Towards accurate EMRI waveforms and parameter estimation for LISA

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CRG and B.L. Hu, PRD **79**, 064002 (2009)
CRG, Ph.D. Thesis
CRG and M. Tiglio, PRD **79**, 124027 (2009)

Motivations and goals

- Understand the effect of 2nd order self-force corrections on the gravitational waveform/orbital dynamics

  To what extent are 2nd order self-force effects needed for precise parameter estimation with LISA?

  What if the small body has spin angular momentum?

- Use (black hole) perturbation theory techniques to describe binaries having not-so-small mass ratios

  e.g., For what mass ratios is 2nd order perturbation theory an adequate description?

  Can perturbation theory be useful for comparable mass binaries (e.g. LIGO sources)?

- Describe compact binary systems within a single, comprehensive framework for arbitrary masses and velocities

  Is it possible to do so? If so, can it be used for practical computations?

Outline

- Higher order corrections

    Overview of the EFT approach

    Second order waves, second order self-force, and spin

- A nonlinear scalar model

    Some exact results for conservative self-force on circular orbits in Schwarzschild

- Another numerical approach *(in progress with Manuel Tiglio)*

    Computing the retarded propagator

    Solving the self-force equations

- Conclusions

Introduction

EMRIs have two very different length scales:

Size of the body

Background curvature scale

Small compact object is an extended body that is often treated with a point particle description, which leads to problems (e.g., divergences)

Divergences arise, as in any field theory coupled to point sources, implying that the theory needs to be supplemented by a more complete model.

The effective field theory framework cleanly and systematically separates the two length scales in the problem and borrows techniques from quantum field theory to streamline the perturbation theory calculations.

Effective field theory approach

EFT philosophy:

> *"Rather than trying to resolve the point particle singularities by using a specific model of the short distance physics [black hole, neutron star with a given equation of state, etc.], in an EFT framework we systematically parameterize our ignorance of this structure..."*

Goldberger & Rothstein, PRD **73**, 104029 (2006)

Add to the point particle Lagrangian all terms consistent with the symmetries of the large distance theory (ie., general relativity)

General coordinate transformations

Reparameterization of worldline parameter

[SO(3) rotations]

Provides a model-independent description for the motion of any small extended body in a background (supermassive black hole) spacetime

Effective point particle action:

$$S_{epp}[z^\nu] = -m\int d\tau + c_R\int d\tau\, R + c_V\int d\tau\, R_{\alpha\beta}u^\alpha u^\beta + c_E\int d\tau\, \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta} + c_B\int d\tau\, \mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta} + \cdots$$

"Wilson coefficients" or "non-minimal coupling constants"

Coefficients of these extra terms are related to the properties of the structure of the extended body.

Example: Non-spinning neutron star

External, adiabatic quadrupole tidal field induces a quadrupole moment

$$Q^{(n)}_{ij} = -\lambda_n \mathcal{E}_{ij}$$

Thorne, PRD **58**, 124031 (1998)

In weak-field region of a black hole spacetime

Flanagan & Hinderer, PRD **77**, 021502(R) (2008)

$$S[Q_{ij}] = -\frac{1}{2}\sum_n \int dt\, Q^{(n)}_{ij}\mathcal{E}_{ij} + \sum_n \int dt\, \frac{\dot{Q}^{(n)}_{ij}\dot{Q}^{(n)} - \omega_n^2 Q^{(n)}_{ij}Q^{(n)}_{ij}}{4\lambda_n^2\omega_n^2}$$

$$= \frac{k_2 r_m^5}{6G_N}\int dt\, \mathcal{E}_{ij}\mathcal{E}^{ij}$$

If the properties of the extended body are known then these coefficients can be calculated from a matching procedure.

Effective point particle action:

$$S_{epp}[z^\nu] = -m\int d\tau + c_R\int d\tau\, R + c_V\int d\tau\, R_{\alpha\beta}u^\alpha u^\beta + c_E\int d\tau\, \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta} + c_B\int d\tau\, \mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta} + \cdots$$

Notice that there are an *infinite* number of terms that can be added. Therefore, all divergences that may appear in the perturbation theory can be absorbed by *renormalizing* the Wilson coefficients:

The theory is renormalizable.

The theory has predictive power.

In addition, the Wilson coefficients can exhibit *classical renormalization group running*.

<u>Example</u>:

At 3PN in metric of a single particle at rest there is a log divergence that renormalizes $c_{R,V}$

$$\mu\frac{dc_R}{d\mu} = -\frac{1}{6}G_N^2 m^3 \qquad \mu\frac{dc_V}{d\mu} = \frac{2}{3}G_N^2 m^3$$

Goldberger & Rothstein, PRD **73**, 104029 (2006)

These coefficients can be gauged away as would be expected by Birkhoff's theorem.

And a couple more things...

- Power counting rules allow one:

  To find the order at which some relevant quantity first contributes

  To determine which terms enter a given order before any calculation is done

- In the EFT approach, perturbation theory is done at the level of the action instead of the equations of motion. This suggests using "Feynman diagrams" to systematically aid in calculations, particularly at high orders.

  Self-forces, gravitational waveforms, spin precession, power loss,...

How does one incorporate retarded boundary conditions on the gravitational perturbations within an action principle?

A scalar field example

Naive way:

$$S[\phi, J] = \frac{1}{2}\int d^4x\, g^{1/2}\phi_{,\alpha}\phi^{,\alpha} + \int d^4x\, g^{1/2}(S + J)\phi$$

Physical source

Auxiliary source

$$\Box\phi = J + S$$

$$\phi_{ret}(x) = \int d^4y\, D_{ret}(x,y)\big(S(y) + J(y)\big)$$

$$S_{eff}[J] = S[\phi_{ret}, J] = \frac{1}{2}\int d^4x\, g^{1/2}(x)\int d^4y\, g^{1/2}(y)\big(S(x)+J(x)\big)D_{ret}(x,y)\big(S(y)+J(y)\big)$$

$$\phi(x) = \left.\frac{\delta S_{eff}[J]}{\delta J(x)}\right|_{J=0} = \frac{1}{2}\int d^4y\, g^{1/2}\big(D_{ret}(x,y) + D_{adv}(x,y)\big)S(y)$$

This isn't consistent with the retarded solution!

