Comments on First and Second Order Gravitational Self-force

or:

some stuff I'd love to have discussions about at this meeting

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Outline

1. Perturbative versus Self-consistent formulation
   - Overview
   - Role of geodesic deviation term
   - Gauge-dependence

Aside: the “natural” class of gauges

2. First-order and (hopefully) Second-order Self-force
   - Equation for the metric perturbations
   - Center of mass definition
   - Gauge-independent self-force formula

Perturbative Formulation

<u>My definition of perturbation theory is a Taylor series.</u>

For gravitational self-force, one Taylor expands exact solutions in mass M and radius R (or uses a dimensionless parameter proportional to mass and radius).

Every term in this series is evaluated at M=R=0; i.e., the background and all perturbations live on the same "background space".

Our "background space" here is a manifold with a preferred worldline (the place where the body "disappeared to").

The mathematical objects describing the perturbations are

1. Rank-2 tensors ("metric perturbations") defined on the background manifold minus the background worldline
2. Rank-1 tensors ("deviation vectors") defined on the background worldline

Once suitable assumptions on a family of exact solutions are found, item 1 follows from Einstein's equation and item 2 follows from a center of mass definition.

That's all there is to the perturbative formulation

Self-consistent Formulation

Going to higher order in perturbation theory can only help so much. Eventually, the true motion will have deviated from the background motion by so much that the series will "not converge". At this point it makes sense to switch to a new perturbation series, based off of a new background motion that is close to the present motion.

One often imagines that this process of "switching to a new background" should happen constantly, and correspondingly makes up an equation whose appearance suggests those words. We call such an equation a **self-consistent perturbative equation**. Examples are the harmonic-gauge-relaxed linearized Einstein equation, the MiSaTaQuWa equation, the blackbody-cooling equation, and many other equations.

These equations go beyond Taylor's theorem and have fewer "rules" governing how they are derived. However, in many cases they are obviously better.

Self consistent equations are like English class: there are no right answers, but there are definitely wrong answers.

Examples of Perturbative and Self-consistent Equations

	Perturbative result	Self-consistent equation(s)
Blackbody cooling (box with hole of area A)	$\frac{dE^{(1)}}{dt} = -\sigma A^{(1)} T_0^4$	$\frac{dE}{dt} = -\sigma A T^4(t)$ (and $E = E(T)$)
Gravitational self-force	$u^c\nabla_c(u^b\nabla_b Z^a) = -R_{bcd}{}^a u^b Z^c u^d - (g^{ab}+u^a u^b)(\nabla_d h^{\rm tail}_{bc} - \frac{1}{2}\nabla_b h^{\rm tail}_{cd})u^c u^d$ (tail on geodesic)	$u^b\nabla_b u^a = -(g^{ab}+u^a u^b)(\nabla_d h^{\rm tail}_{bc} - \frac{1}{2}\nabla_b h^{\rm tail}_{cd})u^c u^d$ (tail on self-consistent motion <u>or</u> tangent geodesic)
Metric perturbations	$\nabla^c\nabla_c \tilde{h}_{ab} - 2R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt}$, $\nabla^b \tilde{h}_{ab} = 0.$ (source on geodesic)	$\nabla^c\nabla_c \tilde{h}_{ab} - 2R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt}$ (source on self-consistent motion)
EM self-force	$\vec{a}^{(1)} = \frac{2}{3}\frac{q^2}{m}\dot{\vec{a}}^{(0)}$	$\vec{a} = \frac{2}{3}\frac{q^2}{m}\frac{\dot{\vec{F}}_{\rm ext}}{m}$

ALD equation not allowed!

Role of geodesic deviation term

Our perturbative derivation came with a surprise,

$$u^c\nabla_c(u^b\nabla_b Z^a) = \frac{1}{2M}R_{bcd}{}^a S^{bc}u^d - R_{bcd}{}^a u^b Z^c u^d - (g^{ab}+u^a u^b)(\nabla_d h^{\text{tail}}_{bc} - \frac{1}{2}\nabla_b h^{\text{tail}}_{cd})u^c u^d$$

spin force

self-force

Geodesic deviation!

But it is clear in hindsight that this term should be here. Consider a one-parameter-family wherein the initial position of a spinless body is smoothly translated with lambda. In the limit as M→0 we have a family of geodesics, and the correct perturbative description of motion is the geodesic deviation equation.

