Self-force, generalized symmetries, and inertia

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AIH, CQG 25: 205008, 2008 (arXiv: 0805.4259)
AIH, CQG 25: 235020, 2008 (arXiv: 0807.1150)
AIH, arXiv: 0903.0167
SE Gralla, AIH, RM Wald, arXiv: 0905.2391

Outline

- Motivation
- Deriving laws of motion using generalized Killing fields
- Detweiler-Whiting(-Dirac) axiom
- Approximate equations of motion
- Higher-order effects

Introduction

There are two broad classes of self-force derivations

Point particle or axiomatic	Brute-force perturbation theory
Physically insightful	Not much physics
Simple derivations	Complicated
Mathematical issues	Can be rigorous and general
No derivation from first principles	Not easily generalized

- Despite their problems, axiomatic methods work. Why? What happens if you take (some of) their assumptions seriously?

- Can similar physical statements about self-interacting systems be rigorously derived without much trouble?

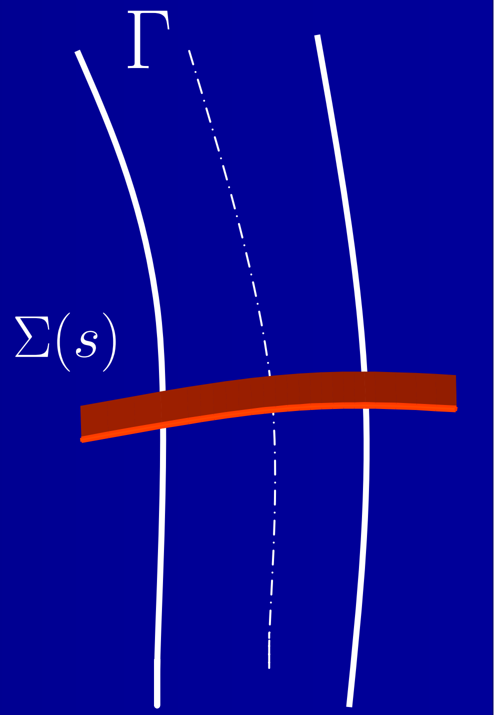
- Yes! The Detweiler-Whiting axiom becomes a very simple theorem if you look at continuum mechanics in the right way.

- It is mathematically equivalent to a statement regarding the degree to which the underlying Green function (or field equation) is preserved with respect to certain approximate Killing fields.

- This has a direct interpretation as the degree to which Newton's 3rd law is violated by the S-type self-field.

Mechanics in curved spacetime

What can be learned solely from stress-energy conservation + (say) Maxwell's equations?



- Choose "good" state variables: linear and angular momentum.
- These are traditionally associated with Killing fields and conservation laws.

$$\mathcal{P}_\xi(s) = \int_{\Sigma(s)} T^a{}_b \xi^b \mathrm{d}S_b$$

- No Killing fields exist in general, so define something similar.
- Need 10 vector fields for the 4 components of linear and 6 components of angular momentum.

Properties of the Generalized Killing fields (GKFs)

- Form a “generalized Poincaré group” with respect to a given frame. Momentum is then a map

$$\mathcal{P}_\xi : GP \times \mathbb{R} \rightarrow \mathbb{R}$$

- Working with scalars is easier than 1 and 2-forms. It also allows linear and angular momentum to be considered simultaneously.

- Translations are easy because any GKF is fixed by $\{\xi^a, \nabla_{[a}\xi_{b]}\}$.

$$\mathcal{P}_\xi(s) = \left(\xi_a p^a + \frac{1}{2}\nabla_{[a}\xi_{b]} S^{ab}\right)_{\gamma(s)} \qquad \{\mathcal{P}_\xi | \forall\, \xi^a\} \Leftrightarrow \{p_a, S_{ab}\}.$$

- Any real Killing fields that may exist are also GKFs.