A scalar field example

Better way: *Double the degrees of freedom* Galley & Hu (2009), Galley & Tiglio (2009)

$$S[\phi_{1,2}, J_{1,2}] = \int d^4x\, g^{1/2} \left[\frac{1}{2} \phi_{1,\alpha} \phi_1^{;\alpha} + (S_1 + J_1)\phi_1 \right] - \int d^4x\, g^{1/2} \left[\frac{1}{2} \phi_{2,\alpha} \phi_2^{;\alpha} + (S_2 + J_2)\phi_2 \right]$$

Set J_1=J_2=0 and S_1=S_2=S at end of calculations

$$X_{1,2} = X_+ \pm \frac{1}{2} X_-$$

$$S[\phi_\pm, J_\pm] = \int d^4x\, g^{1/2} \Big[\phi_{+,\alpha} \phi_-^{;\alpha} + (S_+ + J_+)\phi_- + (S_- + J_-)\phi_+ \Big]$$

$$\Box \phi_\pm = J_\pm + S_\pm$$

$$\phi_+^{ret}(x) = \int d^4y\, g^{1/2} D_{ret}(x,y) \big(J_+(y) + S_+(y) \big)$$

$$\phi_-^{?}(x) = \int d^4y\, g^{1/2} D_{?}(x,y) \big(J_-(y) + S_-(y) \big)$$

$$S_{eff}[J_\pm] \equiv S[\phi_+^{ret}, \phi_-^{?}, J_\pm]$$

$$D_?(x,y) = D_{adv}(x,y) \longrightarrow \left. \frac{\delta S_{eff}[J_\pm]}{\delta J(x)} \right|_{J_1 = J_2 = 0, S_1 = S_2 = 0} = \int d^4y\, g^{1/2} D_{ret}(x,y) S(y)$$

*A classical mechanics for open systems?

- Build Lagrangian and Hamiltonian formalisms based on doubling the degrees of freedom

  Classically describe dissipative systems with a Hamiltonian formalism, for example.

- Possibly useful for:

  **Perturbation theories with time-asymmetric boundary conditions (e.g., radiating systems)*

  **More accurate EOB waveforms? (no grafting on of radiation reaction, less fitting & phenomenology)*

  Non-equilibrium statistical mechanics

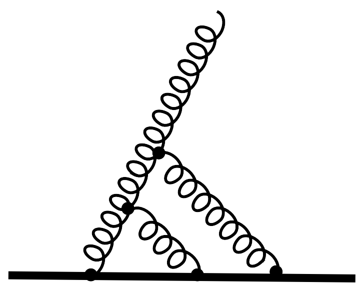
  Plasma physics

  etc...

...and some more things...

- Systematize the expansion using "Feynman diagrams" and "Feynman rules" to construct the effective action order-by-order

  <u>Example</u>: A (connected and tree) Feynman diagram



  <u>Example</u>: Two of the Feynman rules:

  i) Associate a Green's function $D_{ret/adv}(x,y)$ for every curly line connecting two points x, y

  ii) Associate the appropriate "vertex function" for each vertex in the diagram

$$V_{(3)}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(x_1,x_2,x_3) = \frac{\delta}{\delta h_{\mu_1\nu_1}(x_1)}\frac{\delta}{\delta h_{\mu_2\nu_2}(x_2)}\frac{\delta}{\delta h_{\mu_3\nu_3}(x_3)}\left[S_{(3)}^{EH}\right]$$

- Which diagrams do you construct? Those diagrams that are connected and with no loops of curly lines.

$$S_{eff}[z^\mu] = -m\int d\tau + \left(\begin{array}{c}\text{Sum of all connected}\\ \text{tree diagrams}\end{array}\right)$$

...and finally divergences

Any theory of point sources and fields contains divergences. Dimensional regularization is particularly well-suited for EFT calculations:

Divergences that scale as a positive power of a high frequency cut-off vanish for a massless field ['t Hooft and Veltman]

<u>Example</u>: (see also Birrell & Davies, p. 170)

$$I(3) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d^4k \frac{e^{ik^0 s}}{-(k^0)^2 + \mathbf{k}^2} = \int_{-\infty}^{\infty} \frac{d^3k}{\mathbf{k}^2}$$

Infrared regulator

$$I(d) = \mu^{3-d} \int_{S^{d-1}} d\Omega \int_0^{\infty} dk \frac{k^{d-1}}{k^2 + m^2} = \frac{m^{d-2}\pi^{1+d/2}\mu^{3-d}\csc(d\pi/2)}{\Gamma(d-1/2)}$$

$$I(d = 3-\epsilon) = -\frac{4m\pi^2}{3} + O(\epsilon) \to 0 \text{ as } m \to 0$$

Therefore, only log divergences can potentially renormalize the coupling constants of the theory (which are the "Wilson coefficients") and imply a possible renormalization group running of the non-minimal coupling constants of the effective point particle theory.

Quasi-local behavior of Green's function

Ultraviolet divergences come about from the well-known singular structure of the Green's function near coincidence. We use a momentum space representation for the Green's function near coincidence. [Bunch & Parker]

<u>*Example*</u>: *Compute the quasi-local expansion of the momentum space representation for the Feynman Green's function, say, for a massive scalar field in a curved spacetime.*

$$S[\phi, J] = \int_{\mathcal{V}_d} d^d y\, g^{1/2} \left[\frac{1}{2} \left(g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + m^2 \phi^2 + \xi R \phi^2 \right) + J\phi \right]$$

$$\phi = g^{-1/4} \bar{\phi}, \quad J = g^{-1/4} \bar{J}$$

$$S[\bar{\phi}, \bar{J}] = \int_{\mathcal{V}_d} d^d y \left[-\frac{1}{2} \left(g^{\alpha\beta} \bar{\phi}_{,\alpha} \bar{\phi}_{,\beta} + m^2 \bar{\phi}^2 + \mathcal{M}(x) \bar{\phi}^2 \right) + \bar{J}\bar{\phi} \right]$$

$$\mathcal{M}(x) \equiv \xi R + g^{-1/4} \left(g^{\alpha\beta} g^{1/4}{}_{,\alpha} \right)_{,\beta}$$

$$S[\bar{\phi}, \bar{J}] = \int_{\mathcal{N}(P)} d^d \hat{y} \left\{ -\frac{1}{2} \left[\eta^{ab} \bar{\phi}_{,a} \bar{\phi}_{,b} + m^2 \bar{\phi}^2 \right] - \frac{1}{2} \left[\mathcal{K}^{ab}(\hat{y}, P) \bar{\phi}_{,a} \bar{\phi}_{,b} + \mathcal{M}(\hat{y}, P) \bar{\phi}^2 \right] + \bar{J}\bar{\phi} \right\} + \cdots$$

$$\mathcal{K}^{ab}(\hat{y}, P) \equiv g^{ab}(\hat{y}, P) - \eta^{ab}$$

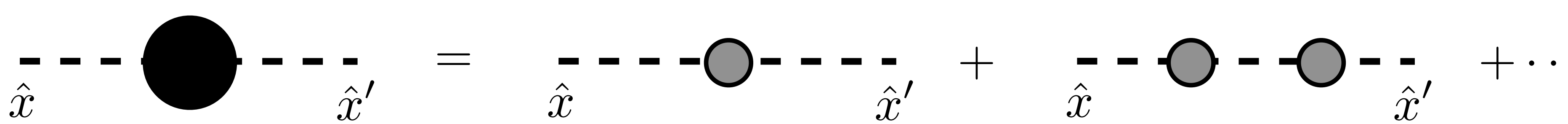
Now, the part of the action in the normal neighborhood of P is written as a background piece plus some small corrections that scale as powers of the interval from P divided by the background curvature scale.

