Discontinuous Galerkin Method for Computing Perturbations of a Schwarzschild Spacetime

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Organization of Presentation

- Overview of Perturbation Treatment of the Extreme Mass Ratio Binary (EMRB) Problem
- Discontinuous Galerkin Method
- Numerical Scheme for Perturbation Equations
- Results
- Conclusion and Future Work

Extreme Mass Ratio Binary

Physical Relevance

- EMRBs expected to be primary source of gravitational waves detected by LISA
  - "Small" star of mass m_p orbiting a super-massive black hole of M where $\mu = m_p / M << 1$
- Motivation for considering new high-order (in this particular case spectral) numerical methods: Construction of quality gravitational waveforms in the shortest amount of time.

Background and Assumptions

- Point particle in stable orbit of SMBH
  - Schwarzschild $ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \qquad f = 1 - 2M/r$
  - Particle follows a time-like geodesic in equatorial plane
  - Stress energy tensor $T^{\alpha\beta} = m_p \int \frac{d\tau}{\sqrt{-g^{Sch}}} u^\alpha u^\beta \delta^4[x - z(\tau)]$
- Particle causes small metric perturbations $h_{\alpha\beta}$
  - Metric $g_{\alpha\beta} = g^{Sch}_{\alpha\beta} + h_{\alpha\beta} \quad \Rightarrow \quad \delta G^{Lin}_{\alpha\beta}(h) = 8\pi T_{\alpha\beta}$
  - Perturbations can be combined into master-functions which evolve according to

$$(-\partial_t^2 + \partial_x^2 - V^{A/P}(r))\psi^{A/P}(x,t) = G^{A/P}(t,r)\delta(r - rp) + F^{A/P}(t,r)\partial_r \delta(r - rp)$$

Equations and Quantities

- In the distant wave-zone and at BH horizon:

$$h_+ - ih_x = \frac{1}{2r}\sum_{l\geq 2,m}\sqrt{\frac{(l+2)!}{(l-2)!}}\{\psi^P_{lm} + i\psi^A_{lm}\}_{-2}Y^{lm}$$

$$\dot{E}_{lm} = \frac{1}{64\pi}\frac{(l+2)!}{(l-2)!}|\dot{\psi}_{lm}|^2 \qquad \dot{L}_{lm} = \frac{im}{64\pi}\frac{(l+2)!}{(l-2)!}\psi^*_{lm}\dot{\psi}_{lm}$$

- Metric perturbations can be reconstructed everywhere in space

Discontinuous Galerkin Method

Numerical Simulations of EMRBs

- Numerous proposed methods, such as…
  - Frequency-domain
    - Fourier decomposition
    - Becomes numerically expensive when one must sum over many modes – e.g. eccentric orbits
  - Time-domain
    - Typically spatial derivatives are approximated by finite differences
      - Errors fall off as a power
      - Long time integrations can produce phase errors in low order methods
    - Distributional source term can be troublesome
    - Large computational domain-size
      - Waveform/BC issues

DG: A Hybrid of Methods

- Finite element method: if $D^k = \Omega$ and basis functions are "coupled" (ie span multiple elements) we get a finite element method
- Legendre collocation, "Pseudospectral," on each element
- When basis functions are constants, dG formally becomes a finite volume method

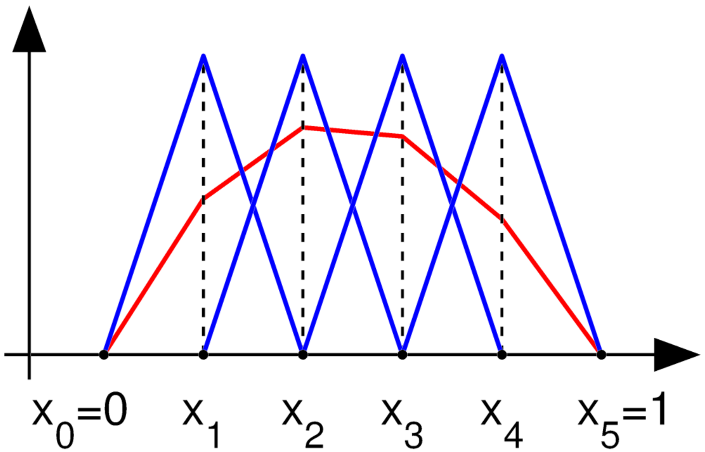


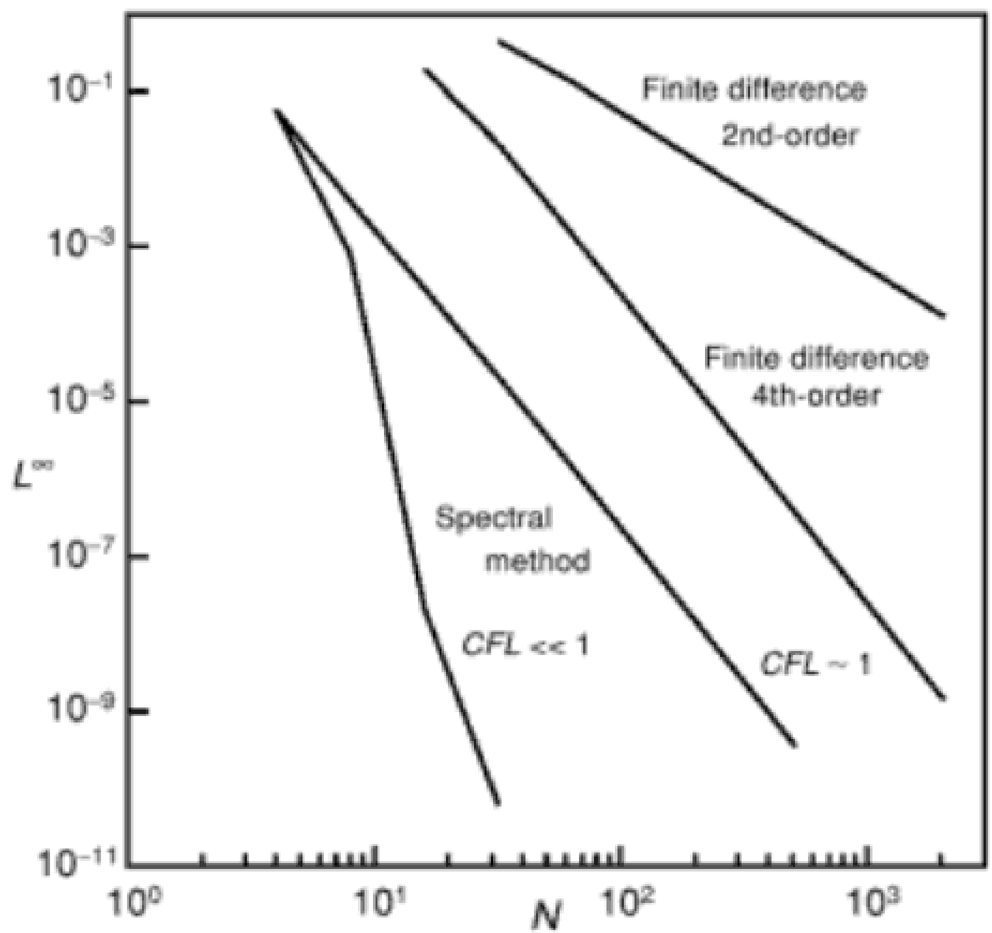
Figure created by Oleg Alexandrov

Spectral and Finite Difference Errors

- 1D advection equation with speed 2pi, on a domain 0 to 2pi. Integration with RK4

$$u(x) = e^{\sin(x - 2\pi t)}$$

Spectral Methods for Time Dependent Problems by Jan Hesthaven & David Gottlieb



Advection: Phase Errors

- When evolving hyperbolic PDEs for long times, phase errors can become significant
- An example with the advection equation