- Always satisfy $\mathcal{L}_\xi g_{ab}|_\Gamma = \nabla_a \mathcal{L}_\xi g_{bc}|_\Gamma = 0$. This means that

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{P}_\xi = (\dot{p}^a - \frac{1}{2} S^{bc}\dot{\gamma}^d R_{bcd}{}^a)\xi_a + \frac{1}{2}(\dot{S}^{ab} - 2p^{[a}\dot{\gamma}^{b]})\nabla_{[a}\xi_{b]}$$

- Details can be found in AIH, CQG 25: 205008, 2008 (arXiv: 0805.4259)

Laws of motion

Using stress-energy conservation, the final (exact) result is that

"Self-momentum"

External force

S-field self-force

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{P}_\xi+\mathcal{E}_\xi) = \int_{\Sigma(s)} \left[\frac{1}{2}T^{ab}\mathcal{L}_\xi g_{ab} + \rho\mathcal{L}_\xi(\Phi^{\mathrm{ext}} + \Phi^{\mathrm{R}}) + \frac{1}{2}\int_W \rho\rho'\mathcal{L}_\xi G^{\mathrm{S}}\mathrm{d}V'\right] t^c \mathrm{d}S_c$$

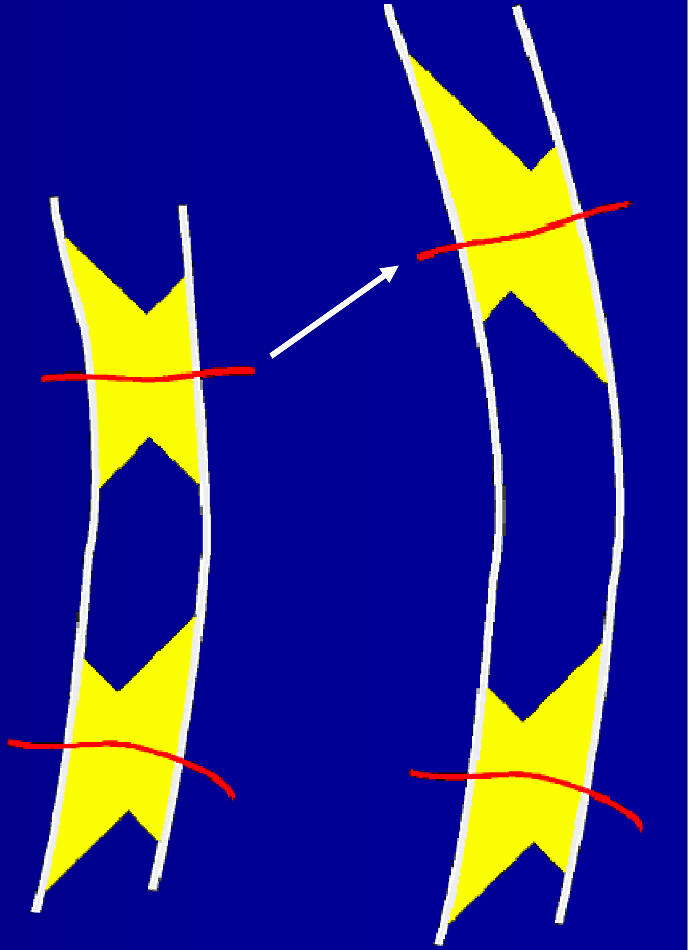
"Bare" momentum

Gravitational force

Typical scalar self-force

Self-momentum

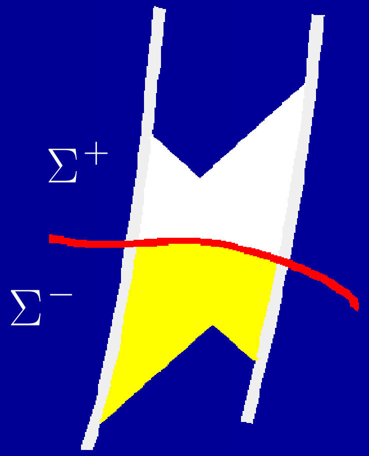
- Look at changes in $\mathcal{P}_\xi$ over finite times. Find that there's a part of the self-force that only depends on regions right next to the bounding hypersurfaces.
- This can be associated directly with these hypersurfaces, and is naturally identified as the linear and angular momenta of the S-type self-field
- This only happens for DW self-fields.
- Probably could not have been identified perturbatively or by considering only instantaneous rates of change.