$$S[\bar{\phi}, \bar{J}] = \int_{\mathcal{N}(P)} d^d\hat{y} \left\{ -\frac{1}{2}\left[\eta^{ab}\bar{\phi}_{,a}\bar{\phi}_{,b} + m^2\bar{\phi}^2\right] - \frac{1}{2}\left[\mathcal{K}^{ab}(\hat{y},P)\bar{\phi}_{,a}\bar{\phi}_{,b} + \mathcal{M}(\hat{y},P)\bar{\phi}^2\right] + \bar{J}\bar{\phi} \right\} + \cdots$$

"Flat space background Lagrangian"

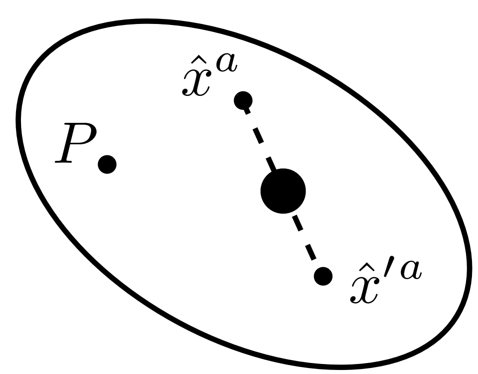
"Interaction Lagrangian"

We can develop the quasi-local expansion for the Green's function expressed in Riemann normal coordinates using diagrammatic methods.

$\bar{G}_F(\hat{x}, \hat{x}'; P) =$ 

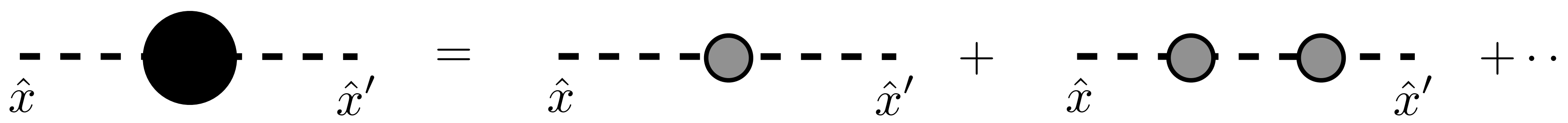
This is the quasi-local expansion of the Feynman Green's function in Riemann normal coordinates centered on P. Note that with this method, neither x nor x' have to be at P.

$$G_F(x, x', P) = g^{-1/4}(x)\, \bar{G}_F(x, x', P)\, g^{-1/4}(x')$$



*Green's function miscellanea

$$\bar{G}_F(\hat{x},\hat{x}';P) =$$



These terms can be formally resummed since this is essentially a geometric series:

$$\boxed{\begin{gathered}\bar{G}_F(\hat{x},\hat{x}';P) = \int \frac{d^d k}{(2\pi)^d}\, e^{ik_a(\hat{x}^a - \hat{x}'^a)}(k^2+m^2)\frac{1}{k^2+m^2+\mathcal{M}(iD^a,P)}\,\frac{-i}{k^2+m^2} \\ iD^a = \hat{x}^a - i\frac{\partial}{\partial k_a} = \hat{x}'^a + i\frac{\partial}{\partial k_a} = \cdots\end{gathered}}$$

Some interesting properties:

i) V(x,x') can be identified as (and expanded in a quasi-local expansion...)

$$\theta(-\sigma)\bar{V}(\hat{x},\hat{x}',P) = -2\,\mathrm{Im}\int \frac{d^d k}{(2\pi)^d}\, e^{ik_a(\hat{x}^a-\hat{x}'^a)}\left\{(k^2+m^2)\frac{1}{k^2+m^2+\mathcal{M}(iD,P)}\,\frac{-i}{k^2+m^2} + \frac{i}{k^2+m^2}\right\}$$

ii) For x' at P one can show that

$$\bar{G}_F(\hat{x};P) = \int \frac{d^d k}{(2\pi)^d}\, e^{ik_a\hat{x}^a}\,\frac{-i}{k^2+m^2+\xi R(P)}(1+\cdots)$$

[DeWitt (1967), Parker & Toms (1985)]

Recap of basic ingredients

- Power counting
- Retarded boundary conditions in an action principle

  Double the number of variables

  *Get an "effective action" from "integrating out" the field

- Feynman rules
- Quasi-local expansions of:

  Green's function in a momentum representation

  van Vleck determinant on an arbitrary (i.e., accelerated) worldline

Equations of motion & radiation

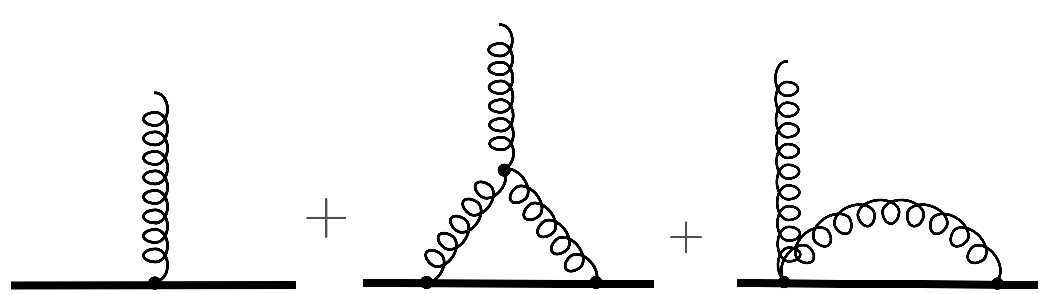
- Worldline equations of motion follow by varying the effective action (through the desired order in the expansion) with respect to the worldline coordinates

$$0 = \frac{\delta S_{eff}[z_\pm^\mu]}{\delta z^{+\mu}(\tau)}\bigg|_{z_-=0, z_+=z} = W_\mu[z^\alpha(\tau)]$$

  Notice that the worldine coordinates are arbitrary until we impose the equations of motion, which chooses the self-consistent solution for the worldline through that order in the expansion.

  i.e., The appropriate solution is the one that extremizes the effective action.

- Gravitational waves are computed directly from the radiation diagrams and the source is the self-consistent solution of the above worldline equations of motion through the given order in the mass ratio.