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0 \qquad \rightarrow \qquad \psi(x,t) = \exp[ik(x - ct)]$$

$$IC: \ \psi(x,0) = \exp(ikx)$$

$$\frac{\partial \psi_j}{\partial t} + cD_{Num}\psi_j = 0 \qquad \rightarrow \qquad \psi_j(t) = \exp[ik(x_j - c_{num}t)]$$

$$IC: \ \psi_j = \exp(ikx_j)$$

- **Numerical solution propagate at wrong speed**

Advection: Phase Errors

- If using second order central difference

$$D_{Num}\psi_j = \frac{\psi_{j+1} - \psi_{j-1}}{2(\Delta x)}$$

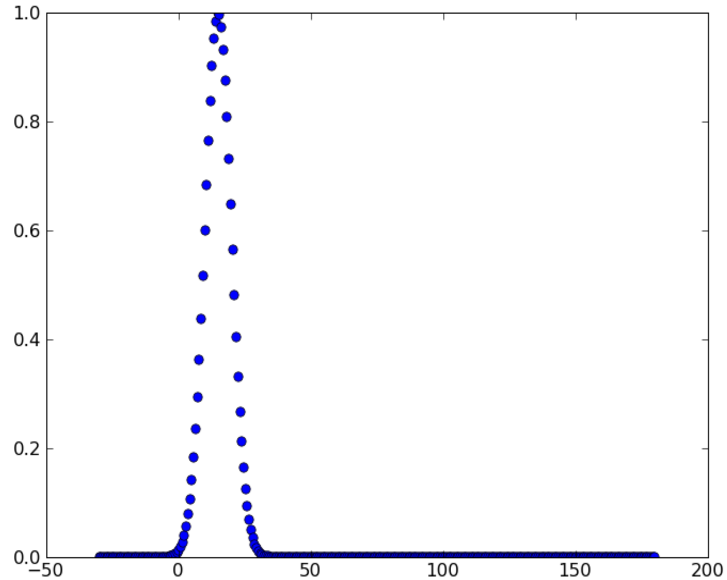
we have a phase error of

$$\left|\exp(-ikct) - \exp(-ikc_{num}t)\right| = ktc\left(1 - \frac{\sin k\Delta x}{k\Delta x}\right) = \frac{ktc(k\Delta x)^2}{6}$$

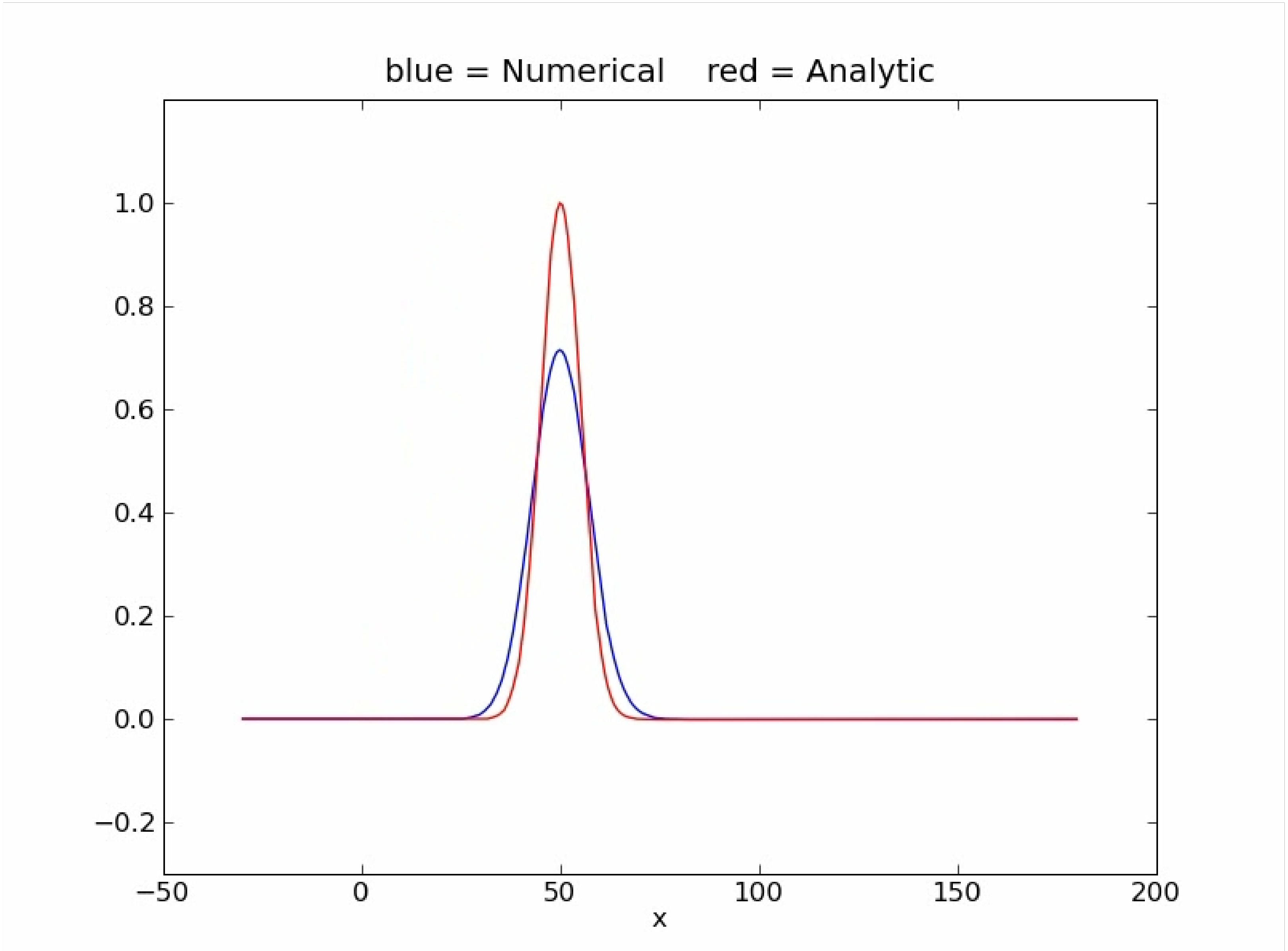
- Typically phase errors scale as $\propto t(k\Delta x)^{order}$
- To achieve small phase error one must:
  - Run for short times
  - Use a fine grid
  - High order method

Advection: Phase Errors

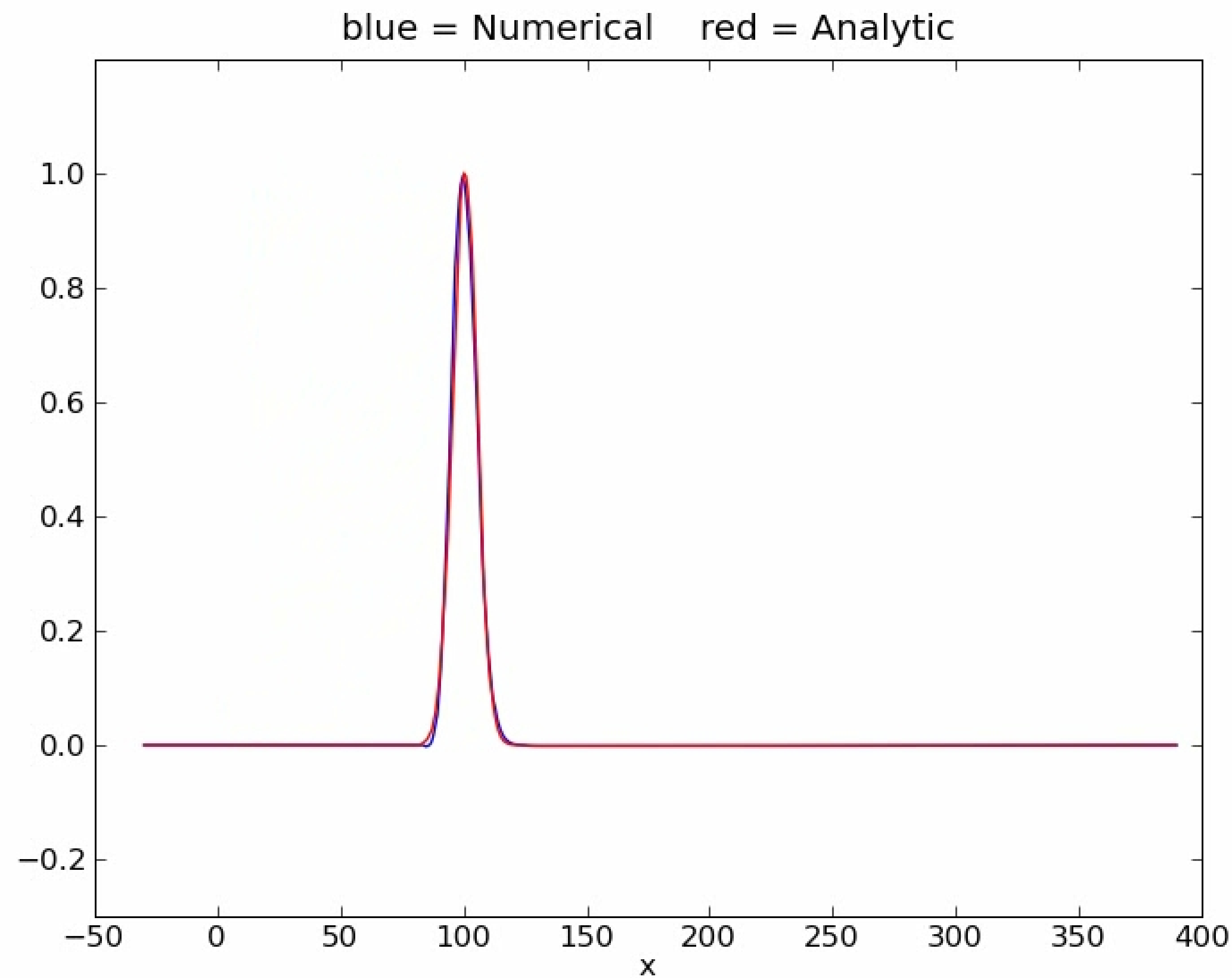
- Using a 4th order Runge-Kutta time integrator, evolve a Gaussian packet using first and second order approximations to $\frac{\partial}{\partial x}$
- In each example we have chosen $\Delta x \sim .5$



