Harmonic Gauge Perturbations
of the
Schwarzschild Metric

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Black hole perturbation theory: developed by

Regge-Wheeler (1957) and Zerilli (1970)

Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi \tfrac{G}{c^4} T_{\mu\nu} \qquad \text{(c=G=1)}$$

Schwarzschild metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Perturbed metric (small mass m_0, large mass M)

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} + O((m_0/M)^2)$$

Perturbed field equations

$$-\left[h_{\mu\nu;\alpha}{}^{;\alpha} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} h_{\alpha\beta} - (h_{\mu\alpha}{}^{;\alpha}{}_{;\nu} + h_{\nu\alpha}{}^{;\alpha}{}_{;\mu}) + h_{;\mu;\nu} \right]$$

$$- g_{\mu\nu}(h_{\lambda\alpha}{}^{;\alpha;\lambda} - h_{;\lambda}{}^{;\lambda}) = 16\pi T_{\mu\nu}$$

Stress energy tensor (point mass)

$$T^{\mu\nu} = m_0 \int_{-\infty}^{\infty} \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \, d\tau$$

Solve perturbed field equations by separation of variables: Fourier transform, tensor harmonics

$$h_{\mu\nu}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty} e^{-i\omega t}\left(h_{\mu\nu}^{o,lm}(\omega,r,\theta,\phi) + h_{\mu\nu}^{e,lm}(\omega,r,\theta,\phi)\right) d\omega$$

Odd parity perturbation $(-1)^{l+1}$; three radial factors

$$h_{\mu\nu}^{o,lm}(\omega,r,\theta,\phi) = \begin{pmatrix} 0 & 0 & h_0^{lm}(\omega,r)\csc\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\phi} & -h_0^{lm}(\omega,r)\sin\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta} \\ * & 0 & h_1^{lm}(\omega,r)\csc\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\phi} & -h_1^{lm}(\omega,r)\sin\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta} \\ * & * & -h_2^{lm}(\omega,r)X_{lm}(\theta,\phi) & h_2^{lm}(\omega,r)\sin\theta W_{lm}(\theta,\phi) \\ * & * & * & h_2^{lm}(\omega,r)\sin^2\theta X_{lm}(\theta,\phi) \end{pmatrix}$$

Even parity perturbation $(-1)^{l}$; seven radial factors

$$h_{\mu\nu}^{e,lm}(\omega,r,\theta,\phi) =$$

$$\begin{pmatrix} \left(1-\frac{2M}{r}\right)H_0^{lm}(\omega,r)Y_{lm} & H_1^{lm}(\omega,r)Y_{lm} & h_0^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\theta} & h_0^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\phi} \\ * & \frac{H_2^{lm}(\omega,r)Y_{lm}}{\left(1-\frac{2M}{r}\right)} & h_1^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\theta} & h_1^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\phi} \\ * & * & r^2\left(K^{lm}(\omega,r)Y_{lm} + G^{lm}(\omega,r)W_{lm}\right) & r^2\sin\theta\, G^{lm}(\omega,r)X_{lm} \\ * & * & * & r^2\sin^2\theta\left(K^{lm}(\omega,r)Y_{lm} - G^{lm}(\omega,r)W_{lm}\right) \end{pmatrix}$$

(notation is from Ashby)

$$W_{lm}(\theta,\phi) = \frac{\partial^2 Y_{lm}(\theta,\phi)}{\partial\theta^2} - \cot\theta \frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}(\theta,\phi)}{\partial\phi^2}$$

$$X_{lm}(\theta,\phi) = \frac{2}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta} - \cot\theta\, Y_{lm}(\theta,\phi)\right)$$

Odd and even parity field equations decouple.

To solve perturbed field equations, find three odd parity and seven even radial functions.

Gauge is choice of coordinates

$$x^{\mu}_{\rm new} = x^{\mu}_{\rm old} + \xi^{\mu} \quad \text{leads to}$$

$$h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$$

Field equations are gauge invariant.

Other quantities also separate.

Stress energy tensor

$$T_{\mu\nu}(t,r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty}e^{-i\omega t}\left(T_{\mu\nu}^{o,lm}(\omega,r,\theta,\phi)+T_{\mu\nu}^{e,lm}(\omega,r,\theta,\phi)\right)d\omega$$

Odd parity

$$T_{\mu\nu}^{o,lm}(\omega,r,\theta,\phi)=\begin{pmatrix} 0 & 0 & So_{02}^{lm}(\omega,r)\csc\theta\frac{\partial Y_{lm}}{\partial\phi} & -So_{02}^{lm}(\omega,r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ * & 0 & So_{12}^{lm}(\omega,r)\csc\theta\frac{\partial Y_{lm}}{\partial\phi} & -So_{12}^{lm}(\omega,r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ * & * & -So_{22}^{lm}(\omega,r)X_{lm} & So_{22}^{lm}(\omega,r)\sin\theta W_{lm} \\ * & * & * & So_{22}^{lm}(\omega,r)\sin^2\theta X_{lm} \end{pmatrix}$$

