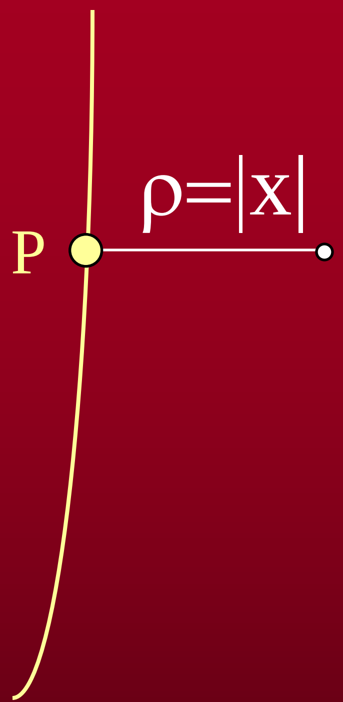
Self-force in a gauge appropriate to separable wave equations

John Friedman

Abhay Shah, Toby Keidl, Dong-Hoon Kim, Larry Price

UNIVERSITY of WISCONSIN
UWMILWAUKEE

Form of singular metric $\approx$ flat metric of particle



$\rho=|x|$, geodesic distance from particle's trajectory

Can choose

$$h_{\alpha\beta}^{\text{sing}} = (g_{\alpha\beta} + 2u_\alpha u_\beta)\frac{2\mu}{\rho}$$

The self-force per unit mass is the acceleration with respect to the background metric:

$$u^\beta \nabla_\beta u^\alpha = -(g^{\alpha\beta} + u^\alpha u^\beta) u^\gamma u^\delta \left(\nabla_\gamma h^{\text{ren}}{}_{\beta\delta} - \frac{1}{2} \nabla_\beta h^{\text{ren}}{}_{\gamma\delta}\right) = f^\alpha$$

Difficulties:

1. h_{singular} is known explicitly only for Lorenz gauge, and in a Lorenz gauge one needs to solve 10 coupled PDE's to find h_{ret}

2. The decoupled, separable Teukolsky equation gives ψ_0 and ψ_4 but not h.
   In vacuum, there is a prescription for finding h (Chrzanowski, Cohen-Kegeles, Wald, Lousto-Whiting, Ori, Price-Shankar-Whiting)
   but it has 5 gauge conditions and they cannot all be satisfied when a particle is present. (Barack, Ori)

•Radiation Gauge (RG)

$$h_{\alpha\beta} n^{\beta} = 0 \qquad\qquad h = 0$$

5 constraints, agreeing for outgoing waves in flat space with a transverse-tracefree gauge.

$$\mathrm{h} = \mathrm{L}_{\mathrm{RG}}\ \psi$$

Where L_{RG} is a 2nd-order differential operator involving only $\tilde{\partial}$ and and ∂_{t} .

Two theorems:

I. Let h be given by h = $L_{RG}\,\Psi$ with Ψ an RG potential obtained from $\delta\Psi_4$ ($\delta\Psi_0$) . If Ψ satisfies the sourcefree Teukolsky equation then h satisfies the vacuum Einstein equation.

(Chrzanowski;Cohen, Kegeles; Wald)

IIa. Given a solution $\delta\Psi_4$ ($\delta\Psi_0$) to the sourcefree Teukolsky equation on a globally hyperbolic type D vacuum spacetime, one can find a local RG gauge:

There is a perturbed metric $h_{\alpha\beta}$ in an RG gauge that satisfies the linearized vacuum Einstein equation, and for which $\delta\Psi_4$ ($\delta\Psi_0$) is the perturbed Weyl tensor component.