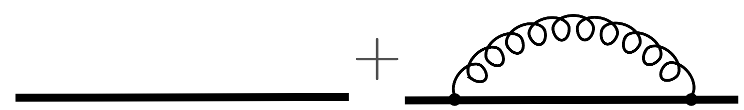


A linear scalar field & charge

Total action [Quinn, PRD **62**, 064029 (2000)]

$$S[z,\phi] = -\frac{1}{2}\int d^4x\, g^{1/2}\phi_{,\alpha}\phi^{,\alpha} - m\int d\tau + q\int d\tau\,\phi(z)$$

Effective action

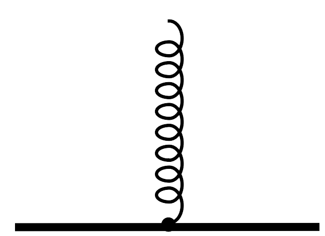


Self-force

$$m_{eff}(\tau)a^{\mu} = \frac{1}{4\pi}\left(\frac{q^2}{3}w_{\mu\alpha}\frac{Da^{\alpha}}{d\tau} + \frac{q^2}{6}w_{\mu\alpha}R^{\alpha\beta}u_{\beta}\right) + q^2\lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon} d\tau' w_{\mu}{}^{\nu}\nabla_{\nu}D_{ret}(z^{\mu}, z^{\mu'})$$

$$m_{eff}(\tau) = m - q^2\lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon} d\tau'\, D_{ret}(z^{\mu}, z^{\mu'})$$

Scalar waves



$$= q^2\int_{-\infty}^{\infty} d\tau'\, D_{ret}(x, z^{\mu})$$

Gravitational self-force @ 0th order

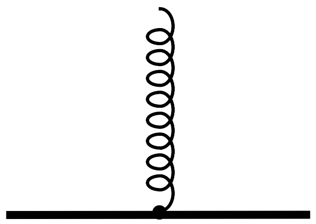
There is only one diagram that contributes to the effective action through *O(m/M)^0*

Varying to find the worldline equations of motion gives the geodesic equation

$$a^{\mu}(\tau) = 0$$

Gravitational waves @ at leading order

There is only one diagram that contributes to the gravitational wave emission at leading order

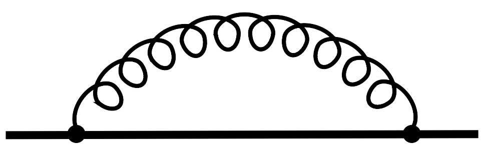


Using the Feynman rules it is straightforward to show that this equals

$$h_{\mu\nu}^{ret}(x) = -16\pi \int_{-\infty}^{\infty} d\tau D_{\mu\nu\alpha\beta}^{ret}(x, z^\mu) u^\alpha u^\beta$$

Gravitational self-force @ 1st order...

Only one diagram contributes to the effective action at *O(m/M)*



Using the Feynman rules to evaluate this diagram we see that there is a divergence, as expected. The divergent integral is

$$\propto \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d^4k \, \frac{e^{ik^0 s} k^a}{-(k^0)^2 + \mathbf{k}^2} = \int_{-\infty}^{\infty} d^3k \, \frac{\eta^a{}_i k^i}{\mathbf{k}^2}$$

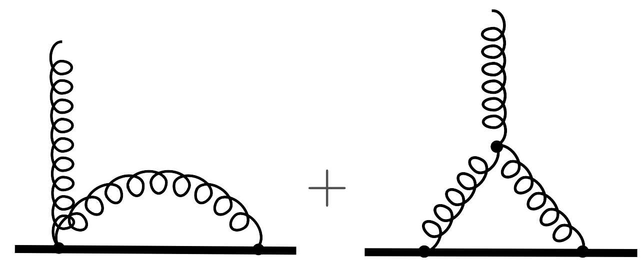
which vanishes by symmetry.

Varying the resulting regular effective action gives the MSTQW equation

$$m\, a^\mu = \frac{m}{2m_{pl}^2} w^{\mu\alpha\beta\nu} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau' \, \nabla_\nu D^{ret}_{\alpha\beta\gamma'\delta'}(z^\mu, z^{\mu'}) u^{\gamma'} u^{\delta'}$$

Gravitational waves @ NLO

At next-to-leading order there are two diagrams contributing to the gravitational wave emission



The Feynman rules imply that the radiation, including the leading order waves, is given by

$$h^{ret}_{\mu\nu}(x^\alpha) = 16\pi m \int_{-\infty}^{\infty} d\tau'\, D^{ret}_{\mu\nu\alpha\beta}(x^\alpha, z^\mu) u^\alpha u^\beta + 128\pi^2 m^2 \int_{-\infty}^{\infty} d\tau'\, D^{ret}_{\mu\nu\alpha\beta}(x^\alpha, z^\mu) u^\alpha u^\beta u^\gamma u^\delta h^{tail}_{\gamma\delta}(\tau)$$

$$-128\pi^2 \int d^4y\, g^{1/2}(y) A(y) D_{ret}(x,y) \int_{-\infty}^{\infty} \nabla D_{ret}(y, z^\mu) uu \lim_{\epsilon\to 0^+} \int_{-\infty}^{y^0_{ret}-\epsilon} d\tau'\, \nabla D_{ret}(y, z^{\mu'}) u' u'$$

$$-128\pi^2 \int d^4y\, g^{1/2}(y) B(y) \nabla D_{ret}(x,y) \int_{-\infty}^{\infty} d\tau \nabla D_{ret}(y, z^\mu) uu \int_{-\infty}^{\infty} d\tau'\, D_{ret}(y, z^{\mu'}) u' u'$$

$$h^{tail}_{\alpha\beta}(\tau) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau'\, D^{ret}_{\alpha\beta\gamma'\delta'}(z^\mu, z^{\mu'}) u^{\gamma'} u^{\delta'}$$