1st Order Upwind Differencing

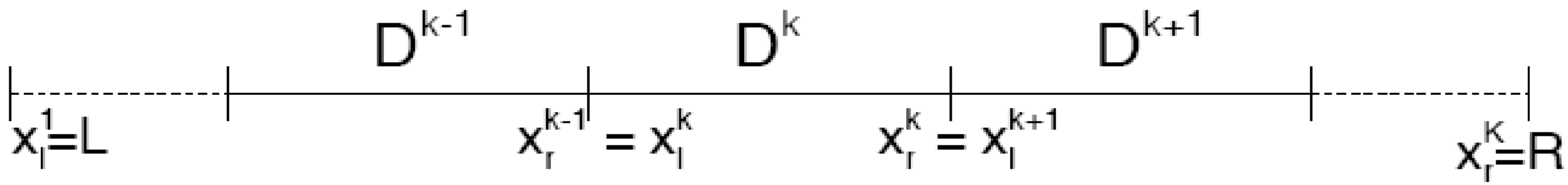


2nd Order Central Differencing



Discontinuous Galerkin

- Domain Ω approx. by local elements D^k $\quad \Omega \approx \Omega_h = \bigcup_{k=1}^{K} D^k$



- Local solution is a linear combination of basis functions

$$x \in D^k : u_h^k(x,t) = \sum_{n=0}^{N} \hat{u}_n^k(t) P_n(x) = \sum_{i=0}^{N} u_h^k(x_i,t)\, l_i^k(x)$$

  - P_n Form a basis for space of polynomials in D^k of degree at most N

$$Span\{P_n : 0 \le n \le N\} = P^N(D^k)$$

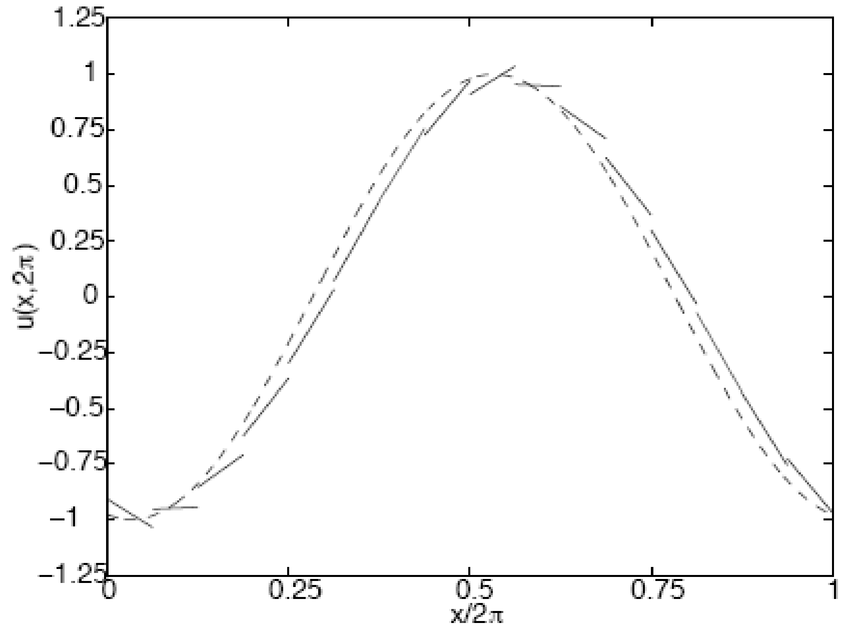
- Global solution is a direct sum of local solutions

$$u(x,t) \approx u_h(x,t) = \bigoplus_{k=1}^{K} u_h^k(x,t)$$

Discontinuous Galerkin

- Given an operator L such that $Lu = 0$ can be cast in integral form $\int_{D^k} (Lu)v\,dx = 0$ on elements D^k
  - Require of numerical solution u_h $\int_{D^k} (Lu_h^k)v_n^k\,dx = 0;\ 0 \le n \le N$
  - Galerkin condition: test and basis functions taken to be the same

➢ To resolve: coupling between elements?



Nodal Discontinuous Galerkin Methods by Jan Hesthaven and Tim Warburton

Numerical Flux

- To couple elements first perform IBPs

$$\int_{D^k} (\partial_t u_h^k + \partial_x f(u_h^k)) v\,dx = 0 \rightarrow \int_{D^k} (v \partial_t u_h^k - f(u_h^k) \partial_x v)\,dx = -\oint_{\partial D^k} \hat{n} \cdot (f^*(u_h)) v\,dx$$

- Numerical flux $f^*(u_h) = f^*(u_h^+, u_h^-)$
  - u_h^+ and u_h^- are the exterior and interior solutions
  - passes information between elements, ensures stability of scheme, implements boundary conditions
  - Choice of $f^*(u_h)$ related to dynamics of PDE system
    - Consistency condition: $f(u_h) = f^*(u_h, u_h)$
- Example:
  - Central flux $f^*(u_h) = \dfrac{f(u_h^-) + f(u_h^+)}{2}$

Example: Advection with a Distributional Source Term

Problem Statement

- Each $(\ell, m, parity)$ sector of the perturbation equations similar in spirit to 2 copies of advection
- Equation: $\frac{1}{c}\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} - G(t)\delta(x) = 0$
- Domain: $\Omega = [a,b]$ s.t. $a < 0 < b$
- BC: $\psi(a,t) = 0$
- Initial data: $\psi(x,t_0) = \Theta(x)G(t_0 - x/c)$
- Jump condition: $\psi(0^+) - \psi(0^-) = G(t)$
- Sends information G(t) off to the right at speed c

Numerical Scheme

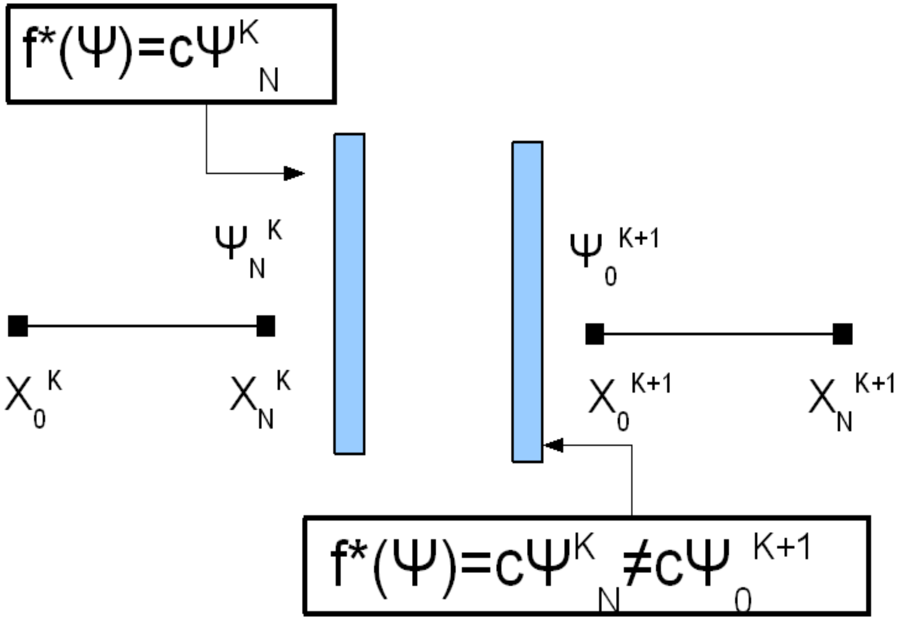
- Basis space: Lagrange interpolating polynomials of order N on defined on each element