Why call it a self-momentum?

Explicitly,

$$\mathcal{E}_\xi = \frac{1}{2}\left(\int_{\Sigma^+} J^a \mathcal{L}_\xi A^{\rm S}_a[\Sigma^-]{\rm d}V - \int_{\Sigma^-} J^a \mathcal{L}_\xi A^{\rm S}_a[\Sigma^+]{\rm d}V\right) + \int_\Sigma \xi^b A^{\rm S}_b J^a {\rm d}S_a$$



If everything is stationary,

$$\mathcal{E}_\xi \to \int_\Sigma T^{ab}_{A^{\rm S}} \xi_a {\rm d}S_b$$

This limit could have been guessed, and is sufficient to derive the lowest-order self-force (see Gralla, Harte, and Wald). It is problematic in general. The more complicated expression always works.

Why is this hard?

- Extended bodies are complicated.
- A mass renormalization alone is sufficient only in very special cases.
- The bare and self-momenta are not necessarily parallel in general. This is true even for stationary charge distributions in flat spacetime.
- The "center-of-charge" and center-of-mass needn't coincide. Different mass centers arise when using $\mathcal{P}_\xi$ vs $\mathcal{P}_\xi + \mathcal{E}_\xi$. Only the latter have simple evolution equations.
- All of this is taken into account automatically. There was no fiddling with definitions. Mathematically natural choices really do simplify life.

Other effects of the S-field

$$f_\xi^{\mathrm{S}} = \frac{1}{2}\int_\Sigma t^a \mathrm{d}S_a \int_W \mathrm{d}V'\rho\rho'\mathcal{L}_\xi G^{\mathrm{S}}$$

- Measures the failure of Newton's 3rd law for the S-type field in "direction" ξ^a :

$$\begin{aligned}\rho(x)\mathrm{d}V\xi^a\nabla_a[\rho(x')\mathrm{d}V'G^{\mathrm{S}}(x,x')] + \rho(x')\mathrm{d}V'\xi^{a'}\nabla_{a'}[\rho(x)\mathrm{d}VG^{\mathrm{S}}(x',x)]\\ = \rho(x)\rho(x')\mathcal{L}_\xi G^{\mathrm{S}}(x,x')\mathrm{d}V\mathrm{d}V'\end{aligned}$$

- It exactly vanishes if the GKF is Killing. This true for all forces and torques in Minkowski and de Sitter spacetimes.

- Smaller than the typical ("regular") self-force for sufficiently small, slowly-varying charge distributions.

Generalized Detweiler-Whiting axiom

There exist linear and angular momenta $\hat{\mathcal{P}}_\xi = \mathcal{P}_\xi + \mathcal{E}_\xi$ whose evolution is not affected by the DW singular component of the self-field.

- This is exactly true for all charge distributions in Minkowski and de Sitter spacetimes
- Given any Killing vector K^a, it is also exact for $\hat{p}^a K_a + \frac{1}{2}\hat{S}^{ab}\nabla_{[a}K_{b]}$.
- It's true to leading order for shrinking charge distributions $\rho = \lambda^{-\alpha}\tilde{\rho}(\lambda, t, \mathbf{r}/\lambda)$ in every fixed spacetime.

The ordinary Lorentz-Dirac (or Dewitt-Brehme) term is simply added to test-body equations of motion if the center-of-mass is defined using the renormalized momentum. But test body equations aren't trivial!

What if the singular self-force isn't negligible?

- Terms like this occur even in nonrelativistic physics (with elliptic field equations).

$$F^a_{\mathrm{self}}\xi_a + \frac{1}{2}N^{ab}_{\mathrm{self}}\nabla_{[a}\xi_{b]} = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y\rho(x)\rho(y)\mathcal{L}_\xi G(x,y)$$

- Old arguments demonstrated that the singular self-field renormalizes a particle's mass. More recent ones showed that it renormalizes the full linear and angular momenta.