Even parity

$$T_{\mu\nu}^{e,lm}(\omega,r,\theta,\phi)=$$

$$\begin{pmatrix} Se_{00}^{lm}(\omega,r)Y_{lm} & Se_{01}^{lm}(\omega,r)Y_{lm} & Se_{02}^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\theta} & Se_{02}^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\phi} \\ * & Se_{11}^{lm}(\omega,r)Y_{lm} & Se_{12}^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\theta} & Se_{12}^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\phi} \\ * & * & \begin{matrix}Ue_{22}^{lm}(\omega,r)Y_{lm}\\ +Se_{22}^{lm}(\omega,r)W_{lm}\end{matrix} & Se_{22}^{lm}(\omega,r)\sin\theta X_{lm} \\ * & * & * & \sin^2\theta\left(\begin{matrix}Ue_{22}^{lm}(\omega,r)Y_{lm}\\ -Se_{22}^{lm}(\omega,r)W_{lm}\end{matrix}\right) \end{pmatrix}$$

Regge-Wheeler gauge: ten radial factors, set four equal to zero (1 odd parity, 3 even parity). Solutions can be written in terms of two scalar functions that satisfy second order differential equations. Solved by Regge, Wheeler and Zerilli. Odd parity: Regge-Wheeler equation - a gravitational wave interacting with a potential

$$\frac{d^2\psi_2}{dr_*^2}+\omega^2\psi_2-\left(1-\frac{2M}{r}\right)\left(\frac{2(\lambda+1)}{r^2}-\frac{6M}{r^3}\right)\psi_2=S_2$$

$$S_2=-\frac{16\pi}{r}\left(1-\frac{2M}{r}\right)^2 So_{12}+\frac{32\pi(6M^2-5Mr+r^2)}{r^4}So_{22}-\frac{16\pi}{r}\left(1-\frac{2M}{r}\right)^2 So_{22}'$$

$$2(\lambda+1)=l(l+1)$$

"tortoise coordinate"

$$dr_*=\frac{dr}{\left(1-\frac{2M}{r}\right)}\ ,\ r_*=r+2M\ln\left(\frac{r}{2M}-1\right)$$

$$(2\ M < r < \infty, -\infty < r_* < \infty)$$

$$h_0^{\mathrm{RW}}=\frac{(2M-r)}{i\omega}\psi_2'-\left(1-\frac{2M}{r}\right)\frac{\psi_2}{i\omega}-\left(1-\frac{2M}{r}\right)\frac{16\pi}{i\omega}So_{22}$$

$$h_1^{\mathrm{RW}} = \frac{r^2}{r-2M}\psi_2 \qquad\qquad h_2^{\mathrm{RW}} = 0.$$

Regge-Wheeler function in terms of radial functions:

$$\psi_2 = \left(1 - \frac{2M}{r}\right)\left(\frac{h_1}{r} - \frac{2h_2}{r^2} + \frac{h_2'}{r}\right)$$

Above expression is gauge invariant. It applies in any gauge (Moncrief, 1974).

Even parity: solved in terms of Zerilli function

$$\frac{d^2\psi_2}{dr_*^2} + \omega^2\psi_2 + \frac{2(2M-r)\left(9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2(1+\lambda)r^3\right)}{r^4(3M+\lambda r)^2}\psi_2 = S_2$$

$$\begin{aligned} S_2 = -\frac{8\pi r^2}{3M+\lambda r}Se_{00} - \frac{16\lambda\pi(-2M+r)^2}{i\omega(3M+\lambda r)^2}Se_{01} + \frac{16\pi(-2M+r)^2}{r(3M+\lambda r)}Se_{12} \\ + \frac{32M\pi(2M-r)(3M-(3+\lambda)r)}{i\omega r^2(3M+\lambda r)^2}Se_{02} + \frac{8\pi(-2M+r)^2}{3M+\lambda r}Se_{11} \\ + \frac{32\pi(2M-r)}{r^2}Se_{22} + \frac{16\pi(-2M+r)^2}{i\omega r(3M+\lambda r)}Se_{02}' \end{aligned}$$

Is also gauge invariant (Moncrief). Related to Regge-Wheeler function by differential operators (Chandrasekhar).

In Regge-Wheeler gauge, solutions in terms of two quantities, the Regge-Wheeler and Zerilli functions.

Harmonic gauge (also Lorentz, Lorenz gauge)

Trace-reversed metric:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h \qquad h = g^{\alpha\beta} h_{\alpha\beta}$$

Harmonic gauge condition:

$$\bar{h}_{\mu\nu}{}^{;\nu} = 0$$

Harmonic gauge field equations:

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} \bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$$

Gauge changes which preserve harmonic gauge:

$$\xi_{\mu;\nu}{}^{;\nu} = 0$$

Unlike Regge-Wheeler gauge, none of the radial pertubation functions are zero.