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{x - \eta_j}{\eta_i - \eta_j}$$

  - Legendre-Gauss-Lobatto nodal points η
- Integrate with fourth order Runge-Kutta
- Numerical flux:
  - PDE shifts solution to

the right, so use an “upwinded” form

$$f^*(\psi_h) = c\left(\frac{\psi^+ + \hat{n}^+ \cdot \psi^+}{2} + \frac{\psi^- + \hat{n}^- \cdot \psi^-}{2}\right)$$



A Generalized DG (GDG) Method

- GDG extends DG to solutions (analytically) discontinuous at an interface
- Idea: treat the δ function in the PDE as an additional numerical flux term in an appropriate way, ie mimic the physics of the PDE
  - Require usual delta property over Ω $\int_{\Omega} \delta(x) v(x) dr = v(0)$
  - Freedom to choose how to "split" the delta function between adjacent elements

$$\int_{D^i \bigcup D^{i+1}} \delta(x) v(x) dr = \int_{D^i} \delta(x) v(x) dr + \int_{D^{i+1}} \delta(x) v(x) dr = a v(0) + b v(0) = v(0)$$

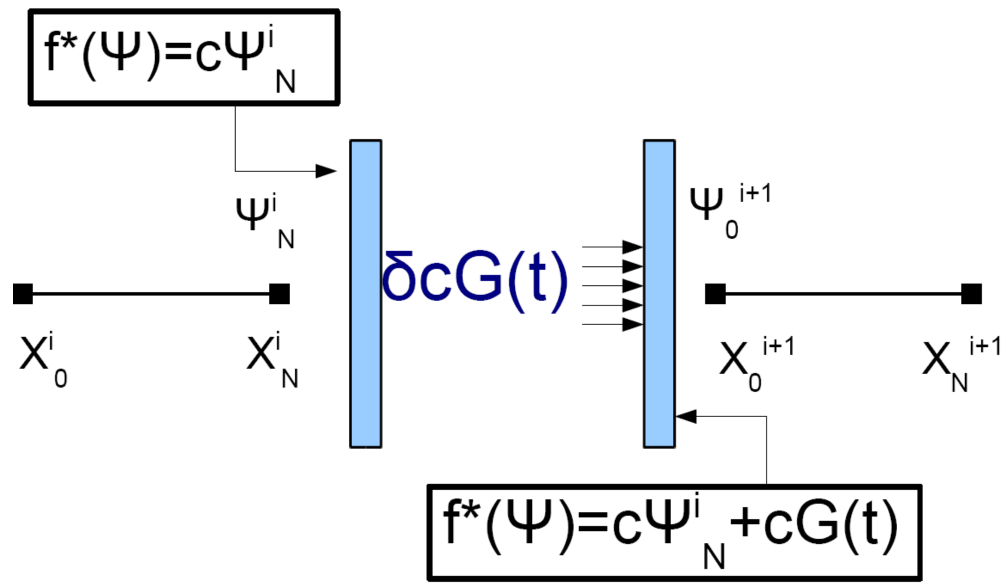
K. Fan, W. Cai, X. Ji, *A generalized discontinuous Galerkin (GDG) method for Schrodinger equations with nonsmooth solutions*, J. Comp. Phys., 227 (2008) 2387-2410.

Treating an Advected δ Source

- Upwind the source:

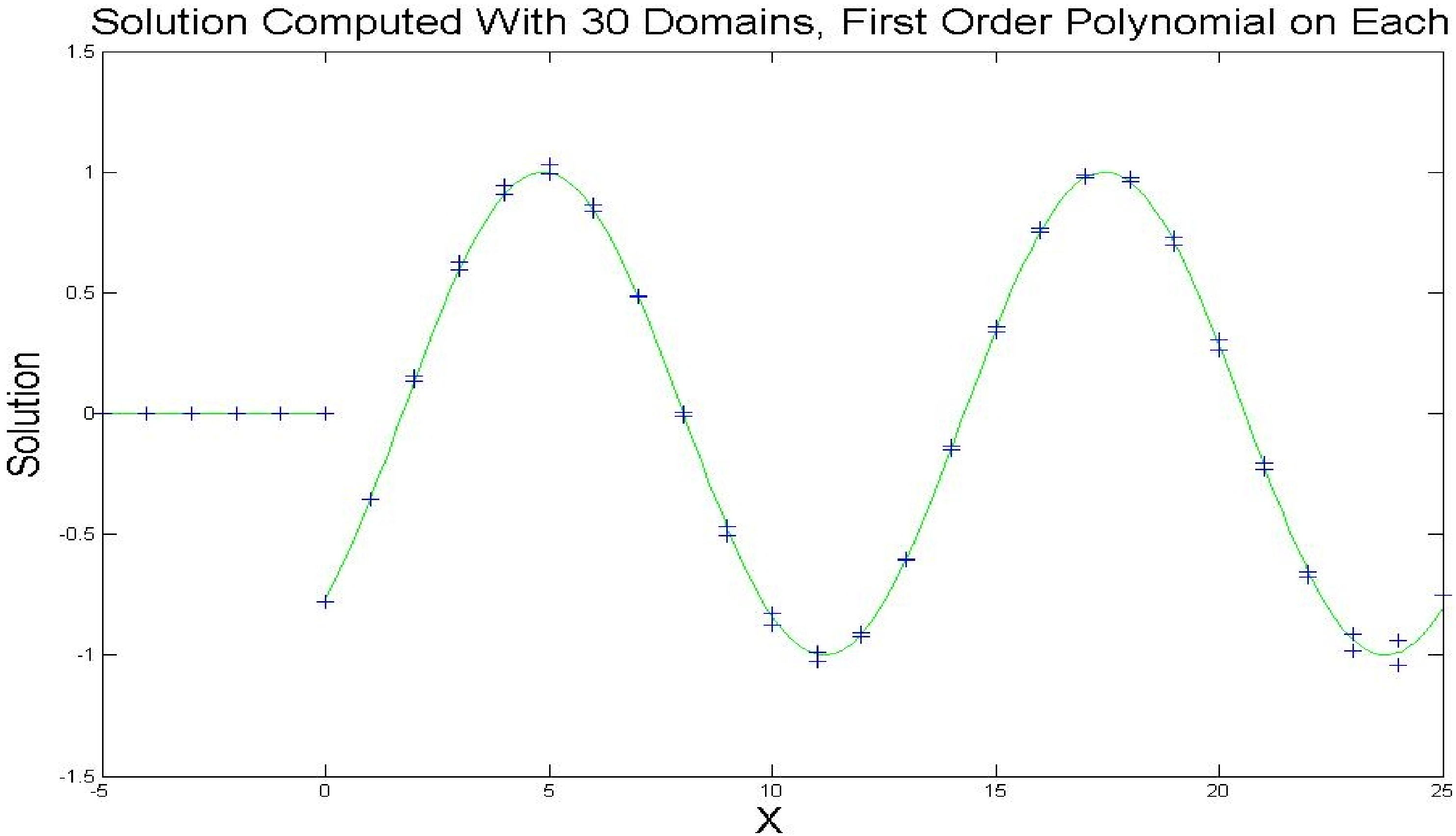
$$\int_{D^i} \vec{l}(x)[cG(t)\delta(x)]^* dx = 0 \qquad \int_{D^{i+1}} \vec{l}(x)[cG(t)\delta(x)]dr = cG(t)\vec{l}(0)$$

- GDG principle suggests we add a flux term $(cG)^{*,\delta}_{i+1,L}$ to the existing flux term $[c\psi(x,t)]^*$ on the right hand side of the interface