(L. Price, Shankar, Whiting)

IIb. Under the conditions of (a) there is a local solution Ψ to the sourcefree Teukolsky equation for which $h = L\Psi$ is a perturbed vacuum metric associated with $\delta\Psi_4$ ($\delta\Psi_0$). (Not yet proved)

A way to handle each difficulty:

1. Renormalize the gauge invariant $\psi_0 := \delta\Psi_0$

$$\psi_0 = Lh$$

$$\psi_0^{\text{renormalized}} = Lh^{\text{ret}} - Lh^{\text{singular}} = \psi_0^{\text{ret}} - \psi_0^{\text{singular}}$$

Known

Computable

A way to handle each difficulty:

1. Renormalize the gauge invariant ψ_0

$$\psi_0 = Lh$$

$$\psi_0^{\text{renormalized}} = Lh^{\text{ret}} - Lh^{\text{singular}} = \psi_0^{\text{ret}} - \psi_0^{\text{singular}}$$

Known

Computable

Because Detweiler-Whiting form of h^{singular} is a local solution to linearized field eqs, $\psi_0^{\text{renormalized}}$ is a local solution to sourcefree Teukolsky equation

A way to handle each difficulty:

2. Because $\psi_0^{\text{renormalized}}$ *is* sourcefree, the metric reconstruction exists and is nonsingular.

Outline of calculation

I. Compute ψ_0^{ret} (or ψ_4^{ret}) from Teukolsky eq

II. Compute h_{singular} near particle in any gauge, and compute corresponding gauge invariant component of perturbed Weyl tensor, ψ_0^{sing} (ψ_0^{sing}).

III. $\psi_0^{\mathrm{reg}} = \psi_0^{\mathrm{ret}} - \psi_0^{\mathrm{sing}}$ is a sourcefree solution to the Teukolsky equation

IV. Use radiation gauge (RG) to compute the perturbed metric of ψ_0^{reg}

A. A potential Ψ is found by 4 integrations of ψ_0;

B. $h_{\alpha\beta}$ is obtained by taking derivatives of Ψ.

I. ψ_0^{ret}: The Green's function for the Teukolsky eqn (Bardeen-Press eqn), yields ψ_0^{ret}:

$$
\begin{aligned}
\psi_0^{\rm ret} = & \mathfrak{m}(u^t)^2 \frac{\Delta_{\rm o}^{5/2}}{r_{\rm o}^3} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} A_l \left[(l-1)l(l+1)(l+2)\right]^{1/2} R^{\rm Hor}_{lm}(r_<) R^{\infty}_{lm}(r_>) {}_2Y_{lm}(\theta,\phi) Y^*_{lm}\left(\frac{\pi}{2},\Omega t\right) \\
& + \frac{2i\mathfrak{m}u^t u^\phi \Delta_{\rm o}^{3/2}}{r_{\rm o}} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} A_l \left[(l-1)(l+2)\right]^{1/2} {}_2Y_{lm}(\theta,\phi) {}_1Y^*_{lm}(\frac{\pi}{2},\Omega t) \\
& \qquad \left\{(2+i\omega r_{\rm o}) r_{\rm o} R^{\rm Hor}_{lm}(r_<) R^{\infty}_{lm}(r_>) - \Delta_{\rm o}\left(\Theta(r-r_{\rm o}) R^{\rm Hor}_{lm}(r_{\rm o}) R'^{\infty}_{lm}(r)\right.\right. \\
& \qquad \left.\left. + \Theta(r_{\rm o}-r) R'^{\rm Hor}_{lm}(r) R^{\infty}_{lm}(r_{\rm o})\right)\right\} \\
& - \mathfrak{m}(u^\phi)^2 \frac{\Delta_{\rm o}^{1/2}}{r_{\rm o}} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} A_l \; {}_2Y_{lm}(\theta,\phi) {}_2Y^*_{lm}\left(\frac{\pi}{2},\Omega t\right) \\
& \qquad \times \left\{ R^{\rm Hor}_{lm}(r_<) R^{\infty}_{lm}(r_>) \; r_{\rm o}^2 \left[4(r_{\rm o}^2 - 2M^2) + 6i\omega r_{\rm o}^2 (r_{\rm o} - M) - r_{\rm o}^4 \omega^2\right] \right. \\
& \qquad + 2\left[\Theta(r-r_{\rm o}) \; R^{\rm Hor}_{lm}(r) R'^{\infty}_{lm}(r_{\rm o}) + \Theta(r_{\rm o}-r) \; R'^{\rm Hor}_{lm}(r_{\rm o}) R^{\infty}_{lm}(r)\right] \\
& \qquad \times \Delta_{\rm o} r_{\rm o}^2 (9 r_{\rm o} - 14M - i\omega (r_{\rm o})^2) \\
& \qquad + r_{\rm o}^2 \Delta_{\rm o}^2 \left[\Theta(r-r_{\rm o}) \; R^{\rm Hor}_{lm}(r) R''^{\infty}_{lm}(r_{\rm o}) + \Theta(r_{\rm o}-r) \; R''^{\rm Hor}_{lm}(r_{\rm o}) R^{\infty}_{lm}(r)\right] \\
& \qquad \left. - r_{\rm o}^2 \Delta_{\rm o}^2 W\left[R^{\rm Hor}_{lm}(r_{\rm o}), R^{\infty}_{lm}(r_{\rm o})\right] \right\}.
\end{aligned}
\tag{5.35}
$$