Generalized Regge-Wheeler equation (Leaver,1986)

$$\frac{d^2\psi_s(r_*)}{dr_*^2} + \omega^2 \psi_s(r_*) - V_{sl}(r)\psi_s(r_*) = S_{slm}(\omega, r_*)\ , \ \ s = 0,1,2$$

effective potential $V_{sl}(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2(\lambda+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right)$

Abbreviate as $\mathscr{L}_s \psi_s = S_s$

Regge-Wheeler function is s=2

Zerilli also s=2, as related by differential operators.

Cases s=0,1 also appear in harmonic gauge

Odd parity harmonic gauge solutions:

Field equations from separation of variables

$$\left(1-\frac{2M}{r}\right)^2 h_0'' + \frac{(-8M^2+4(2+\lambda)Mr - r^2(2+2\lambda+(i\omega)^2r^2))}{r^4} h_0 + \frac{2i\omega M(2M-r)}{r^3} h_1 = -16\pi\left(1-\frac{2M}{r}\right) So_{02}$$

$$\left(1-\frac{2M}{r}\right)^2 h_1'' + \frac{4M}{r^2}\left(1-\frac{2M}{r}\right) h_1' + \frac{2i\omega M}{2Mr-r^2} h_0 + \frac{\lambda(8M-4r)}{r^4} h_2 + \frac{-16M^2+4(5+\lambda)Mr - r^2(6+2\lambda+(i\omega)^2r^2)}{r^4} h_1 = -16\pi\left(1-\frac{2M}{r}\right) So_{12}$$

$$\left(1-\frac{2M}{r}\right)^2 h_2'' - \frac{2(2M-r)(3M-r)}{r^3} h_2' - \frac{2(r-2M)^2}{r^3} h_1 + \frac{16M^2+4(-3+\lambda)Mr - r^2(-2+2\lambda+(i\omega)^2r^2)}{r^4} h_2 = -16\pi\left(1-\frac{2M}{r}\right) So_{22}$$

Harmonic gauge condition $\overline{h}_{\mu\nu}{}^{;\nu} = 0$

$$\left(1-\frac{2M}{r}\right) h_1' - \frac{i\omega r}{2M-r} h_0 - \frac{2(M-r)}{r^2} h_1 + \frac{2\lambda}{r^2} h_2 = 0$$

Gauge changes which preserve harmonic gauge:

$$\xi_{\mu;\nu}{}^{;\nu} = 0 \qquad \frac{d^2 Z}{dr_*^2} + \omega^2 Z - \left(1 - \frac{2M}{r}\right)\frac{2(\lambda+1)}{r^2} Z = 0$$

$$h_0^{\text{new}} = h_0^{\text{old}} + i\omega Z \quad h_1^{\text{new}} = h_1^{\text{old}} + \frac{2}{r} Z - Z \quad h_2^{\text{new}} = h_2^{\text{old}} + Z$$

This is generalized Regge-Wheeler equation for s=1.

Odd parity solutions:

$$h_0 = \frac{1}{i\omega}\left(\psi_1 + \frac{2\lambda}{3}\psi_2\right)$$

$$h_1 = \frac{1}{(i\omega)^2}\left[-\frac{2\lambda}{3}\psi_2' + \frac{2}{r}\psi_1 - \frac{2\lambda}{3r}\psi_2 + 16\pi\left(1 - \frac{2M}{r}\right) So_{12} - \psi_1'\right]$$

$$h_2 = \frac{1}{(i\omega)^2}\left[(r-2M)\psi_2' + \psi_1 + \frac{-6M + (3+2\lambda)r}{3r}\psi_2 + 16\pi\left(1 - \frac{2M}{r}\right) So_{22}\right]$$

$$\mathscr{L}_1\psi_1 = \frac{32(3+\lambda)\pi(r-2M)^2}{3r^3} So_{12} + \frac{32\lambda\pi(r-2M)}{3r^3} So_{22} + \frac{16\pi(r-2M)^3}{r^3} So_{12}' + \frac{32\lambda\pi(r-2M)^2}{3r^3} So_{22}' .$$

$$\psi_1 = i\omega h_0 + \left(1 - \frac{2M}{r}\right)\frac{2\lambda}{3r}\left(-h_1 + \frac{2}{r}h_2 - h_2'\right)$$

Odd parity harmonic gauge solutions are in terms of two generalized Regge-Wheeler functions with s=1 and s=2. The s=1 function is not gauge invariant.

Even parity solutions:

Seven radial functions, with seven field equations

$\overline{h}_{\mu\nu}{}^{;\nu} = 0$ gives three equations

$\xi_{\mu;\nu}{}^{;\nu} = 0$ gives three equations

Solutions are in terms of Zerilli function (spin 2) and three generalized Regge-Wheeler functions (one with s=1 and two with s=0).