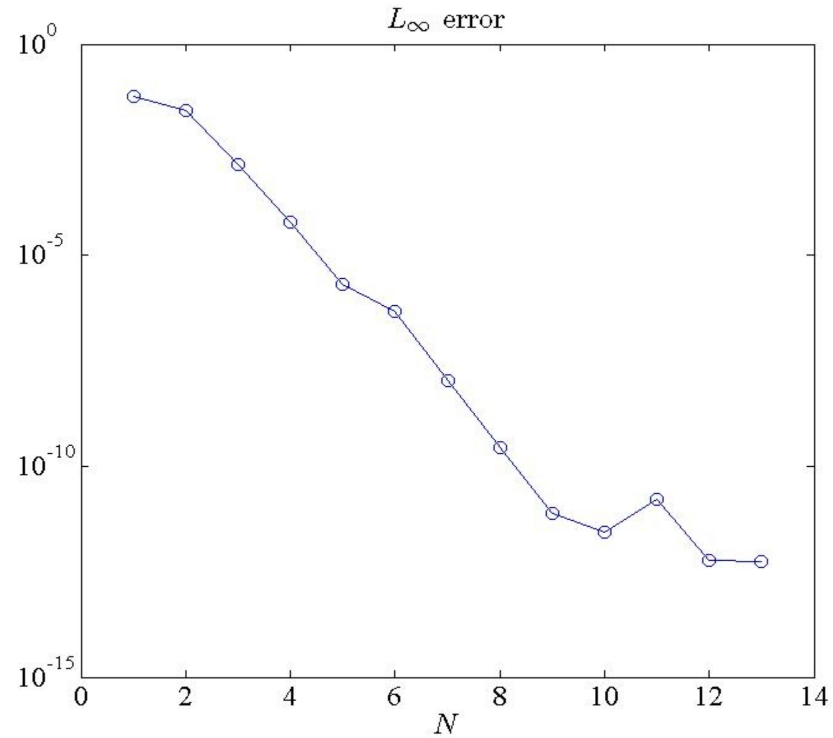
Comparison With Analytic Solution

- Analytic solution is $\psi(x,t) = \Theta(x)G(t - x/c)$
  - Choose c = 2, G(t) = cos(t), final time = 15



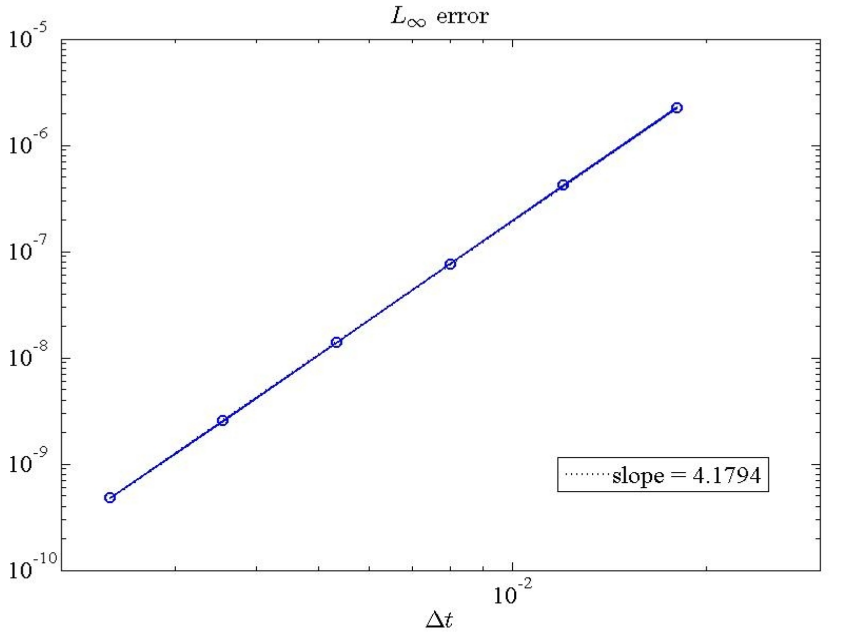
Exponential convergence

- Final time = 30
- 4 domains on [-5,5]
- Time-step set below CFL requirement of largest N



4^{th} order RK

- Final time = 30
- 30 Domains on [-5,25]
- Polynomial Order = 10



A Numerical Scheme
for the Perturbation Equations

Issues and Resolutions

- Particle must be located at an interface for spectral accuracy to be maintained
  - Coordinate transformation
- The derivative of a delta function term differentiates the test functions
  - Rewrite equations in full first order
- Non-trivial boundary conditions
  - Exact outgoing boundary conditions
- Computing waveform at null infinity
  - Use ***flat*** space extraction (Schwarzschild solution asymptotically flat)

Rewriting the Equations in First Order System

$$(-\partial_t^2 + \partial_x^2 - V(r))\psi = G(t,r)\delta(r - r_p) + F(t,r)\partial_r\delta(r - r_p)$$

$$[-\partial_t\psi]|_{x=x_p} = \mathrm{J}_\Pi(\mathrm{t}) \quad [\partial_x\psi]|_{x=x_p} = \mathrm{J}_\Phi(\mathrm{t})$$

- As first order system, and specializing to $r_p(t) = r_p$

$$\partial_t\psi = -\Pi$$

$$\partial_t\Pi = -\partial_x\Phi - V(r)\psi + J_\Phi(t)\delta(x - x_p)$$

$$\partial_t\Phi = -\partial_x\Pi + J_\Pi(t)\delta(x - x_p)$$

- Using properties of distributions, one can show this is the first order form of the original equation

Semidiscrete Form

$$\partial_t \begin{pmatrix} \psi \\ \Pi \\ \Phi \end{pmatrix} = -\partial_x \begin{pmatrix} 0 \\ \Phi \\ \Pi \end{pmatrix} + \begin{pmatrix} -\Pi \\ -V\psi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ J_\Phi(r,t)\delta(r-rp) \\ J_\Pi(r,t)\delta(r-rp) \end{pmatrix}$$

↓

$$\partial_t \vec{u} = -\partial_x \vec{f}(\vec{u}) + \vec{H}(\vec{u}) + \vec{J}$$

↓ LGL Nodal Points + $\vec{u}_h^k(x,t) = \sum_{i=0}^{N} \vec{u}_h^k(x_i,t) l_i^k(x)$

$$\int_{D^k} l_i(x)\partial_t \vec{u}_h(x,t)dx = \int_{D^k} \vec{f}(\vec{u}_h)\partial_x l_i(x)dx - \oint_{\partial D^k} \hat{n}\cdot l_i(x)\vec{f}^*(\vec{u}_h)dx + \int_{D^k} l_i(x)\vec{J}dx + \int_{D^k} l_i(x)\vec{H}(\vec{u}_h)dx$$

- upwind numerical flux $\vec{f}^*(\vec{u}_h)$ (generalization of advection example)

Treating the Source Terms

- Diagonalize the sourced sector of the system

$$\Phi = \omega + \lambda \qquad \partial_t \omega + \partial_x \omega = (J_\Pi + J_\Phi)\delta(x - x_p)/2$$
$$\Pi = \omega - \lambda \quad \longrightarrow \quad \partial_t \lambda - \partial_x \lambda = (J_\Pi - J_\Phi)\delta(x - x_p)/2$$

- GDG suggests adding $(.5(J_\Pi + J_\Phi))_R^{*,\delta} = (\omega)_R^{*,\delta}$ to $(\omega)^*$ on the RHS of the interface and $(.5(J_\Pi - J_\Phi))_L^{*,\delta} = (\lambda)_L^{*,\delta}$ to $(\lambda)^*$ on the LHS of the interface

- Numerical flux at particle interface is changed as

$$\Phi^* \to \Phi^* + (\omega)_R^{*,\delta} + (\lambda)_L^{*,\delta} \qquad \Pi^* \to \Pi^* + (\omega)_R^{*,\delta} - (\lambda)_L^{*,\delta}$$

Boundary Conditions

- Non-reflecting BC
  - Reduces domain size, especially useful for eccentric orbits where many periods required
  - Sommerfeld BC works well near BH horizon as $V \sim e^{x/2M}$
  - At the right boundary Sommerfeld fails as $V \sim 1/r^2$ instead

$$(\partial_t + \partial_x)\psi = F(t, r_b, \psi)$$

  - $F(t, r_b, \psi)$ depends on form of potential
  - Approximated using known techniques to satisfy a specifiable error tolerance, here $\varepsilon < 10^{-10}$

Stephen Lau. *Rapid Evaluation of Radiation Boundary Kernels for Time-domain Wave Propagation on Blackholes.* gr-qc/0401001

Sketch of BC in Flat 3+1 Space

- For given ell, write down an arbitrary outgoing solution in the Laplace frequency domain
- Apply Sommerfeld operator $s + \partial_r$
- Rewrite result in form

$$(s + \partial_r)\Psi_\ell^{out}(s,r) = \frac{1}{r}\hat{\Omega}_\ell(s,r)\Psi_\ell^{out}(s,r)$$