- It actually shifts *all* multipole moments of the body's stress-energy tensor. These couple to the equations of motion via the background curvature.

$$\dot{\mathcal{P}}_\xi = (\ldots) - \frac{1}{6}(\underset{\substack{\nearrow \\ \text{From matter stress-} \\ \text{energy tensor}}}{J^{abcd}} + \delta \underset{\substack{\nwarrow \\ \text{From singular} \\ \text{self-field}}}{J^{abcd}})\mathcal{L}_\xi R_{abcd} + \ldots$$

- The effective field mass is both inertial and (passively) gravitational.

Calculating the quadrupole shift

- It is tedious to compute Lie derivatives of bitensors directly using brute-force perturbation theory. Things are easier using “transport equations.” In general,

$$\mathcal{L}_\xi \sigma(x,x') = \frac{U}{2} \int_0^U \dot{y}^a \dot{y}^b \mathcal{L}_\xi g_{ab}(y) \mathrm{d}u$$

$$\mathcal{L}_\xi \ln \Delta(x,x') = [\sigma^a{}_{a'}]^{-1} \nabla_{a'} \nabla^a \mathcal{L}_\xi \sigma - \frac{1}{2}(g^{ab} \mathcal{L}_\xi g_{ab} + g^{a'b'} \mathcal{L}_\xi g_{a'b'})$$

- For the GKFs,

$$\mathcal{L}_\xi g_{ab} \sim O(r^2)$$

$$\mathcal{L}_\xi \sigma \sim O(r^4) \qquad \mathcal{L}_\xi \ln \Delta \sim O(r^2) \qquad \mathcal{L}_\xi g^{a'}{}_a \sim O(r^2)$$

- Final shift in the effective quadrupole moment of the stress-energy tensor:

$$\delta J^{abcd} \simeq \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y'\frac{\rho(x)\rho(y)}{|x-y|^3}\Big\{\frac{3}{2}n^{[a}e_i^{b]}n^{[c}e_j^{d]}|x-y|^2(2x^ix^j-\delta^{ij}|x-x'|^2)$$
$$-e_i^{[a}e_j^{b]}e_k^{[c}e_l^{d]}(y^ix^jy^kx^l+\frac{1}{2}\delta^{jl}|x-y|^2[(x-y)^i(x-y)^j-\delta^{ik}|x-y|^2])\Big\}$$

$$\delta J^{abcd} = \delta J^{[ab]cd} = \delta J^{cdab}$$

$$\delta J^{[abc]d} = 0$$

- If the background is Ricci-flat, traces decouple. Then everything follows from a shift in the effective mass quadrupole of

$$\delta Q^{ab} \simeq \frac{1}{2}e_i^a e_j^b \int \mathrm{d}^3x\rho\phi^{\mathrm{S}}(x^ix^j-\frac{1}{3}\delta^{ij}|x|^2)$$

- This is expected from the electromagnetic stress-energy tensor.
- The general (finite-trace) quadrupole moment cannot be derived so simply. A sufficiently large background electromagnetic field could couple to the internal structure in a very peculiar way. Maybe some alternative gravities would do this too.
- Explicit formulae for these corrections could not be derived from “external” derivations of the self-force.
- In the gravitational case, these kinds of effects might modify relations between equations of state and measured quadrupole moments.
- The spirit of the DW axiom is retained...

An example: small, slowly-varying electromagnetic charges

- Choose a 1-parameter family such that $D,\, q,\, m \sim O(\lambda)$.
- Then $q/m,\, q^2/mD \sim O(1)$ and $q^2\ddot{u} \sim \mu\nabla B \sim O(\lambda^2)$.

$$\frac{\mathrm{D_F} S_a}{\mathrm{d}s} = \dot{S}_a \overbrace{+\, \dot{\gamma}_a \ddot{\gamma}^b S_b}^{\text{Thomas precession}} = \underbrace{-\epsilon_{ab}{}^{cd}\dot{\gamma}^b Q^f{}_{[c} F^{\mathrm{ext}}_{d]f}}_{\mu \times B + d \times E} + O(\lambda^3)$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(m \overbrace{- \frac{1}{2} Q^{ab} F^{\mathrm{ext}}_{ab}}^{\mu \cdot B + d \cdot E}\right) = -\frac{1}{2}\frac{\mathrm{D_F} Q^{ab}}{\mathrm{d}s} F^{\mathrm{ext}}_{ab} + O(\lambda^3)$$