Only Zerilli function is gauge invariant.

Gauge changes which preserve the harmonic gauge are implemented by adding homogeneous spin 1 and spin 0 solutions.

$$
\begin{aligned}
h_0 = &\frac{\lambda(2M-r)\psi_2'}{3i\omega r} + \frac{(-2M+r)\psi_0}{2i\omega r^2} + \frac{2i\omega\left(f_0\psi_0 + f_{d0}\psi_0'\right)}{r} + \frac{4(1+\lambda)\psi_1}{r} \\
&- \frac{\lambda\left(6M^2 + 3\lambda M r + \lambda(1+\lambda)r^2\right)\psi_2}{3i\omega r^2(3M+\lambda r)} - \frac{16\pi(2M-r)(2M+\lambda r)Se_{02}}{(i\omega)^2 r(3M+\lambda r)} \\
&- \frac{8\pi\left(16M^2 + (-5+6\lambda)Mr - 2\lambda r^2\right)Se_{01}}{(i\omega)^2(3M+\lambda r)} + \frac{16\pi r(-2M+r)Se_{11}}{i\omega} \\
&- \frac{(-2M+r)\psi_0'}{2i\omega r} + \left(1 - \frac{2M}{r}\right)\psi_1' + \frac{2i\omega\psi_{0a}}{r}
\end{aligned}
$$

Special case: l=0 can solve analytically for circular orbits of radius R

My solution for r>R: (1/r relative to background metric)

$$h_{tt} = \frac{2m_0}{r}\tilde{E}\frac{R-3M}{R-2M} + O\left(r^{-2}\right)\ ,\ h_{rr} = \frac{2m_0}{r}\tilde{E}\frac{R-3M}{R-2M} + O\left(r^{-2}\right)\ ,$$

$$h_{\theta\theta} = \frac{h_{\phi\phi}}{\sin^2\theta} = 2m_0 r\tilde{E}\frac{R-3M}{R-2M} + O(1)$$

Detweiler & Poisson (2004) solution for r>R:

$$h^{\mathrm{DP}}_{tt} = \frac{2m_0\tilde{E}}{2M-R} - \frac{2m_0\tilde{E}R}{r(2M-R)} + O\left(r^{-2}\right)\ ,\ h^{\mathrm{DP}}_{rr} = \frac{2m_0\tilde{E}}{r} + O\left(r^{-2}\right)\ ,$$

$$h^{\mathrm{DP}}_{\theta\theta} = \frac{h^{\mathrm{DP}}_{\phi\phi}}{\sin^2\theta} = 2m_0\tilde{E}r\frac{R-3M}{R-2M} + O(1)$$

Both bounded near event horizon in Eddington coordinates. Solutions differ by homogeneous solutions.

Summary: Harmonic gauge solutions in terms of 6 functions which satisfy decoupled differential equations.

Odd parity: s=2 and s=1

Even parity: s=2, s=1 and two s=0

Why?

1. Solutions need to reflect $\xi_{\mu;\nu}{}^{;\nu} = 0$
2. Form of harmonic gauge field equations vs. generalized Regge-Wheeler equation

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$$

$$\frac{d^2\psi_s(r_*)}{dr_*^2} + \omega^2\psi_s(r_*) - V_{sl}(r)\psi_s(r_*) = S_{slm}(\omega, r_*)$$

Radiation: waveforms, energy flux

Outgoing radiation gauge (Chrzanowski, 1975)

$$h_{\mu\nu} n^{\nu} = 0\,,\ h = g^{\mu\nu} h_{\mu\nu} = 0 \qquad n^{\mu} = \frac{1}{2}\left(1, -1 + \frac{2M}{r}, 0, 0\right)$$

Leads to transverse-traceless gauge (MTW)

Two polarizations h_+ and $h_\times$

$$h_+ = h_{\hat\theta\hat\theta} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \int_{-\infty}^{\infty} e^{-i\omega t}\, h_{\hat\theta\hat\theta}^{lm}(\omega, r, \theta, \phi)\, d\omega$$

odd $$h_{\hat\theta\hat\theta}^{lm}(\omega, r, \theta, \phi) = -\frac{e^{i\omega r_*}}{i\omega r} A_{2lm\omega}^{\infty} X_{lm}(\theta, \phi) + O(r^{-2})$$

even $$h_{\hat\theta\hat\theta}^{lm}(\omega, r, \theta, \phi) = \frac{e^{i\omega r_*}}{2r} A_{2lm\omega}^{\infty} W_{lm}(\theta, \phi) + O(r^{-2})$$

$$h_\times = h_{\hat\theta\hat\phi} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \int_{-\infty}^{\infty} e^{-i\omega t}\, h_{\hat\theta\hat\phi}^{lm}(\omega, r, \theta, \phi)\, d\omega$$

odd $$h_{\hat\theta\hat\phi}^{lm}(\omega, r, \theta, \phi) = \frac{e^{i\omega r_*}}{i\omega r} A_{2lm\omega}^{\infty} W_{lm}(\theta, \phi) + O(r^{-2})$$

even $$h_{\hat\theta\hat\phi}^{lm}(\omega, r, \theta, \phi) = \frac{e^{i\omega r_*}}{2r} A_{2lm\omega}^{\infty} X_{lm}(\theta, \phi) + O(r^{-2})$$