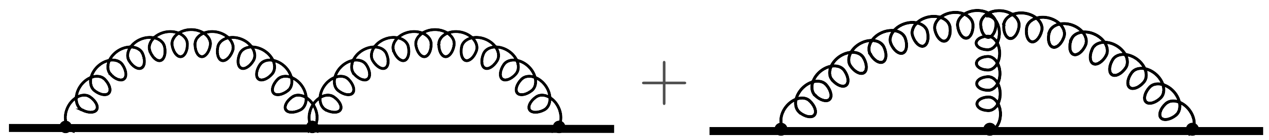
Gravitational waves @ NLO

A and *B* are certain parts of the "3-graviton" vertex

$$
\begin{aligned}
&V^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(g) = \\
&\quad +\frac{1}{4}\, I^{\mu_1\nu_1\mu_2\nu_2}\, I^{\mu_3\nu_3}{}_{\alpha\beta} + \frac{1}{4}\, I^{\mu_1\nu_1\mu_3\nu_3}\, I^{\mu_2\nu_2}{}_{\alpha\beta} + \frac{1}{2}\, I^{\mu_2\nu_2\mu_3\nu_3}\, I^{\mu_1\nu_1}{}_{\alpha\beta} \\
&\quad -\frac{1}{2}\, I_{\alpha\gamma}{}^{\mu_3\nu_3}\, I_{\beta\delta}{}^{\mu_1\nu_1}\, I^{\gamma\delta\mu_2\nu_2} - \frac{1}{2}\, I_{\alpha\gamma}{}^{\mu_2\nu_2}\, I_{\beta\delta}{}^{\mu_1\nu_1}\, I^{\gamma\delta\mu_3\nu_3} - \frac{1}{2}\, I_{\alpha\gamma}{}^{\mu_3\nu_3}\, I_{\beta\delta}{}^{\mu_2\nu_2}\, I^{\gamma\delta\mu_1\nu_1} \\
&\quad -\frac{1}{4}\, I^{\mu_1\nu_1\mu_2\nu_2}\, I^{\mu_3\nu_3}{}_{\alpha\beta} + \frac{1}{2}\, I_{\alpha\gamma}{}^{\mu_2\nu_2}\, I_{\beta}{}^{\gamma\mu_3\nu_3}\, I^{\alpha\beta\mu_1\nu_1} - \frac{1}{2}\, g^{\mu_1\nu_1} I^{\mu_2\nu_2\mu_3\nu_3} \\
&\quad -\frac{1}{8}\, g^{\mu_1\nu_1} I^{\mu_2\nu_2\mu_3\nu_3} + \frac{1}{2}\, g^{\mu_1\nu_1} I_{\alpha}{}^{\gamma\mu_3\nu_3}\, I_{\gamma\beta}{}^{\mu_2\nu_2} + \frac{1}{4}\, g^{\mu_1\nu_1} I_{\alpha}{}^{\gamma\mu_3\nu_3}\, I_{\gamma\beta}{}^{\mu_2\nu_2} \\
&\quad +\frac{1}{4}\, g^{\mu_2\nu_2} I^{\mu_1\nu_1\mu_3\nu_3} - \frac{1}{2}\, g^{\mu_2\nu_2} I_{\alpha}{}^{\gamma\mu_3\nu_3}\, I_{\gamma\beta}{}^{\mu_1\nu_1} - \frac{1}{8}\, g^{\mu_1\nu_1} g^{\mu_2\nu_2} I^{\mu_3\nu_3}{}_{\alpha\beta} \\
&\quad +\frac{1}{8}\, g^{\mu_1\nu_1} g^{\mu_2\nu_2}\, I^{\mu_3\nu_3}{}_{\alpha\beta} - \frac{1}{4}\, g^{\mu_3\nu_3} I^{\mu_1\nu_1\mu_2\nu_2} + \frac{1}{2}\, g^{\mu_3\nu_3} I_{\alpha}{}^{\gamma\mu_2\nu_2}\, I_{\gamma\beta}{}^{\mu_1\nu_1} \\
&\quad +\frac{1}{2}\, g^{\mu_3\nu_3} I_{\alpha}{}^{\gamma\mu_2\nu_2}\, I_{\gamma\beta}{}^{\mu_1\nu_1} - \frac{1}{4}\, g^{\mu_1\nu_1} g^{\mu_3\nu_3} I^{\mu_2\nu_2}{}_{\alpha\beta} - \frac{1}{4}\, g^{\mu_1\nu_1} g^{\mu_3\nu_3} I^{\mu_2\nu_2}{}_{\alpha\beta} \\
&\quad +\frac{1}{8}\, g^{\mu_1\nu_1} g^{\mu_2\nu_2} g^{\mu_3\nu_3} + (\text{permutations of } \{1,2,3\})
\end{aligned}
$$

Gravitational self-force @ 2nd order...

- Potentially needed for precise parameter estimation with LISA
- There are two diagrams contributing to the effective action at *O(m/M)*^2



The first diagram contributes the following to the second order self-force

$$= -128\pi^2 m^3 w^{\mu\alpha\beta\nu} \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau' \, \nabla_\nu D^{ret}_{\alpha\beta\gamma'\delta'}(z^\mu, z^{\mu'}) u^{\gamma'} u^{\delta'} u^{\epsilon'} u^{\eta'} h^{tail}_{\epsilon'\eta'}(\tau')$$

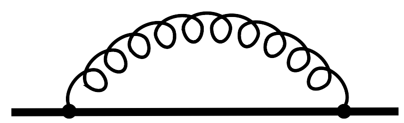
$$+128\pi^2 m^3 w^{\mu\alpha\beta\gamma\delta\nu} h^{tail}_{\alpha\beta}(\tau) \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau' \, \nabla_\nu D^{ret}_{\gamma\delta\epsilon'\eta'}(z^\mu, z^{\mu'}) u^{\epsilon'} u^{\eta'}$$

The contribution of the second diagram is nearly complete... stay tuned!

*Time-dependent mass

Interestingly, there is a third contribution that comes from varying the first order contribution to the effective action and seems to contribute a correction to the particle's mass that depends on the tail

$$\frac{\delta}{\delta z^{\mu}(\tau)}$$



$$= -16\pi m \left(w^{\mu(\alpha} a^{\beta)} + \frac{1}{2} a^{\mu} u^{\alpha} u^{\beta} \right) h^{tail}_{\alpha\beta}(\tau)$$

Factoring out the acceleration and gathering all terms involving the acceleration to the left hand side of the worldline equations of motion (through second order) we find the time-dependent effective mass

$$\frac{m^{\mu\nu}_{eff}(\tau)}{m} = g^{\mu\nu} + 16\pi \left(w^{\mu(\alpha} g^{\beta)\nu} + \frac{1}{2} g^{\mu\nu} u^{\alpha} u^{\beta} \right) h^{tail}_{\alpha\beta}(\tau)$$

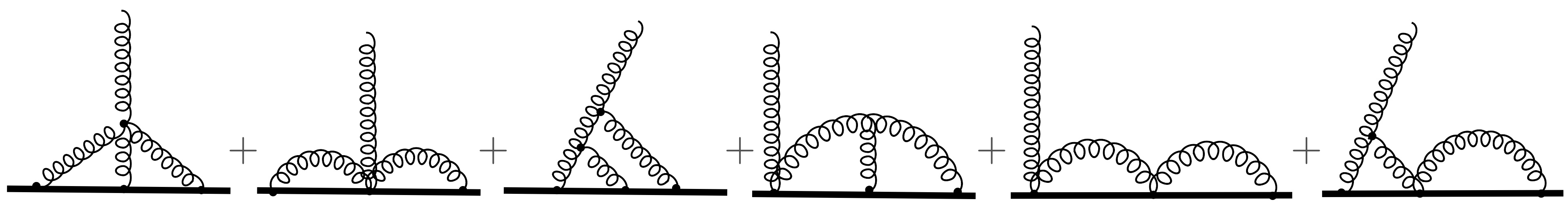
Pound finds a time-dependent effective mass, too

$$\partial_t (2m + \mathscr{K}^{(2,\text{-}1)}) = \tfrac{1}{9} m \left(5 g^{\alpha\beta} + 11 u^{\alpha} u^{\beta} \right) h^{\text{tail}}_{\alpha\beta\gamma} u^{\gamma}$$

Mino, Sasaki & Tanaka also find a time-dependent effective mass...

$$\frac{m^{(1)}_{MST}(\tau)}{m} = 1 + \frac{1}{6} \left(g^{\alpha\beta} + 10 u^{\alpha} u^{\beta} \right) h^{tail}_{\alpha\beta}(\tau)$$

Gravitational waves @ NNNLO



Can we remove diagrams with self-interactions?

