm R}_{ab}$ reconstruct

$$h^{\rm ret}_{ab} = h^{\rm R}_{ab} + h^s_{ab}.$$

3+1 treatment of second order self-force

Let $h^s_{ab} = h^{s1}_{ab} = h^{s2}_{ab}$.

Note that

$$\left\{G^{(1)}(g,h^{s2}) + G^{(2)}(g,h^{s1})\right\} = O(m^2/\mathcal{R}^4) \text{ at } m$$

Assume that $h^s_{ab} = h^{\mathrm{S}}_{ab} + O(mr^3/\mathcal{R}^4) + O(m^2r^2/\mathcal{R}^4)$. The $O(mr^3/\mathcal{R}^4)$ and $O(m^2r^2/\mathcal{R}^4)$ are chosen to provide convenient global behavior for h^s_{ab}

Solve the "main equation:"

$$G(g+h^{\mathrm{ret}}) = G(g+h^{\mathrm{R}}+h^s) = 8\pi T + O(h^3)$$

Initially treat h^{R}_{ab} as part of the "background" metric and expand:

$$G(g+h^{\mathrm{R}}+h^s) \;=\; G(g+h^{\mathrm{R}}) + G^{(1)}(g+h^{\mathrm{R}},h^s) + G^{(2)}(g+h^{\mathrm{R}},h^s) + O(h^3),$$

The first term on the right hand side expands to

$$G(g+h^{\mathrm{R}}) \;=\; G^{(1)}(g,h^{\mathrm{R}}) + G^{(2)}(g,h^{\mathrm{R}}) + O(h^3),$$

The third term is

$$G^{(2)}(g+h^{\mathrm{R}},h^s) \;=\; G^{(2)}(g,h^s) + O(h^3).$$

Now, a recombination of these three parts back together yields

$$\begin{aligned} G(g+h^{\mathrm{R}}+h^s) \;=\; & G^{(1)}(g,h^{\mathrm{R}}) + G^{(2)}(g,h^{\mathrm{R}}) \\ & + G^{(1)}(g+h^{\mathrm{R}},h^s) + G^{(2)}(g,h^s) + O(h^3). \end{aligned}$$

Substitute this result back into main equation and regroup:

$$G^{(1)}(g,h^{\mathrm{R}}) \;=\; -G^{(2)}(g,h^{\mathrm{R}}) - [G^{(1)}(g+h^{\mathrm{R}},h^s) + G^{(2)}(g,h^s) - 8\pi T] + O(h^3).$$

On the left: usual first order perturbed Einstein operator $E_{ab}(h^{\mathrm{R}})$.

On the right: $G^{(2)}(g,h^{\mathrm{R}})$ which includes $\nabla h^{\mathrm{R}}\nabla h^{\mathrm{R}} + h^{\mathrm{R}}\nabla\nabla h^{\mathrm{R}}$.

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Focus on "square brackets"

$$-[G^{(1)}(g + h^{\rm R}, h^s) + G^{(2)}(g, h^s) - 8\pi T]$$

First term of Sq brackets:

$$\begin{aligned} G^{(1)}(g + h^{\rm R}, h^s) &= G^{(1)}(g + h^{\rm R}, h^{s1} + h^{s2}) \\ &= G^{(1)}(g + h^{\rm R}, h^{s1}) + G^{(1)}(g, h^{s2}) + O(h^3) \end{aligned}$$

Rearrange terms and Sq brackets becomes:

$$\underbrace{\Big(G^{(1)}(g + h^{\rm R}, h^{s1}) - 8\pi T\Big)}_{O(mr/\mathcal{R}^4) \text{ is } C^0} \;+\; \underbrace{\Big\{G^{(1)}(g, h^{s2}) + G^{(2)}(g, h^{s1})\Big\}}_{m^2\mathcal{R}^2 \text{ is discontinuous at } r=0} \equiv S_{ab}$$

First order source is C^0. Second-order source is discontinuous.

Final form:

$$G^{(1)}(g, h_{\rm R}) \;=\; -G^{(2)}(g, h_{\rm R}) + S_{ab}$$

where

$$S_{ab} \;=\; \Big(G^{(1)}(g + h^{\rm R}, h^{s1}) - 8\pi T\Big) + \Big\{G^{(1)}(g, h^{s2}) + G^{(2)}(g, h^{s1})\Big\}$$