Teukolsky (1973)

$$\Psi_4 = -\left(\delta R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} - i\,\delta R_{\hat{t}\hat{\theta}\hat{t}\hat{\phi}}\right) = -\frac{\omega^2}{2}\left(h_{\hat{\theta}\hat{\theta}} - i\,h_{\hat{\theta}\hat{\phi}}\right)$$

Energy/angular momentum flux: Isaacson (1967) for large r

$$T^{(\mathrm{GW})}_{\mu\nu} = \frac{1}{32\pi}\left\langle \overline{h}_{\alpha\beta;\mu}\overline{h}^{\alpha\beta}{}_{;\nu} - \frac{1}{2}\overline{h}_{;\mu}\overline{h}_{;\nu} - 2\overline{h}^{\alpha\beta}{}_{;\beta}\overline{h}_{\alpha(\mu;\nu)}\right\rangle$$

$$\dot{E}^{\infty} = \left\langle \frac{dE^{\infty}}{dt}\right\rangle = \frac{1}{16\pi}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{k=-\infty}^{\infty} f_{lmk}|A^{\infty}_{2lm\omega}|^2 l(l+1)(l-1)(l+2)$$

$$\dot{L}_z^{\infty} = \left\langle \frac{dL_z^{\infty}}{dt}\right\rangle = \frac{1}{16\pi}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{k=-\infty}^{\infty} \frac{m}{\omega_{mk}} f_{lmk}|A^{\infty}_{2lm\omega}|^2 l(l+1)(l-1)(l+2)$$

$$f_{lmk} = \begin{cases} 1\text{ , odd parity modes ,} \\ \left(\frac{\omega_{mk}}{2}\right)^2\text{ , even parity modes} \end{cases}$$

For circles, k=0 and $\omega_{mk} = m\Omega_\phi$.

$$\dot{E}^{2M} = \left\langle \frac{dE^{2M}}{dt}\right\rangle = \frac{1}{16\pi}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{k=-\infty}^{\infty} f_{lmk}|A^{2M}_{2lm\omega}|^2 l(l+1)(l-1)(l+2)$$

$$\dot{L}_z^{2M} = \left\langle \frac{dL_z^{2M}}{dt}\right\rangle = \frac{1}{16\pi}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{k=-\infty}^{\infty} \frac{m}{\omega_{mk}} f_{lmk}|A^{2M}_{2lm\omega}|^2 l(l+1)(l-1)(l+2)$$

Gravitational self-force: gives first order perturbative corrections to equations of motion

Test mass follows geodesic of background metric

$$\frac{d^2 z^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau} = 0$$

Two constants of motion, energy and orbital angular momentum. Circular orbit of radius R:

$$\widetilde{E} = \frac{E}{m_0} = \frac{1-\frac{2M}{R}}{\sqrt{1-\frac{3M}{R}}}\,,\ \widetilde{L} = \frac{L_z}{m_0} = \sqrt{\frac{MR}{1-\frac{3M}{R}}}$$

$$\Omega_\phi = \sqrt{\frac{M}{R^3}} \qquad (\omega = m\,\Omega_\phi)$$

Self-force gives perturbative corrections:

$$m_0\left(\frac{d^2 z^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\right) = F^\mu_{\rm self}$$

$$F^\mu_{\rm self} = -m_0\left(\delta^\mu_\gamma + \frac{dz^\mu}{d\tau}\frac{dz_\gamma}{d\tau}\right)\delta\Gamma^\gamma{}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}$$

$$\delta\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2}g^{\gamma\epsilon}\left(h^{\rm reg}_{\epsilon\alpha;\beta} + h^{\rm reg}_{\epsilon\beta;\alpha} - h^{\rm reg}_{\alpha\beta;\epsilon}\right)$$

(Mino, Sasaki & Tanaka and Quinn & Wald (1997))

Problem: perturbation diverges at location of orbiting mass (delta function source). Regularize force by mode-sum regularization.

Mode-sum regularization: Decompose bare force in spherical harmonics. Each l-mode is finite, but sum over l diverges (Barack *et al*., 2002). Subtract singular part for each l-mode.

Regularization is gauge dependent and derived in harmonic gauge.

Self-force has two parts: Dissipative and Conservative (Barack & Sago, 2007)

Dissipative, or radiation reaction. Gravitational waves carry away energy and angular momentum, so these are no longer constants of the motion.