This term represents the "perturbation to a nearby geodesic". As soon as the particle has deviated from the background geodesic, it "wants" to move primarily on the new geodesic to which it is tangent. From the vantage-point of the background geodesic, this constitutes an extra force.

The geodesic deviation effect is real! Unless perturbative calculations of inspiral plan to update the background orbit every timestep, these calculations should include the geodesic deviation term.

Gauge dependence in the Perturbative Formulation

[audience spared from additional gauge-dependence blurb]

Consider just smooth gauge transformations. In this case it is obvious that the description of motion just changes by the gauge vector,

If the coordinates change to O(λ)… $x^\mu \rightarrow \hat{x}^\mu = x^\mu - \lambda A^\mu(x^\nu) + O(\lambda^2)$

the O(λ) position of the body changes. $Z^i(t) \rightarrow \hat{Z}^i(t) = Z^i(t) - A^i(t, x^j = 0)$

If you want the gauge-dependence of the "force" you just take two time derivatives.

$$\ddot{Z}^i(t) \rightarrow \ddot{\hat{Z}}^i(t) = \ddot{Z}^i(t) - \ddot{A}^i(t, x^j = 0)$$

That is all there is to it. The metric perturbations h transform in the standard way, and observables are constructed from Z and h.

(The treatment of non-smooth gauge transformations will be given in part 2 after the discussion of center of mass.)

Gauge dependence in the Self-consistent Formulation (1)

One can extend the notion of gauge to the self-consistent formulation at one's own risk.

Some dangers are:

- Any self-consistent equation for the metric perturbations has to handle non-geodesic motion and will therefore *not* be the linearized Einstein equation. How does one determine how the solutions of this equation change under gauge?

- Some gauges that are perfectly fine perturbatively may not have good associated self-consistent equations. An example is the gauge where the self-force vanishes.

- Would the operations of "changing gauge" and "making up a self-consistent equation" commute?

Of course, any formalism that works would be very interesting!

Gauge dependence in the Self-consistent Formulation (2)

That said, there does appear to be a natural notion of the way the "self-consistent force" changes under gauge transformations away from the Lorenz gauge.

The equations of motion in the new gauge is

$$\frac{d^2\hat{Z}^i}{dt^2} = -R_{0j0}{}^i Z^j - \left({h^{\text{tail}}}^i{}_{0,0} - \frac{1}{2} {h^{\text{tail}}}_{00}{}^{,i} \right) + \delta\ddot{Z}^i$$

two derivatives of the gauge vector for a smooth transformation

which can be written,

$$\frac{d^2\hat{Z}^i}{dt^2} = -R_{0j0}{}^i \hat{Z}^j - \left({h^{\text{tail}}}^i{}_{0,0} - \frac{1}{2} {h^{\text{tail}}}_{00}{}^{,i} \right) + \delta\ddot{Z}^i + R_{0j0}{}^i \delta Z^j$$

This form suggests

$$u^b \nabla_b u^a = -(g^{ab} + u^a u^b)(\nabla_d h^{\text{tail}}_{bc} - \frac{1}{2} \nabla_b h^{\text{tail}}_{cd}) u^c u^d + \underbrace{\delta\ddot{Z}^a + R_{cbd}{}^a u^c u^d \delta Z^b}$$

The Barack and Ori law

Part I Punchline

In my experience…

The perturbative formulation is straightforward and clear.

The self-consistent formulation is confusing, but often necessary.

I find the distinction between perturbative and self-consistent equations of essential use in understanding such hot topics as "gauge invariant observables", second-order perturbations for consistency of the waveform, and runaway solutions in electromagnetism.

Aside: Non-smooth gauge transformations

It is clear that at least *some* non-smooth gauge transformations should be allowed. For example, a gauge vector that is radial near the particle simply changes you from “isotropic” to “Schwarzschild” coordinates (or back).

Why should the position of a black hole be defined for a hole in isotropic coordinates (corresponding to Lorenz gauge), but not for a hole in Schwarzschild coordinates (corresponding to a gauge not smoothly related)?

When dealing with finite quantities it is natural to consider smooth coordinate transformations. When dealing with a singularity, it is natural to consider coordinate transformations that do not change the “degree of singularity”. In the self-force case the singularity remains 1/r as long as the gauge vector is bounded (but it could be direction dependent). This seems to be the natural class of gauges to consider. (It is also what comes out of our formalism.)