- Taking the inverse Laplace transform

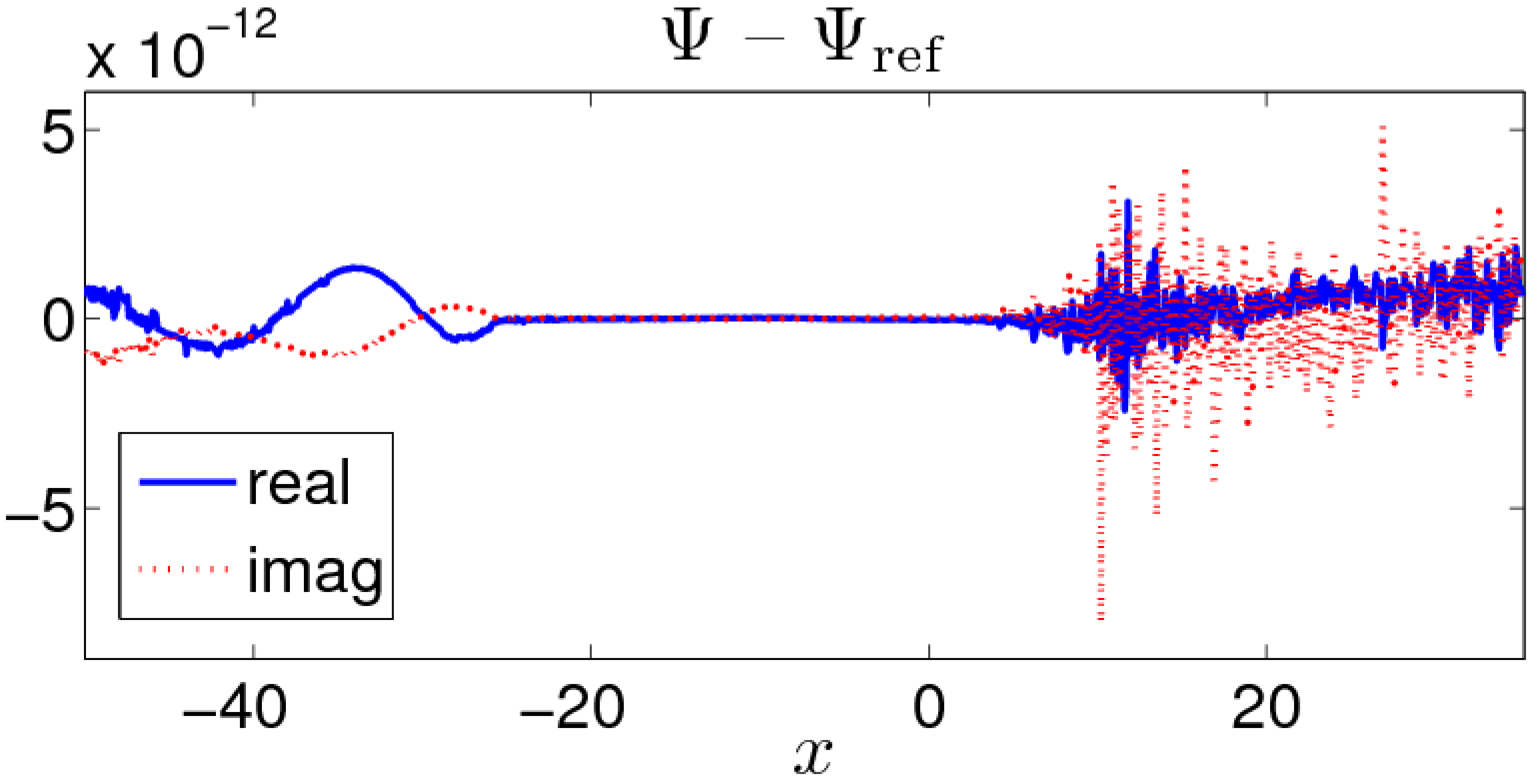
$$(\partial_t + \partial_r)\Psi = C\int_0^T \Omega_\ell(T - T', r_b)\Psi(T', r_b)\,dT'$$

- Theorem: In the time-domain, the kernel is a sum of exponentials

$$\Omega_\ell \cong \sum_{k=1}^{d} \gamma_{\ell,k} \exp\left(t\beta_{\ell,k}\right)$$

Boundary Conditions

Long-time bound on error due to boundary conditions.   Final-Time = 100M



Waveform Matching

- Want to “read off” $\psi^{\text{A/P}}$ at null infinity
- Recording $\psi^{\text{A/P}}$ at x_b introduces an error $O(x_b^{-1})$
- $V^{\text{A/P}} = V^{\text{Flat Space}} + O(r^{-3})$ suggests that $\psi^{\text{A/P}} \sim \psi^{\text{Flat Space}}$
- At right boundary we match waveforms as $(\ell = 2)$

$$\psi_{\ell=2}^{\text{A/P}} \sim \psi_{\ell=2}^{\text{Flat Space}} = f^{(2)}(t - x_b) + \frac{3}{x_b} f^{(1)}(t - x_b) + \frac{3}{x_b^2} f(t - x_b)$$

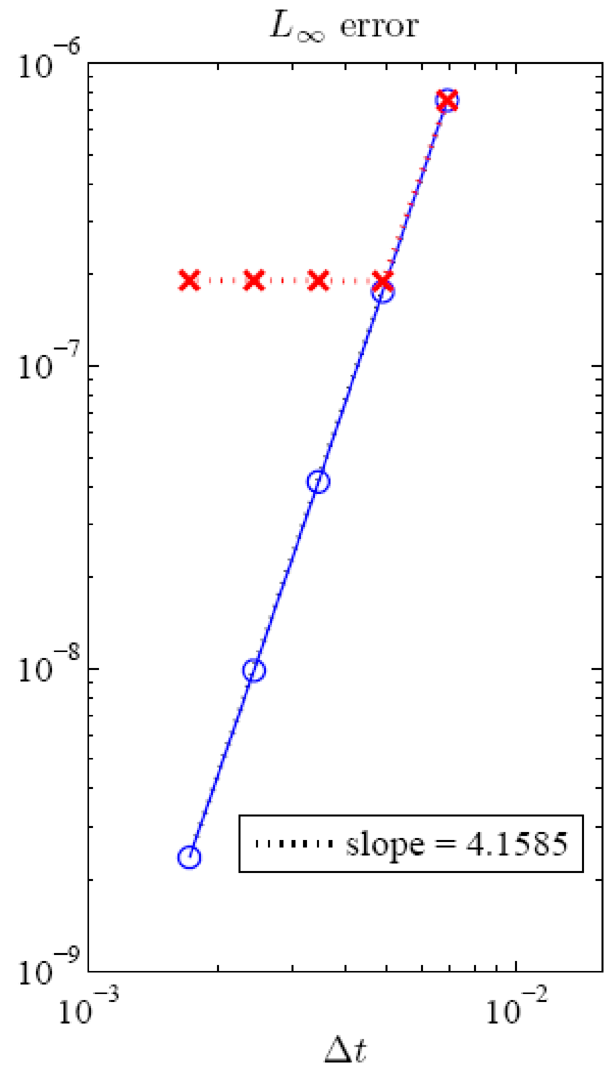
- This is an ODE sourced by $\psi_{\ell=2}^{\text{A/P}}$ and $f^{(2)}(t - x)$ approximates the waveform at infinity
  - Reduces the error to $O(x^{-2})$

Initial Data

- Typically trivial initial data supplied, but this is inconsistent with the PDEs
- "Switch-on" the source terms by multiplying them by a function which smoothly interpolates from 0 to 1
  - We used $\frac{1}{2}\left[erf(\sqrt{\delta}\,(t-\tau))+1\right]$

Initial Data: 1+1 Wave Equation

- Solution to 1+1 wave equation with distribution source term
  - Have analytic solution
- Without smoothing 4th order temporal convergence is abruptly lost (red). It is recovered with a smoother (blue)