II. ψ_0^{sing} for Schwarzschild and Kerr

With locally inertial coordinates

$$T, X, Y, Z,$$

for which $\vec{u} = \vec{\partial}_T$

$$h_{\alpha\beta}{}^{\text{sing}} = (g_{\alpha\beta} + u_\alpha u_\beta)\frac{2\mu}{\rho} + \cdots$$

with $\rho = (X^2+Y^2+Z^2)^{1/2}$. If specialize to THZ chart, the Detweiler-Whiting form of h^{sing} satisfies

$$h_{\alpha\beta}{}^{\text{sing}} = (g_{\alpha\beta} + 2u_\alpha u_\beta)\frac{2\mu}{\rho} + O(\rho).$$

Then

$$\psi_0^{\text{sing}} = \frac{6\mu}{\rho^3}(l^T m^\rho - l^\rho m^T)^2$$

ψ_0^{ret} and ψ_0^{sing} blow up near the particle like ρ^{-3}, two orders faster than the metric.

Invert $\psi_0 = (\partial^4 \overline{\Psi} + 12\rho^{-3}\Psi_2\, \partial_t \Psi)$

to find the potential Ψ.

[(angular derivative)^4 + time derivative] $\Psi = \psi_0$.

With harmonic decomposition in time and angle, the inversion here is algebraic.

Mode sum renormalization (a view of Barack-Ori)

To find a decomposition of h_0^{ret}, can extend h_0^{ret} smoothly to the sphere $r = r_0$

Two smooth extensions differ by a smooth function. The coefficients in their angular harmonic expansion agree – difference between the harmonic expansions disagreement falls off faster than any power of *L*.

That is, the expansion

$$h^{ret}_{Lorenz} = \Sigma[A + \frac{B}{L} + \frac{C}{L^2} + \cdots] + h^R_{Lorenz}$$

is a property of h^{ret}_{Lorenz} itself.

The first three terms renormalize h^{ret}_0 to the order needed to compute the self-force.

Orders in powers of L same as for Lorenz gauge

(A, B, C different constants in each line below)

$$h^{ret}_{Lorenz} = \Sigma[A + \frac{B}{L} + \frac{C}{L^2} + \cdots] + h^R_{Lorenz}$$

$\not{\partial}^2$

$$\psi_0^{ret} = \Sigma[A\,L^2 + B + \frac{C}{L} + \cdots] + \psi_0^R$$

$\not{\partial}^{-4}$

$$\Psi = \Sigma[\frac{A}{L^2} + \frac{B}{L^3} + \frac{C}{L^4} + \cdots] + \Psi^R$$

$\not{\partial}^2$

$$h^{ret}_{rad} = \Sigma[A + \frac{B}{L} + \frac{C}{L^2} + \cdots] + h^R_{rad}$$

$\not{\partial}$

$$F^{ret}_{rad} = \Sigma[A L + B + \frac{C}{L} + \cdots] + F^R_{rad}$$

- But the geodesic equation is known to hold only for gauges related to a Lorentz gauge by a sufficiently regular gauge transformation (Barack-Ori, Gralla-Wald).

- Because of the point-particle source, the retarded metric in a radiation gauge, h_{rad}^{ret}, has a string singularity and is *not* related by a regular gauge transformation.