Conservative. Both masses moving around center of mass. Newtonian frequency shift:

$$\Omega^2 = \frac{M + m_0}{s^3} \quad \text{or} \quad \Omega^2 = \frac{M - 2m_0}{R^3} + O\left(R^{-4}\right)$$

$$\Omega = \Omega_0 \left[1 - \left(\frac{R(R-3M)}{2Mm_0}\right) F_r\right], \; \Omega_0 = \sqrt{\frac{M}{R^3}} = \Omega_\phi$$

For circular orbits, dissipative due to F^t, F^ϕ and conservative due to F^r. Only F^r needs to be regularized.

Radial component of self-force for circular orbits of radius R

R/M	$(M/m_0)^2 F^r$	R/M	$(M/m_0)^2 F^r$
6	4.9685669×10^{-2}	110	1.6237973×10^{-4}
7	3.5624667×10^{-2}	120	1.3664172×10^{-4}
8	2.7112763×10^{-2}	130	1.1657168×10^{-4}
9	2.1452689×10^{-2}	140	1.0061965×10^{-4}
10	1.7454613×10^{-2}	150	8.7731466×10^{-5}
11	1.4507231×10^{-2}	200	4.9508804×10^{-5}
12	1.2263358×10^{-2}	300	2.2075818×10^{-5}
13	1.0511248×10^{-2}	400	1.2438053×10^{-5}
20	4.5872951×10^{-3}	500	7.9682266×10^{-6}
30	2.0912401×10^{-3}	600	5.5371464×10^{-6}
40	1.1929325×10^{-3}	700	4.0700299×10^{-6}
50	7.7022778×10^{-4}	800	3.1172221×10^{-6}
60	5.3811284×10^{-4}	900	2.4636705×10^{-6}
70	3.9708290×10^{-4}	1000	1.9960142×10^{-6}
80	3.0502873×10^{-4}	10000	1.9996001×10^{-8}
90	2.4163987×10^{-4}	100000	$1.9999600 \times 10^{-10}$
100	1.9614005×10^{-4}	1000000	$1.9999960 \times 10^{-12}$

For large R, $F^r \sim \frac{2m_0^2}{R^2}\left(1 - \frac{2M}{R}\right)$

Comparison to results of Barack and Sago (2007), which is a time domain calculation using numerical methods of Barack and Lousto (2005)

R/M	$(M/m_0)^2F^r$ (a)	$(M/m_0)^2F^r$ (b)	Error (c)	$(M/m_0)^2F^r$ (d)
6	4.9685669×10^{-2}	2.44661×10^{-2}	9×10^{-4}	2.4466497×10^{-2}
7	3.5624667×10^{-2}	2.14989×10^{-2}	6×10^{-4}	2.1499068×10^{-2}
8	2.7112763×10^{-2}	1.83577×10^{-2}	5×10^{-4}	1.8357824×10^{-2}
9	2.1452689×10^{-2}	1.56369×10^{-2}	4×10^{-4}	1.5637098×10^{-2}
10	1.7454613×10^{-2}	1.33895×10^{-2}	8×10^{-5}	1.3389470×10^{-2}
11	1.4507231×10^{-2}	1.15518×10^{-2}	6×10^{-5}	1.1551745×10^{-2}
12	1.2263358×10^{-2}	1.00463×10^{-2}	5×10^{-5}	1.0046239×10^{-2}
13	1.0511248×10^{-2}	8.80489×10^{-3}	4×10^{-5}	8.804886×10^{-3}
20	4.5872951×10^{-3}	4.15706×10^{-3}	1×10^{-5}	4.1570550×10^{-3}
30	2.0912401×10^{-3}	1.96982×10^{-3}	5×10^{-6}	1.9698169×10^{-3}
40	1.1929325×10^{-3}	1.14288×10^{-3}	2×10^{-6}	1.1428832×10^{-3}
50	7.7022778×10^{-4}	7.44949×10^{-4}	1×10^{-6}	7.4494860×10^{-4}
60	5.3811284×10^{-4}	5.23613×10^{-4}	2×10^{-5}	5.2361368×10^{-4}
70	3.9708290×10^{-4}	3.88010×10^{-4}	1×10^{-5}	3.8800965×10^{-4}
80	3.0502873×10^{-4}	2.98979×10^{-4}	8×10^{-6}	2.9897883×10^{-4}
90	2.4163987×10^{-4}	2.37406×10^{-4}	7×10^{-6}	2.3740623×10^{-4}
100	1.9614005×10^{-4}	1.93063×10^{-4}	5×10^{-6}	1.9306263×10^{-4}
120	1.3664172×10^{-4}	1.34868×10^{-4}	4×10^{-6}	1.3486847×10^{-4}
150	8.7731466×10^{-5}	8.68274×10^{-5}	2×10^{-6}	8.6827447×10^{-5}

(a) mine

(b) Barack/Sago (using Detweiler/Poisson for l=0)