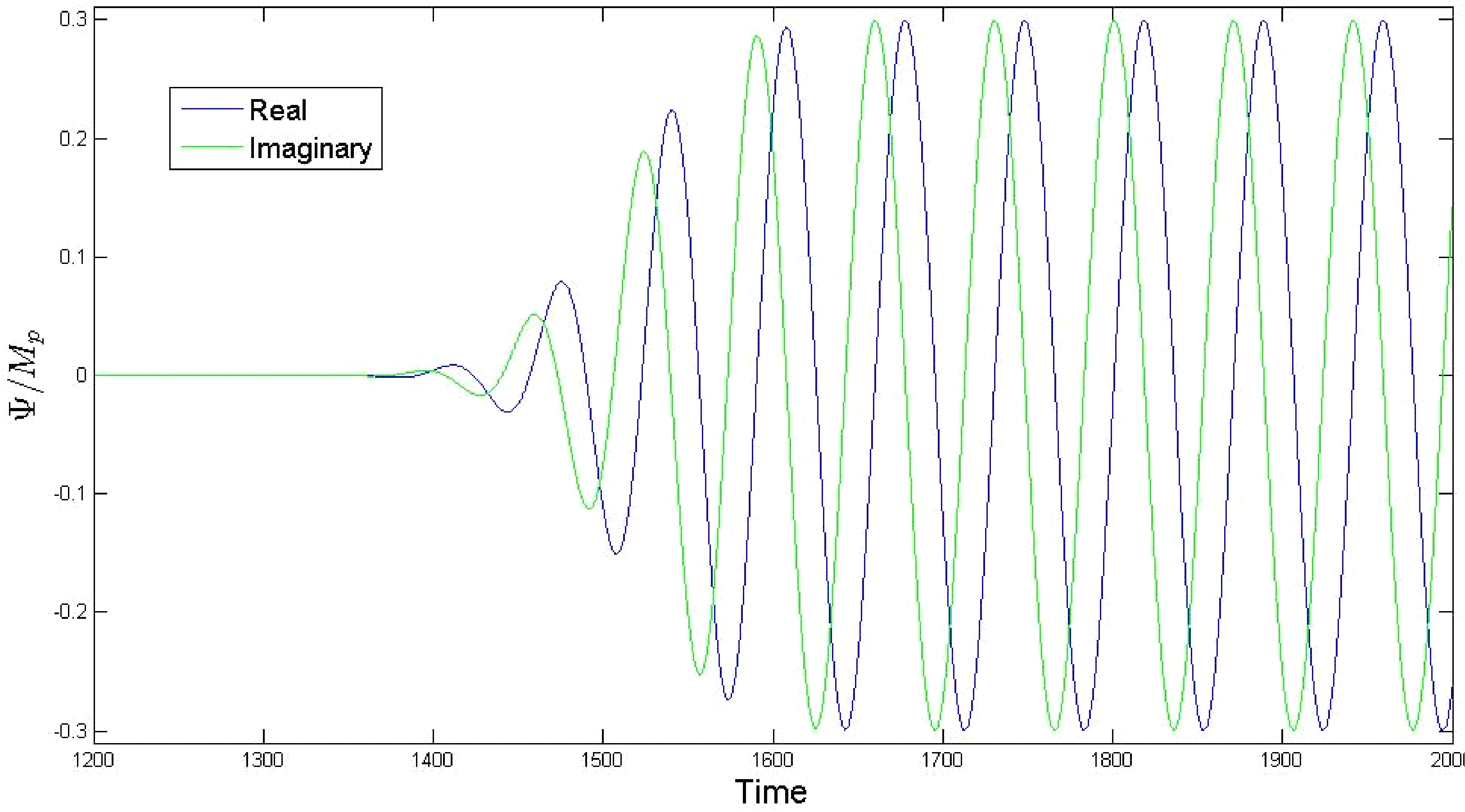
Features of Implementation

- Spectral global accuracy, even at particle
  - Potentially useful for incorporating a self-force
  - Excellent phase resolution for long integrations
- Reduced computational domain without sacrificing accuracy of waveform or introducing reflection
  - Very useful for problems without periodicity
- Particle trajectory need not be *a priori*
- No length scale associated with particle

Results

Results: Circular Orbit

$$(\ell, m) = (2,2),\ M_{BH} = 1,\ r_p = 7.9456,\ \Omega^* = [-200,\ 1000],\ \vec{u}(x,0) = 0$$



Energy and Angular Momentum Luminosity

$$\dot{L}_{lm} = \frac{im}{64\pi}\frac{(l+2)!}{(l-2)!}\overline{\psi}_{\ell m}\dot{\psi}_{lm}$$

(ℓ, m)	$\dot{L}_{DG}$	$\dot{L}_{FE}$	$\dot{L}_{FR}$	$\dot{L}_{FD}$
(2,1)	1.8283×10^{-5}	1.8289[0.04]	1.8283[0.0]	1.8270[0.07]
(2,2)	3.8215×10^{-3}	3.8219[0.01]	3.8215[0.0]	3.8164[0.13]
(3,1)	4.8673×10^{-8}	4.8675[0.004]	4.8670[0.006]	4.8684[0.02]
(3,2)	5.6438×10^{-6}	5.6450[0.02]	5.6439[0.002]	5.6262[0.31]
(3,3)	5.7047×10^{-4}	5.7057[0.02]	5.7048[0.002]	5.6878[0.3]

$$\dot{E}_{lm} = \frac{1}{64\pi}\frac{(l+2)!}{(l-2)!}|\dot{\psi}_{lm}|^2$$

(ℓ, m)	$\dot{E}_{DG}$	$\dot{E}_{FE}$	$\dot{E}_{FR}$	$\dot{E}_{FD}$
(2,1)	8.1633×10^{-7}	8.1662[0.04]	8.1633[0.0]	8.1623[0.01]
(2,2)	1.7062×10^{-4}	1.7064[0.01]	1.7063[0.006]	1.7051[0.07]
(3,1)	2.1732×10^{-9}	2.1732[0.0]	2.1731[0.005]	2.1741[0.04]
(3,2)	2.5199×10^{-7}	2.5204[0.001]	2.5199[0.0]	2.5164[0.14]
(3,3)	2.5471×10^{-5}	2.5475[0.02]	2.5471[0.0]	2.5432[0.15]