First-order and Second-order: Metric Perturbations

Some derivations of self-force begin by considering the (gauged-relaxed) linearized Einstein equation with point particle source. A better approach is to just consider extended bodies in GR under "small size" approximations, and see what comes out. We assumed a kind of "already matched asymptotic expansion",

$$(1) \qquad g_{\mu\nu}(\lambda; t, r, \theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} r^n \left(\frac{\lambda}{r}\right)^m (a_{\mu\nu})_{nm}(t, \theta, \phi) \qquad G[g(\lambda)] = 0 \qquad r > \lambda R_0$$

Denoting the first-order part of the metric by h, we have

$$h_{\alpha\beta} = \frac{c_{\alpha\beta}(t, \theta, \phi)}{r} + O(1) \qquad G^{(1)}[h] = 0 \qquad r > 0$$

The 1/r singularity is integrable and the equation is linear. So, one may regard h as a distribution and calculate the distributional "stress-energy" $T_{ab} \equiv G^{(1)}_{ab}[h_{cd}]/8\pi$. This gives

$$T_{ab} = N_{ab}(t) \frac{\delta^{(3)}(x^i)}{\sqrt{-g}} \frac{d\tau}{dt}$$

Conservation of stress-energy (i.e., the distributional linearized Bianchi identity) fixes $N_{ab} = M u_a u_b$ and requires the background worldline x^i=0 to be a geodesic.

We thus find that the first-order perturbations of (1) satisfy the linearized Einstein equation with point particle source on a geodesic.

Attempting this procedure at second-order, we have

$$j_{ab} = \frac{C_{ab}(t,\theta,\phi)}{r^2} + O(r^{-1}) \qquad\qquad G^{(1)}_{ab}[j] = -G^{(2)}_{ab}[h,h] \quad r>0$$

The 1/r^2 singularity is integrable, so j—and hence G1[j]—make sense as distributions. But G2[h,h] goes as 1/r^4 and does not define a distribution. There is therefore <u>no distributional interpretation</u> of the Einstein equation at second-order. However, homogeneous solutions j^H can still be treated distributionally. The "stress-energy" calculated ~~is~~ will be

$$\frac{1}{8\pi}(G^{(1)}[j^H])^{ab} = \int_\gamma M' u^a u^b \delta_4(x,z(\tau))d\tau + \int_\gamma u^{(a}S^{b)c}\nabla_c \delta_4(x,z(\tau))d\tau$$

background geodesic

where M' is a correction to the mass, and S is the spin of the body.

This observation allows j^H to be found by Green's function techniques.

A particular solution must still be found, of course. (Aside: how does one choose "no incoming radiation" here?)

First-order and Second-order: Center of Mass

The assignment of a “representative worldline” to a body in exact General Relativity is at best highly non-trivial. In the case where the body is a black hole, such a task seems impossible.

However, there is a *perturbative* notion of “position” for black holes and other bodies.

To illustrate our perturbative center of mass definition, consider the Schwarzschild metric of mass $M_0\lambda$ in coordinates that are shifted to $O(\lambda)$. We ask, “Where in flat spacetime is the center of the black hole”? Consider just the time-time component,

$$g_{00}(\lambda) = -1 + \frac{2M_0\lambda}{r + \lambda\vec{P}}$$
$$= -1 + \lambda\frac{2M_0}{r} - \lambda^2\frac{2M_0\vec{P}\cdot\vec{x}}{r^3} + O(\lambda^3)$$

There is no position information is in the zeroth order term.

The first-order term tells you the background position (by being singular at r=0).

The second-order term tells you the first-order position via a dipolar distortion.
It is still singular on the background worldline.

Such an O(λ^2/r^2) dipolar distortion shows up in the "near-zone" at *zeroth* order, where the metric (characterizing the "body at an instant of time") is known to be stationary and asymptotically flat. It becomes the *mass dipole moment*, a quantity well-known to characterize the origin of coordinates.

We define the perturbed center of mass to be (minus) this mass dipole (equivalently, the smooth far-zone gauge transformation required to eliminate the mass dipole).

I want to emphasize: **The information about where the body "is" to first-order is contained in the <u>second</u>-order metric perturbations**.

It <u>turns out</u>, however, that where the body is "accelerating to" to first order (i.e., two time derivatives of the perturbed position) can be found from just the zeroth and first-order metric.

Second Order Center of Mass?

A second-order shift in coordinates will introduce a dipolar dipolar distribution at O(λ^3/r^2), i.e., in the *third*-order metric perturbations.