(c) their error estimate

(d) my figures, adjusted to Detweiler/Poisson for l=0

$$F^r_{(a)} - m_0^2 \widetilde{E} \frac{3M(R-2M)\left(4M^2+2MR+R^2\right)}{(R-3M)R^5} = F^r_{(d)}$$

Comparison of energy flux and self-force energy loss

$$\dot{E}^{\rm sf} = \frac{dE^{\rm sf}}{dt} = \left(1 - \frac{2M}{r}\right)^2 \frac{F^t}{\widetilde{E}}$$

$$\dot{E}^{\infty}, \dot{E}^{2M} \propto h^2, \dot{E}^{\rm sf} \propto h$$

R/M	$(M/m_0)^2 \dot{E}^{\infty}$	$(M/m_0)^2 \dot{E}^{2M}$	$(M/m_0)^2 \dot{E}$	$(M/m_0)^2 \dot{E}^{\rm sf}$
6	$9.37270411 \times 10^{-4}$	$3.06894559 \times 10^{-6}$	$9.40339356 \times 10^{-4}$	$-9.40339356 \times 10^{-4}$
7	$3.99633989 \times 10^{-4}$	$5.29300869 \times 10^{-7}$	$4.00163290 \times 10^{-4}$	$-4.00163290 \times 10^{-4}$
8	$1.95979479 \times 10^{-4}$	$1.25069497 \times 10^{-7}$	$1.96104549 \times 10^{-4}$	$-1.96104549 \times 10^{-4}$
9	$1.05896576 \times 10^{-4}$	$3.66762344 \times 10^{-8}$	$1.05933252 \times 10^{-4}$	$-1.05933252 \times 10^{-4}$
10	$6.15037255 \times 10^{-5}$	$1.25912943 \times 10^{-8}$	$6.15163168 \times 10^{-5}$	$-6.15163168 \times 10^{-5}$
11	$3.77867502 \times 10^{-5}$	$4.87560894 \times 10^{-9}$	$3.77916258 \times 10^{-5}$	$-3.77916258 \times 10^{-5}$
12	$2.42896246 \times 10^{-5}$	$2.07631371 \times 10^{-9}$	$2.42917009 \times 10^{-5}$	$-2.42917009 \times 10^{-5}$
13	$1.62065198 \times 10^{-5}$	$9.55161446 \times 10^{-10}$	$1.62074749 \times 10^{-5}$	$-1.62074749 \times 10^{-5}$
20	$1.87145474 \times 10^{-6}$	$1.61665964 \times 10^{-11}$	$1.87147091 \times 10^{-6}$	$-1.87147091 \times 10^{-6}$
30	$2.48647170 \times 10^{-7}$	$3.80318286 \times 10^{-13}$	$2.48647550 \times 10^{-7}$	$-2.48647550 \times 10^{-7}$
40	$5.95015183 \times 10^{-8}$	$2.73219859 \times 10^{-14}$	$5.95015456 \times 10^{-8}$	$-5.95015456 \times 10^{-8}$
50	$1.96245750 \times 10^{-8}$	$3.57741633 \times 10^{-15}$	$1.96245786 \times 10^{-8}$	$-1.96245786 \times 10^{-8}$
60	$7.92644417 \times 10^{-9}$	$6.82440618 \times 10^{-16}$	$7.92644485 \times 10^{-9}$	$-7.92644485 \times 10^{-9}$
70	$3.68188111 \times 10^{-9}$	$1.68566659 \times 10^{-16}$	$3.68188127 \times 10^{-9}$	$-3.68188127 \times 10^{-9}$
80	$1.89453586 \times 10^{-9}$	$5.02733130 \times 10^{-17}$	$1.89453591 \times 10^{-9}$	$-1.89453591 \times 10^{-9}$
90	$1.05411228 \times 10^{-9}$	$1.73092826 \times 10^{-17}$	$1.05411230 \times 10^{-9}$	$-1.05411230 \times 10^{-9}$
100	$6.23820341 \times 10^{-10}$	$6.67326986 \times 10^{-18}$	$6.23820347 \times 10^{-10}$	$-6.23820347 \times 10^{-10}$
120	$2.51576768 \times 10^{-10}$	$1.28399905 \times 10^{-18}$	$2.51576769 \times 10^{-10}$	$-2.51576769 \times 10^{-10}$
150	$8.27445791 \times 10^{-11}$	$1.71112004 \times 10^{-19}$	$8.27445793 \times 10^{-11}$	$-8.27445793 \times 10^{-11}$