Results: Eccentric Orbit

$$(\ell, m) = (2,2),\ M_{BH} = 1,\ \Omega^* = [-200,\ 1000],\ \vec{u}(x,0) = 0$$

$eccentricity = .63517$

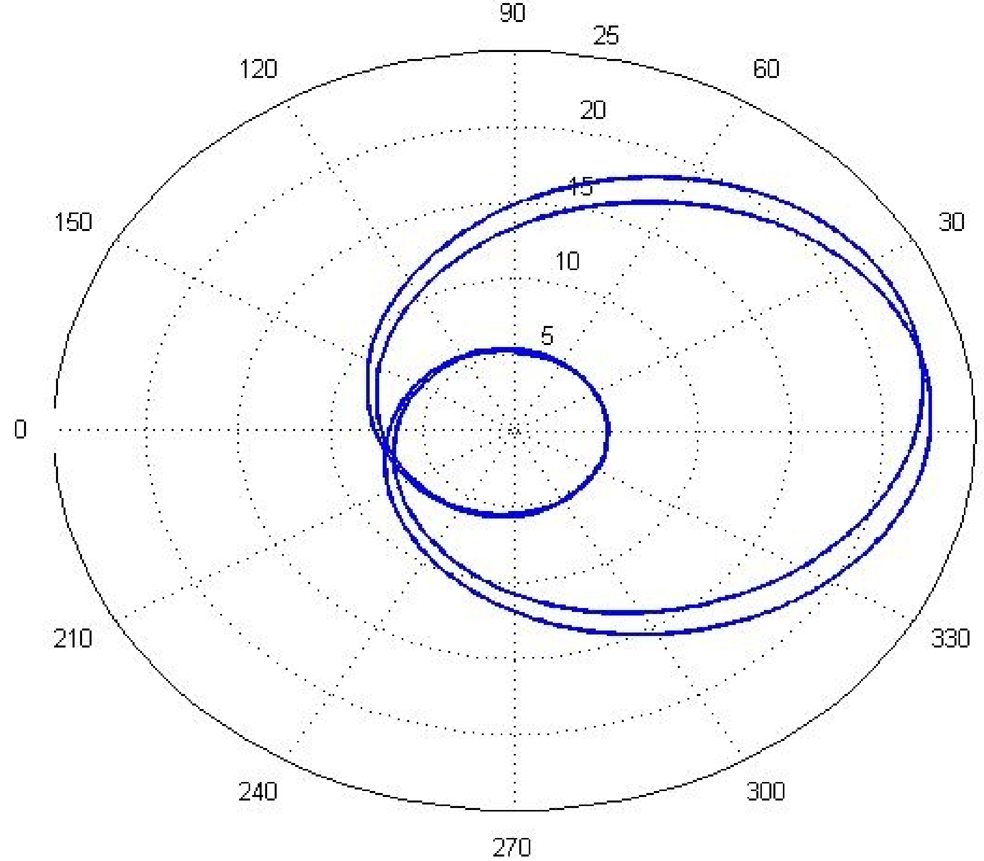
$semi-latus = 8.2873$

$r_{\min} = 5.0682$

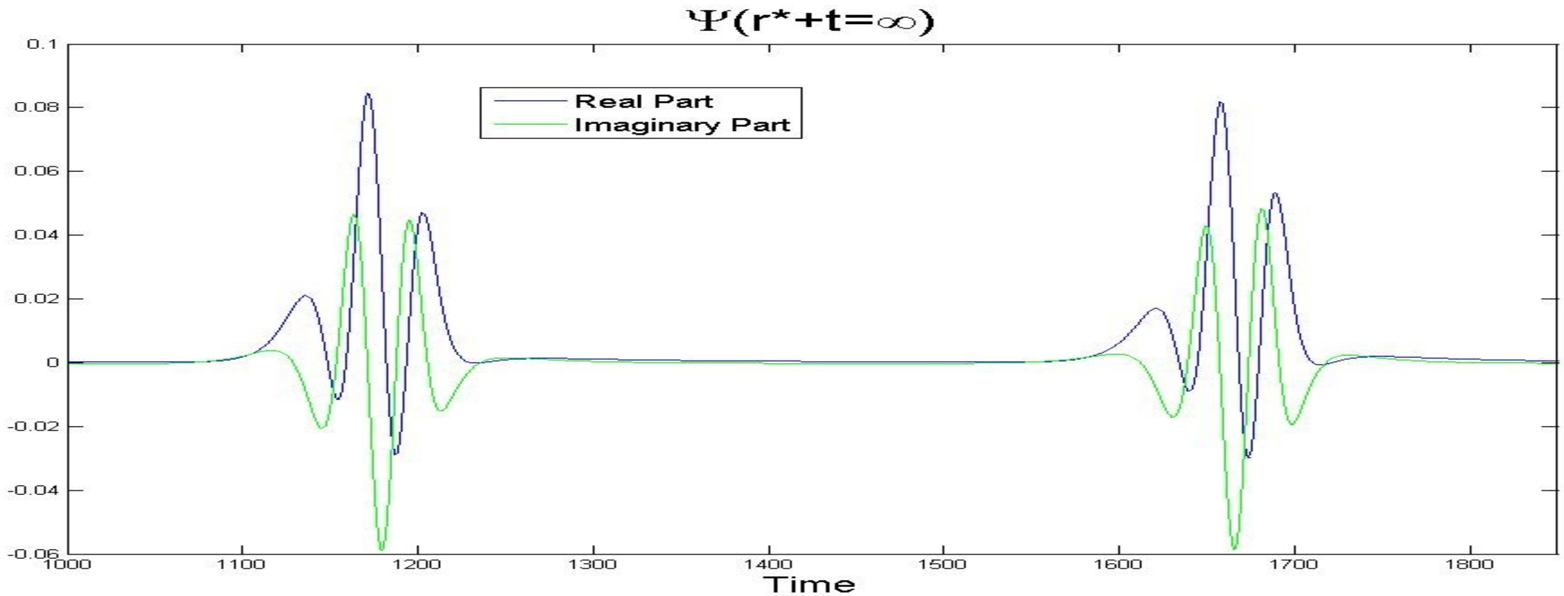
$r_{\max} = 22.7153$

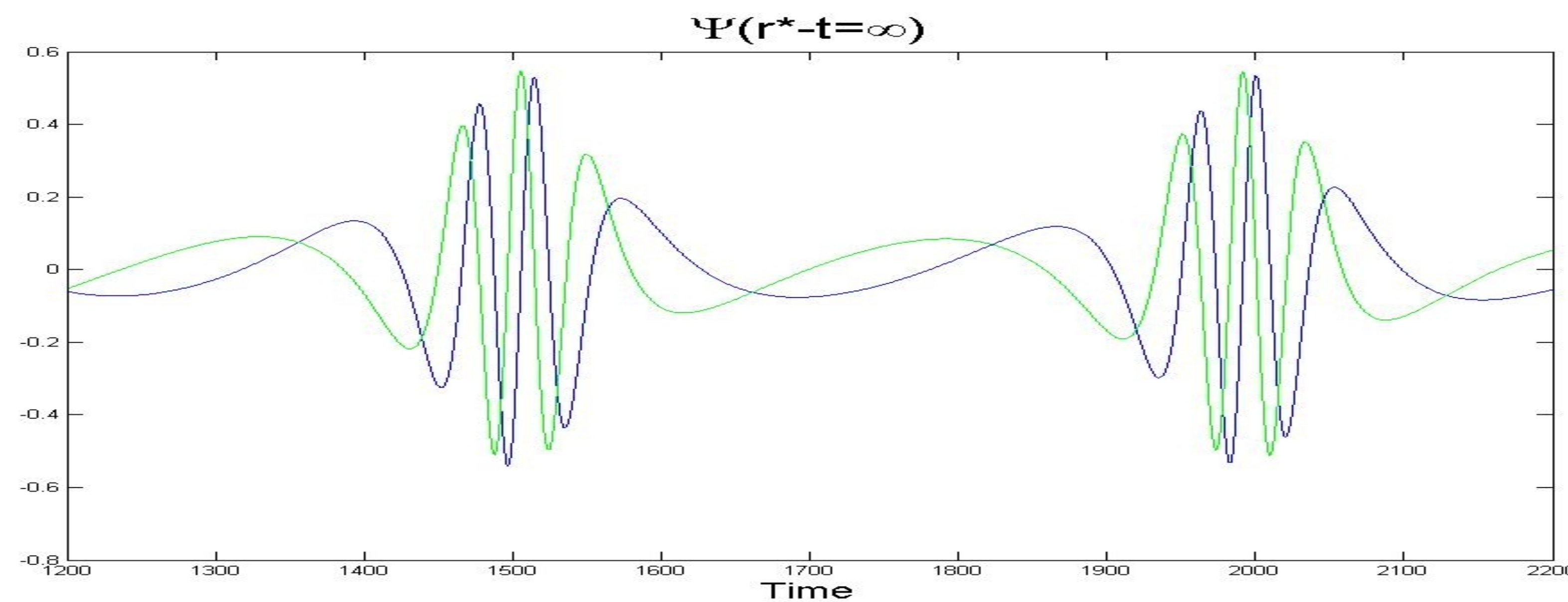
$T_r = 488.6$

$T_\varphi = 243$



Eccentric Orbit Waveforms $(\ell, m) = (2,2)$





Eccentric Orbit Quantities

Selected energy and angular momentum flux calculated at null infinity

e	p	$\dot{E}_{\ell=2,DG}$	$\dot{E}_{\ell=2,FR}$	$\dot{L}_{\ell=2,DG}$	$\dot{L}_{\ell=2,FR}$
0.63517	8.2873	2.4974×10^{-4}	2.49688[.02]	3.4615×10^{-3}	3.45985[.05]

- Averages computed according to

$$\langle \dot{E}_{\ell m} \rangle = \frac{1}{T_f - T_0} \int_{T_0}^{T_f} \dot{E}_{\ell m} dt \qquad T_f - T_0 = 4T_r$$

What's Next?

- Short term:
  - Calculate self force
    - Done for $\dot{E}_{lm}$ and $\dot{L}_{lm}$ when orbit quasi-circular
    - For quasi-circular inspiral orbits would like to calculate waveforms due to dissipation
- Long term:
  - Exact BC for left side of computational domain
    - Arbitrarily short domain size
      - Could calculate up-to ell=25 for 200 orbits of particle initially at r=7.9M in ~3 weeks
        - » On Dell laptop, coded in Matlab
        - » Optimistically on a GPU with python code one might achieve a speed up factor of 50 or more
  - Waveform extraction
    - Weak link, suggestions?

Summary and Kerr

- Purposed a high order DG method for EMRBs
  - Equations as a first order system, exact outgoing radiative BCs, waveform matching, and smooth startup
  - Results agree with literature to stated accuracy
  - Global spectral accuracy
- EMRB problem for Kerr solution
  - 2+1 dimensional
  - 2D problems with distributional sources have been considered in the problem by Fan et al