Introduce an extended Lorenz gauge by making second order transformation

Use a field redefinition of gravitational perturbations

... and with spin

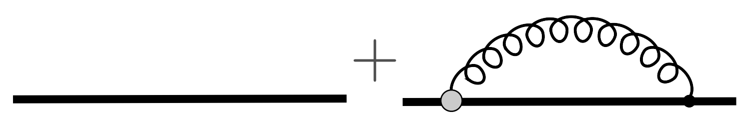
Preliminary!

- Use Routhian approach (Lagrangian for worldline, Hamiltonian for spin)

$$\int d\tau\, \mathcal{R} = -m \int d\tau + \frac{1}{2} \int d\tau\, S^{ab} \omega_{ab\mu} u^\mu + \cdots$$

Like a Lagrangian $\dfrac{\partial \mathcal{R}}{\partial z^\mu(\tau)} = 0$ *Like a Hamiltonian* $\dfrac{DS^{ab}}{d\tau} = \{S^{ab}, \mathcal{R}\}$

- Two diagrams through second order for self-force



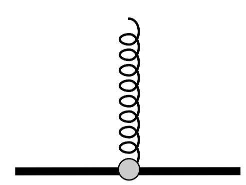
$S \sim I\omega_{rot} \sim \varepsilon L$

Worldline equations of motion

$$m\, a^\mu = MSTQW + (\text{nonspinning 2nd order}) + \frac{1}{2} R^\mu{}_{\alpha\beta\gamma} u^\beta S^{\gamma\delta} + \cdots$$

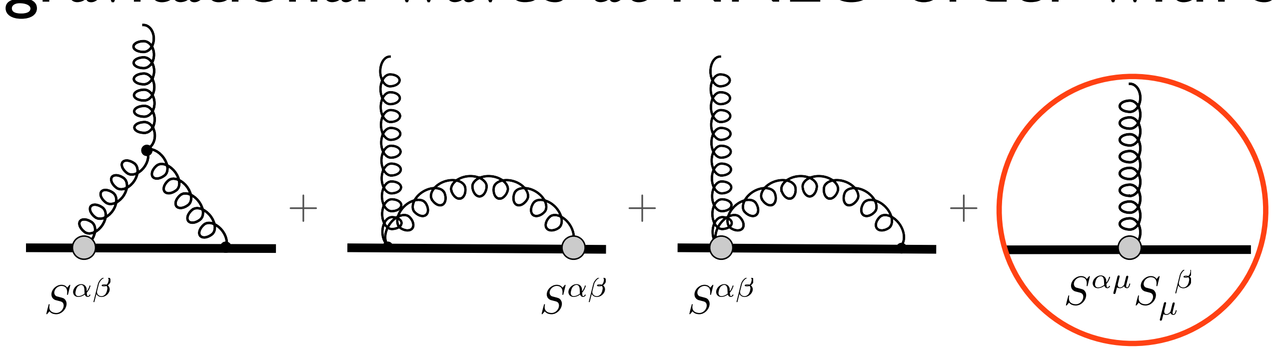
Spin equations of motion $$\frac{DS^{\mu\nu}}{d\tau} = p^\mu u^\nu - u^\mu p^\nu + \cdots$$

- Gravitational waves at NLO order with spin



$$= \frac{1}{2m_{pl}^2} \int_{-\infty}^{\infty} d\tau\, \nabla_\gamma D^{ret}_{\mu\nu\alpha\beta}(x, z^\mu) u^\alpha S^{\beta\gamma}$$

- Diagrams for gravitational waves at NNLO order with spin



Spin-induced finite size effect

$$\sim c_{ESS} \int d\tau\, \mathcal{E}_{\alpha\beta} S^{\alpha\mu} S_\mu{}^\beta$$

A nonlinear scalar model

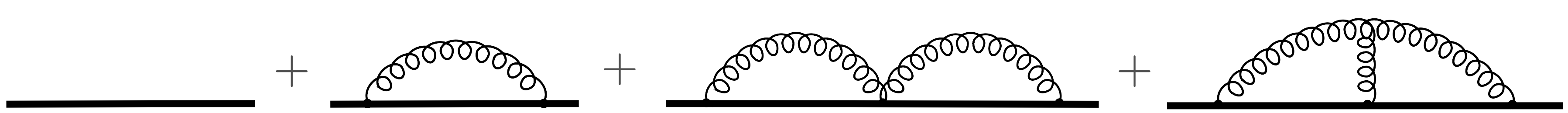
- Build a nonlinear scalar theory analogous to perturbed general relativity

$$S[\phi, z^\mu] = -\frac{1}{2}\int d^4x\, g^{1/2}\phi_{,\alpha}\phi^{,\alpha}A^2(\phi) - m\int d\tau\, B(\phi)$$

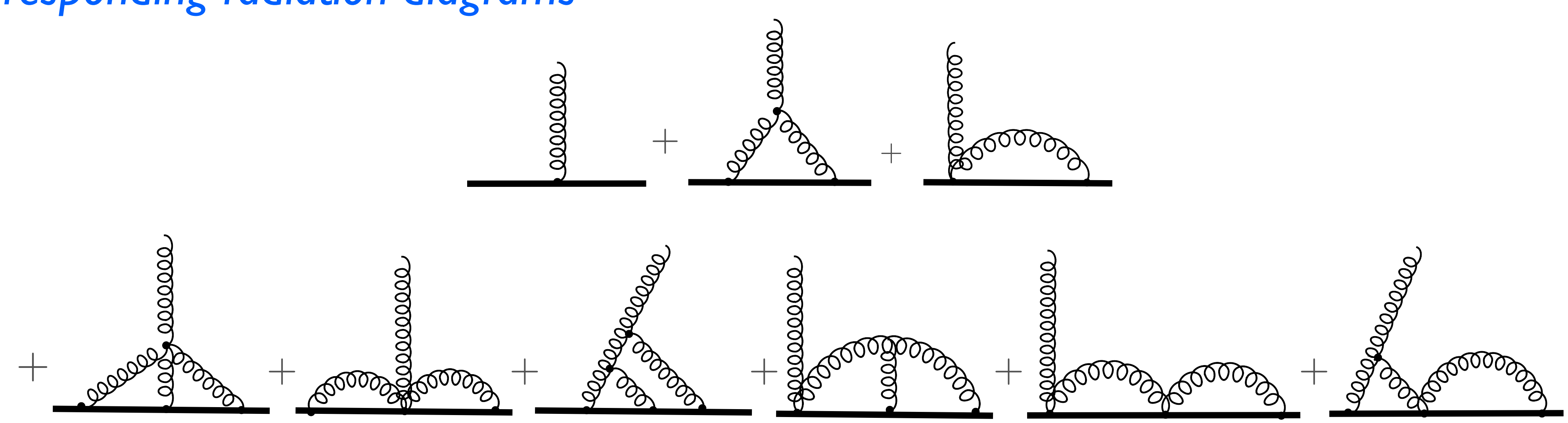
$$A(\phi) = \sum_{n=0}^{\infty}\frac{a_n}{n!}\phi^n \qquad B(\phi) = \sum_{n=0}^{\infty}\frac{b_n}{n!}\phi^n$$