$$p_a = m\dot{\gamma}_a + \underbrace{(\overbrace{2Q^c{}_{[a} F^{\mathrm{ext}}_{b]c}}^{\mu \times E + d \times B} \overbrace{- qS_a{}^c F^{\mathrm{ext}}_{bc}/m}^{\ddot{\gamma} \times S})\dot{\gamma}^b}_{\text{“Hidden momentum”}} + O(\lambda^3)$$

$$\dot{p}_a = [q(F^{\rm ext}_{ab} + F^{\rm R}_{ab}) - \frac{1}{2}R_{abcd}S^{cd}]\dot{\gamma}^b - \frac{1}{2}Q^{bc}\nabla_a F^{\rm ext}_{bc} + O(\lambda^3)$$

$$\Downarrow$$

$$\begin{aligned}\frac{\rm D}{{\rm d}s}(m\dot{\gamma}_a) \quad &= \quad [q(F^{\rm ext}_{ab} + F^{\rm R}_{ab}) - \frac{1}{2}R_{abcd}S^{cd})]\dot{\gamma}^b - \frac{1}{2}Q^{bc}\nabla_a F^{\rm ext}_{bc} \\ & \quad -S_{ab}\dddot{\gamma}^b - 2\left[Q^c{}_{[a}F^{\rm ext}_{b]c}\ddot{\gamma}^b + \frac{{\rm D_F}}{{\rm d}s}(Q^c{}_{[a}F^{\rm ext}_{b]c}\dot{\gamma}^b)\right] + O(\lambda^3)\end{aligned}$$

$$\alpha = e^2/\hbar \simeq 1/137$$

$$F^{\rm R}_{ab}\dot{\gamma}^b = q\left[\frac{2}{3}h_{ab}(\dddot{\gamma}^b + \frac{1}{2}R^b{}_c\dot{\gamma}^c) + ({\rm tail})\right]$$

All of this has been reproduced in flat spacetime using a completely independent method (SE Gralla, AIH, and RM Wald arXiv: 0905.2391).

An “electron”

Set $d^a = 0$ and $\mu^a = \frac{gq}{2m} S^a$.

$$\begin{aligned}(m^* - \mu_b B^b)\ddot{\gamma}_a \quad = \quad & q(E_a^{\mathrm{ext}} + E_a^{\mathrm{R}}) - \frac{1}{2} R_{abcd}\dot{\gamma}^b S^{cd} + \overbrace{\frac{1}{2}\epsilon^{bcdf}\dot{\gamma}_b \mu_c D_a F_{df}^{\mathrm{ext}}}^{\nabla(\mu \cdot B)} \\ & + \frac{(g-2)q}{2m^*}\left[\underbrace{\epsilon_{abcd}\dot{\gamma}^b S^c \dot{E}^d}_{\dddot{\gamma} \times S} + \underbrace{2\mu_{[a}B_{b]}E^b}_{\ddot{\gamma} \times (\mu \times B)}\right]\end{aligned}$$

$$\frac{\mathrm{D_F} S_a}{\mathrm{d}s} = \overbrace{\epsilon_{abcd}\dot{\gamma}^b B^c \mu^d}^{\mu \times B}$$

Conclusions and future directions

- Nontrivial linear and angular self-momenta were derived without using any perturbation theory.
- Detweiler-Whiting axiom was derived (not postulated) very easily. Its correction has a simple physical interpretation.
- Higher-order effects seem to have simple physical interpretations (verifying effective field theory viewpoints)
- Notion of classical renormalizibility? Can all potential complications be reduced to terms involving $\mathcal{L}_\xi G(x,x'),\ \mathcal{L}_\xi [G(x,x'')G(x'',x')],\ \ldots ?$
- How much freedom is there in defining GKFs and a CM? What's observationally interesting?
- Do this for GR. There's currently no local derivation of the gravitational self-force. The required types of bitensor perturbation theory might also be useful for other purposes.