Comparison to Fujita and Tagoshi (2007) calculations for circular orbit (R=10 M)

l	m	$(M/m_0)^2\dot{E}^\infty$ Thesis	$(M/m_0)^2\dot{E}^\infty$ Fujita and Tagoshi
2	1	$1.93160935115669\times10^{-7}$	$1.93160935115669\times10^{-7}$
2	2	$5.36879547910210\times10^{-5}$	$5.36879547910214\times10^{-5}$
3	1	$5.71489891261480\times10^{-10}$	$5.71489891261478\times10^{-10}$
3	2	$4.79591646159026\times10^{-8}$	$4.79591646159025\times10^{-8}$
3	3	$6.42608275624719\times10^{-6}$	$6.42608275624724\times10^{-6}$
4	1	$1.45758564229714\times10^{-13}$	$1.45758564229713\times10^{-13}$
4	2	$5.26224530895924\times10^{-10}$	$5.26224530895930\times10^{-10}$
4	3	$8.77875752521502\times10^{-9}$	$8.77875752521507\times10^{-9}$
4	4	$9.53960039485201\times10^{-7}$	$9.53960039485188\times10^{-7}$
5	1	$2.36763718744954\times10^{-16}$	$2.36763718744955\times10^{-16}$
5	2	$3.81935323719895\times10^{-13}$	$3.81935323719893\times10^{-13}$
5	3	$1.82910132522830\times10^{-10}$	$1.82910132522831\times10^{-10}$
5	4	$1.49211627485282\times10^{-9}$	$1.49211627485280\times10^{-9}$
5	5	$1.52415476457987\times10^{-7}$	$1.52415476457990\times10^{-7}$
6	1	$3.59779535991180\times10^{-20}$	$3.59779535991173\times10^{-20}$
6	2	$1.97636895352003\times10^{-15}$	$1.97636895352005\times10^{-15}$
6	3	$2.12388274763689\times10^{-13}$	$2.12388274763686\times10^{-13}$
6	4	$4.66333988474111\times10^{-11}$	$4.66333988474121\times10^{-11}$
6	5	$2.47463869472717\times10^{-10}$	$2.47463869472724\times10^{-10}$
6	6	$2.51821315681017\times10^{-8}$	$2.51821315681016\times10^{-8}$
7	1	$3.29136294915892\times10^{-23}$	$3.29136294915887\times10^{-23}$
7	2	$9.08415089084877\times10^{-19}$	$9.08415089084875\times10^{-19}$
7	3	$2.03736275096858\times10^{-15}$	$2.03736275096860\times10^{-15}$
7	4	$6.99409365020717\times10^{-14}$	$6.99409365020741\times10^{-14}$
7	5	$1.03409891279350\times10^{-11}$	$1.03409891279349\times10^{-11}$
7	6	$4.06799480917117\times10^{-11}$	$4.06799480917109\times10^{-11}$
7	7	$4.23452267128467\times10^{-9}$	$4.23452267128478\times10^{-9}$

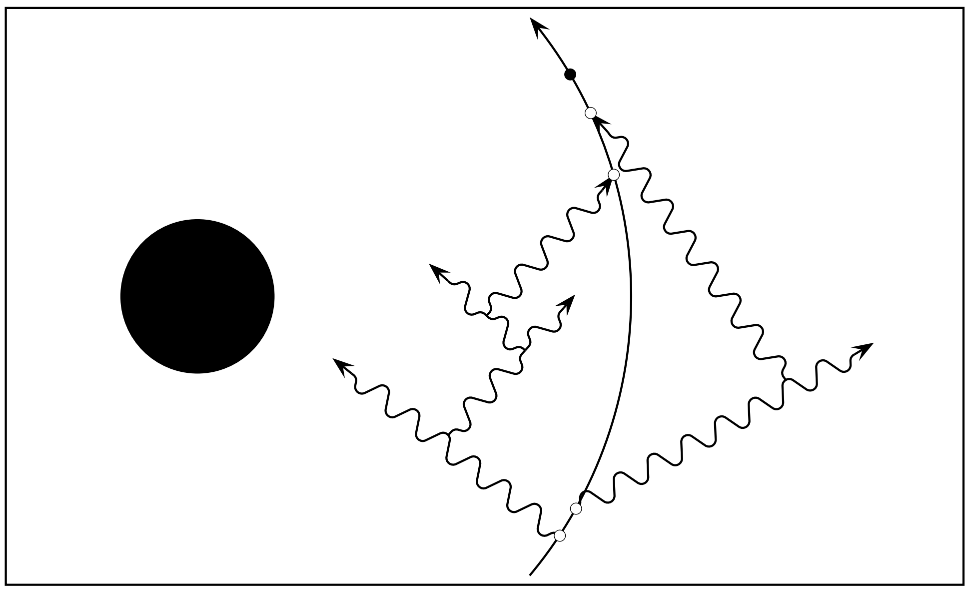

Figure 4.1: Schematic of self-force for circular orbit. A small mass m_0, which is represented by the small solid circle, orbits a much larger black hole. Hollow circles represent previous positions of m_0. The wavy lines represent four-dimensional gravitational waves which scatter off the background spacetime curvature. Part radiates to infinity or into the central mass, and part returns to m_0, giving rise to the tail term of the self-force.