- What's going on?

- Claim: h^R_{rad} is the renormalized field of a metric related to h^{ret}_{Lorenz} by a smooth gauge transformation.

  Proof: The two renormalized fields, h^R_{rad} and h^R_{Lorenz}, yield the same regularized vacuum solution ψ^R_0 and each are smooth. The the two renormalized metrics therefore differ by a smooth gauge transformation (once h^R_{rad} is completed by addition of an *l=0* part with the same value of δM as h^R_{Lorenz}):

$$h^R_{rad} = h^R_{Lorentz} + 2\nabla_{(\alpha}\xi_{\beta)} \ ,$$

with ξ^α smooth.

Then h^R_{rad} is the renormalized field of a perturbed metric

$$h^{ret}_{Lorentz} + 2\nabla_{(\alpha}\xi_{\beta)}$$

that *is* related to $h^{ret}_{Lorentz}$ by a smooth gauge transformation. $\square$

Because the first three terms in the harmonic expansion of ψ_0^{ret} determine the first three terms in the harmonic expansion of h^{ret} one can in principle find ψ_0^{R} from ψ_0^{ret}

- Abhay Shah finds the first two terms of the singular field analytically, but a match to the asymptotic harmonic expansion of h^{ret} gives the expected agreement. To obtain more rapid convergence of the expansion for the renormalized field, he finds by numerical matching the next set of terms in the asymptotic harmonic expansion of ψ_0^{ret}.

Three more comments

- Although the argument required the order

$$\begin{array}{ccc} \psi^{ret}_{rad} & \longrightarrow & \psi^{R}_{rad} \\ & & \downarrow \\ & & h^{R}_{rad} \end{array}$$

The diagram commutes

$$\begin{array}{ccc} \psi^{ret}_{rad} & \longrightarrow & \psi^{R}_{rad} \\ \downarrow & & \downarrow \\ h^{ret}_{rad} & \longrightarrow & h^{R}_{rad} \end{array}$$

because of the algebraic character of the operations

- The $l=0$ part of the metric can be written in any gauge related by a smooth $l=0$ gauge transformation to perturbed Schwarzschild in a Schwarzschild gauge (or in a Lorenz gauge).

- The "Lorenz" gauge, $\nabla_\beta(h^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} h) = 0,$

  was first used by Einstein and Hilbert in 1916 and its existence proved by Hilbert

Die Grundlagen der Physik.

(Zweite Mitteilung.)

Von

David Hilbert.

Vorgelegt in der Sitzung vom 23. Dezember 1916.

deDonder and Fock introduced and used its nonlinear version, the harmonic gauge. Outside the EMRI community, the linearized gauge is variously referred to as the deDonder, harmonic, Fock, Lorenz, Lorentz[sic], Einstein, or Hilbert gauge

$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \cdots, \tag{37}$$

wo ε eine gegen Null konvergierende Größe und $h_{\mu\nu}$ Funktionen der w_s sind. Über die Maßbestimmung (37) mache ich die folgenden zwei Annahmen:

I. Die $h_{\mu\nu}$ mögen von der Variabeln w_4 unabhängig sein.

II. Die $h_{\mu\nu}$ mögen im Unendlichen ein gewisses reguläres Verhalten zeigen.

Soll nun die Maßbestimmung (37) für alle ε die Differentialgleichungen (36) erfüllen, so folgt, daß die $h_{\mu\nu}$ notwendig gewisse lineare homogene partielle Differentialgleichungen zweiter Ordnung erfüllen müssen. Diese Differentialgleichungen lauten, wenn man nach Einstein [1])

$$h_{\mu\nu} = k_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu} \sum_s k_{ss}, \qquad (k_{\mu\nu} = k_{\nu\mu}) \tag{38}$$

einsetzt und zwischen den 10 Funktionen $k_{\mu\nu}$ die vier Relationen

$$\sum_s \frac{\partial k_{\mu s}}{\partial w_s} = 0, \qquad (\mu = 1, 2, 3, 4) \tag{39}$$

annimmt, wie folgt:

$$\Box k_{\mu\nu} = 0, \tag{40}$$