Equations of motion $\Box\phi = -\frac{A'}{A}\phi_{,\alpha}\phi^{,\alpha} + m\int d\tau\frac{\delta^4(x-z)}{g^{1/2}}\frac{B'}{A^2} \qquad a^\mu = -w^{\mu\nu}\nabla_\nu \ln B(\phi)$

Self-force diagrams through 2nd order



Corresponding radiation diagrams

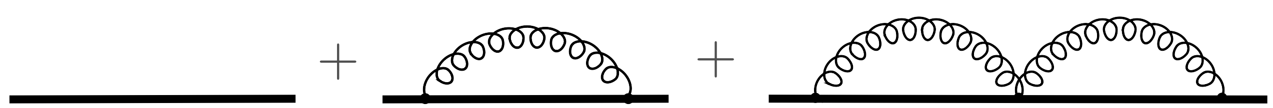


- But a field redefinition removes self-interaction terms

$$\sigma_{,\alpha} = \phi_{,\alpha} A(\phi) \qquad \sigma = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n!} \phi^n$$

$$S[z,\sigma] = -\frac{1}{2}\int d^4x\, g^{1/2} \sigma_{,\alpha}\sigma^{,\alpha} - m\int d\tau\, C(\sigma/m_{pl}) \qquad C(\sigma) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\sigma^n = B(\phi)$$

Self-force through second order

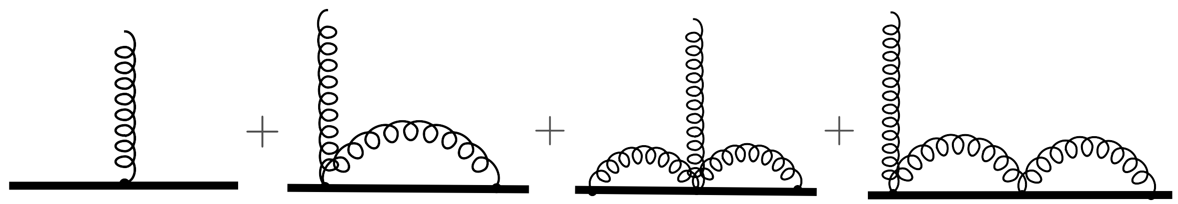


$$\begin{aligned} m_{eff}(\tau) a^\mu =& \frac{1}{4\pi}\left(\frac{1}{3} w^\mu{}_\nu \frac{Da^\nu}{d\tau} + \frac{1}{6} w^{\mu\nu} R_{\nu\alpha} u^\alpha\right)\left(\frac{m^2 c_1^2}{m_{pl}^2} - \frac{2m^3 c_1^2 c_2}{m_{pl}^4} F(\tau)\right) \\ &+ \left(\frac{m^2 c_1^2}{m_{pl}^2} - \frac{m^3 c_1^2 c_2}{m_{pl}^4} F(\tau)\right) \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau'\, w^{\mu\nu} \nabla_\nu D_{ret}(z^\mu, z^{\mu'}) \\ &- \frac{m^3 c_1^2 c_2}{m_{pl}^4} \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau'\, w^{\mu\nu} \nabla_\nu D_{ret}(z^\mu, z^{\mu'}) F(\tau') \end{aligned}$$

$$m_{eff}(\tau) = m - \frac{m^2 c_1^2}{4\pi m_{pl}^2} F(\tau) - \frac{m^3 c_1^2 c_2}{4\pi m_{pl}^4} \frac{DF(\tau)}{d\tau} + \frac{m^3 c_1^2 c_2}{2 m_{pl}^4} F^2(\tau) + \frac{m^3 c_1^2 c_2}{m_{pl}^4} \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau'\, D_{ret}(z^\mu, z^{\mu'}) F(\tau')$$

$$F(\tau) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau'\, D_{ret}(z^\mu, z^{\mu'})$$

Radiation carrying information of second order perturbed motion to a distant observer



$$\sigma_{rad}(x^\alpha) = -\frac{mc_1}{m_{pl}^2}\int_{-\infty}^{\infty} d\tau\, D_{ret}(x, z^\mu)\Big\{1 - \frac{mc_2}{m_{pl}^2}F(\tau) - \frac{m^2c_2^2}{4\pi m_{pl}^4}\frac{DF(\tau)}{d\tau} + \frac{m^2c_1c_3}{2m_{pl}^4}F^2(\tau)$$

$$+\frac{m^2c_2^2}{m_{pl}^4}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\tau-\epsilon} d\tau'\, D_{ret}(z^\mu, z^{\mu'})F(\tau')\Big\}$$

- A clever choice for *B* results in a *fully linear* scalar theory for the *sigma* variable

$$B(\phi) = C(\sigma) = 1 + c_1\sigma \qquad\qquad b_n = c_1 a_{n-1}$$

- Therefore, one can use this linear theory for sigma to perturbatively reconstruct the self-force in the nonlinear theory for phi

$$\sigma = \sum_{n=1}^{\infty}\frac{a_{n-1}}{n!}\phi^n$$

Exact conservative self-force

For a linear scalar charge moving on a circular geodesic of a Schwarzschild black hole, the regularized field and self-force have already been computed

[Diaz-Rivera, Messaritaki, Whiting & Detweiller (2004)]

Use the linear theory (in *sigma*), for which the self-force on these orbits is already computed, to determine the second order self-force in the original nonlinear theory (in *phi)*

Consider the following model: $A(\phi) = e^{\alpha\phi}$

$$\sigma_{,\mu} = \phi_{,\mu} e^{\alpha\phi} \qquad \phi_{,\mu} = \frac{\sigma_{,\mu}}{1+\alpha\sigma}$$

$$\alpha\sigma = e^{\alpha\phi} - 1 \qquad \phi = \frac{1}{\alpha}\ln(1+\alpha\sigma)$$

Equations of motion in sigma variable

$$\Box\sigma = mc_1 \int d\tau \frac{\delta^4(x-z)}{g^{1/2}}$$

$$a^\mu = -\frac{c_1(g^{\mu\nu} + u^\mu u^\nu)\sigma_{,\nu}}{1+c_1\sigma} \equiv f^\mu$$