The extra power of λ makes this translate to the near-zone first-order metric instead of the background. Here one deals with a solution to the linearized Einstein equation that can a priori be neither stationary nor asymptotically flat.

However, it turns out that the solution is stationary to O(1/r^2) (this is related to the constancy of the lowest-order mass and spin), and also asymptotically flat (this is related to geodesic background motion). Thus, it appears reasonable to generalize the “mass dipole moment” notion to this setting.

However, there is a new subtlety: for certain (far-zone) gauge choices the near-zone perturbations do not come out in coordinates adapted to the (approximate) timelike killing field. It is not clear whether it makes sense to speak of the “spatial position” of a (stationary!) body expressed in such coordinates.

I do hope to have this all resolved soon!

Gauge dependence at first order

The mass dipole (i.e., perturbed position) can be extracted with the formula

$$Z^i = -\frac{3}{8\pi} \lim_{r\to 0} \int g_{00}^{(2)} n^i d\Omega \qquad n^i = x^i/r$$

We can now ask how the perturbed position changes under changes of gauge.
We need the second-order gauge transformation law,

$$\delta g^{(2)} = \left(\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^2_{\xi^{(1)}}\right) g^{(0)} + 2\mathcal{L}_{\xi^{(1)}} g^{(1)}$$

According to our "already matched expansion" assumption, the gauge vectors behave as

$$\xi_\mu^{(1)} = O(1) = F_\mu(t,\theta,\phi) + O(r)$$

$$\xi_\mu^{(2)} = O(1/r)$$

Notice that second-order gauge changes don't affect what we've claimed is the first-order position (phew!).

One can then compute that

$$\delta g_{00}^{(2)} = -\frac{2M}{r^2} F^i n_i + O(1/r)$$

and plug to the top in to find

$$\delta Z^i = \frac{3}{4\pi} \lim_{r\to 0} \int \xi_j n^j n^i d\Omega$$

This formula holds for all allowed gauges.

Gauge-independent self-force formula (first order)

We had

$$\delta Z^i = \frac{3}{4\pi}\lim_{r\to 0}\int \xi_j n^j n^i d\Omega$$

The force is just two derivatives, and we manipulate…

$$\begin{aligned}\delta \ddot{Z}^i &= \frac{3}{4\pi}\lim_{r\to 0}\int \partial_0\partial_0\xi_j n^j n^i d\Omega \\ &= \frac{3}{4\pi}\lim_{r\to 0}\int \left(\partial_0\partial_0\xi_j + \partial_0\partial_j\xi_0 - \partial_j\partial_0\xi_0\right) n^j n^i d\Omega \\ &= \frac{3}{4\pi}\lim_{r\to 0}\int \left(\partial_0\delta g^{(1)}_{j0} - \frac{1}{2}\partial_j\delta g^{(1)}_{00}\right) n^j n^i d\Omega\end{aligned}$$

…to get a formula in terms of the change in the metric perturbations. This function of a rank-2 tensor also gives the correct self-force when the Lorenz gauge metric perturbations are inserted. Thus it in fact gives the self-force in an arbitrary (allowed) gauge,

$$\boxed{\ddot{Z}^i = \frac{3}{4\pi}\lim_{r\to 0}\int \left(\partial_0 g^{(1)}_{j0} - \frac{1}{2}\partial_j g^{(1)}_{00}\right) n^j n^i}$$

Gauge at second-order

Now we need the *third*-order gauge-transformation formula,

$$\delta g^{(3)} = (\mathcal{L}_{\xi^{(3)}} + 3\mathcal{L}_{\xi^{(1)}}\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^3_{\xi^{(3)}})g^{(0)} + 3(\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^2_{\xi^{(1)}})g^{(1)} + 3\mathcal{L}_{\xi^{(1)}}g^{(2)}$$

The gauge vectors go like

$$\xi^{(1)}_\mu = O(1)$$

$$\xi^{(2)}_\mu = O(1/r)$$

$$\xi^{(3)}_\mu = O(1/r^2)$$

and the expression for the change in the mass dipole is much more complicated. Some apparently pathological terms correspond directly to redefining the time coordinate away from the (approximate) timelike killing field. Other terms are not yet understood.

Once this is understood, I should be able to derive the analogous formula for the change in second-order self-force under a change of gauge. Hopefully there is also an argument that lets me leap to a gauge-independent second-order self-force formula!

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