In terms of the phi variable (of the original nonlinear theory), reconstruct the perturbative expressions for the self-force

$$F^{\mu}_{(1)} = f^{\mu}(\sigma)\frac{1+c_1\sigma}{1+\alpha\sigma}$$

$$F^{\mu}_{(2)} = f^{\mu}(\sigma)\left(1-\frac{c_1}{\alpha}\right)\frac{1+c_1\sigma}{1+\alpha\sigma}\ln(1+\alpha\sigma) = \left(1-\frac{c_1}{\alpha}\right)\ln(1+\alpha\sigma)F^{\mu}_{(1)}$$

$$a^{\mu} = F^{\mu}_{(1)} + F^{\mu}_{(2)} + \cdots$$

From Burko (2000) and Diaz-Rivera, etal (2004) we know that the first-order self-force for the sigma variable (i.e., linear theory) falls off as R^{-5} and that the regular field falls off with radius as R^{-3}. Therefore, at large values of the orbital radius R,

$$F^{r}_{(1)} \sim \frac{1}{R^5} \qquad\qquad F^{r}_{(2)} \sim \frac{1}{R^8}$$

In general, the nth order radial component of the self-force for circular geodesics in Schwarzschild falls off with radius R as

$$\boxed{F^{r}_{(n)} \sim \frac{1}{R^{2+3n}}}$$

Similar kinds of results can be made for energy, angular momentum, and orbital frequency. Also, the conclusions don't change for generic choices for A or B.

A new numerical approach for self-force computations: the main ideas

(in progress with Manuel Tiglio, UMD)

The overall approach

- Compute the retarded Green's function in Schwarzschild/Kerr

  Once computed, the Green's function is known once and for all

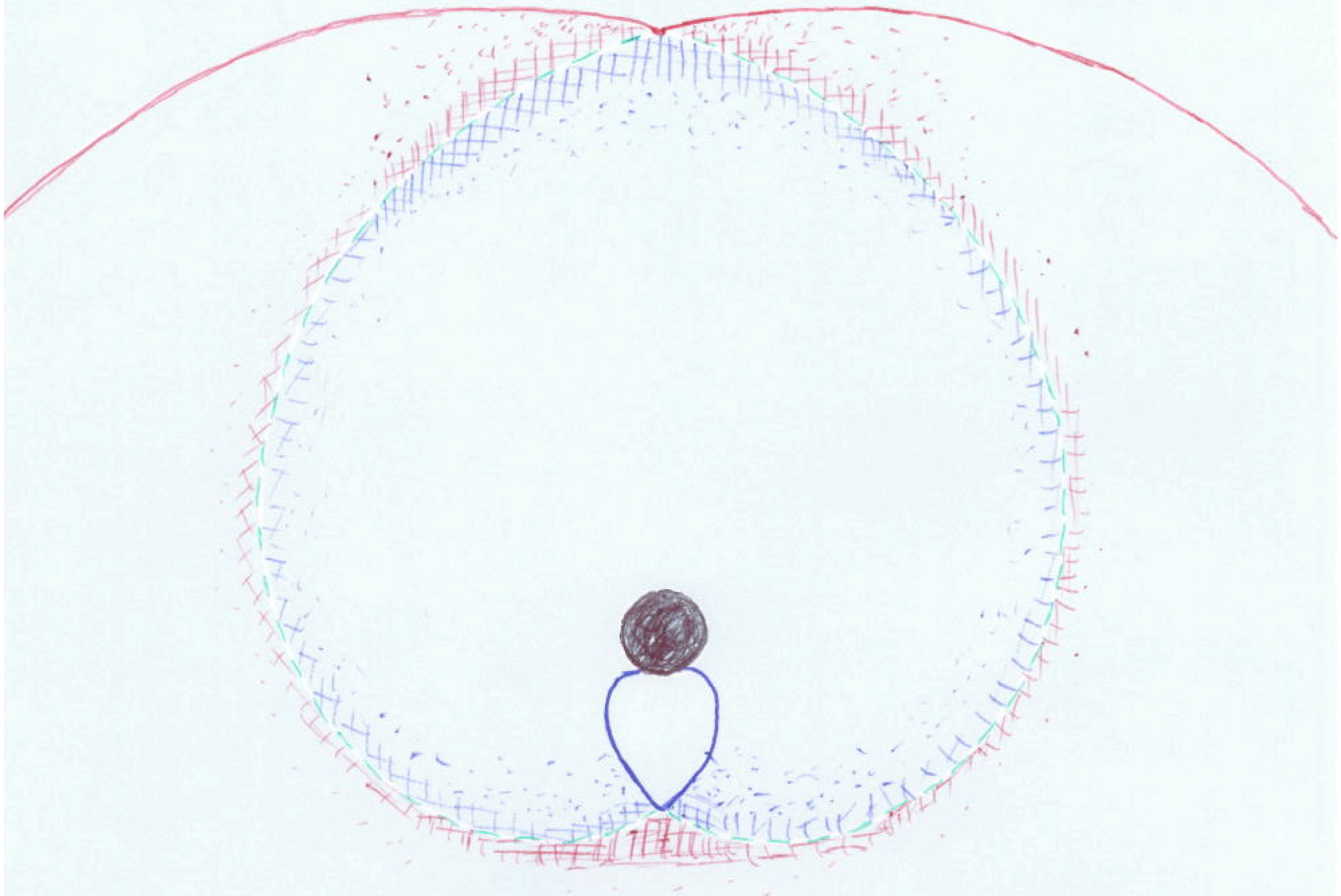
  Analytic expressions for the self-force (e.g., MSTQW equation) are given in terms of the integral of a retarded Green's function

  Higher order analytic self-force equations will be available in the near future implying that these higher order corrections can be computed relatively quickly and easily.

- Solve the integrodifferential equation numerically

  Want to compute the solution to the self-consistent worldline equations of motion

  Need a high order IDE solver to track 10^5 orbits of inspiral



Quasi-local expansion of $V(x,x')$

= 0 outside

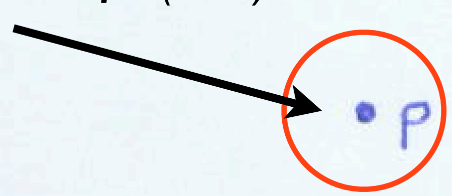


Image courtesy of S. Dolan

Conclusions

- Higher order analytic expressions for:

  Second order gravitational wave emission

  Second order self-force is nearly complete... stay tuned

  Self-force on and GWs from a spinning compact object... in progress

- A nonlinear scalar model

  Found exact (conservative) self-force for a nonlinear scalar model

  Found that nth order self-force falls with radius as $1/R^{2+3n}$

  Can describe binaries with arbitrary mass ratio

- Our numerical approach

  Still in the early development phase... but comments, criticisms, and shoe-throwing are welcome