Black holes in tidal environments:

The nonlinear regime

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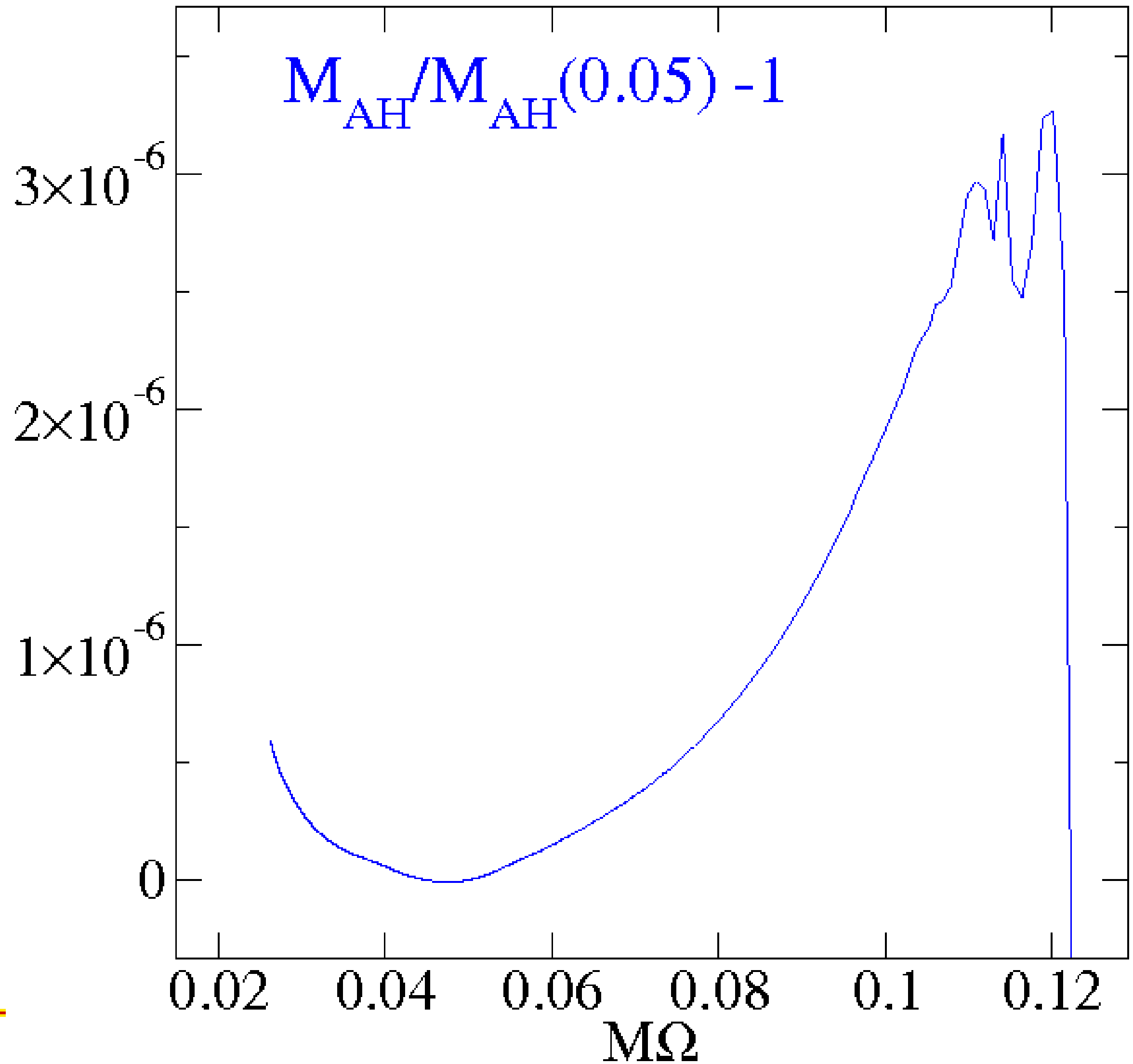
Outline

- Newtonian problem
- Black-hole problem
- Dipole and acceleration
- Deformed horizon
- Conclusions

Tidal heating of a black hole

There are hints that tidal heating is just about to be measured in numerical simulations of black-hole inspirals.



Courtesy of
Harald Pfeiffer

Separation of scales

The problem involves two length scales:

- the black-hole mass M
- the tidal radius (local radius of curvature) $\mathcal{R}$

We assume that the **black hole is well isolated**, so that

$$M/\mathcal{R} \ll 1$$

We work in the **black-hole zone**, so that

$$r \ll \mathcal{R}$$

Newtonian problem (1)

Step 1: Introduce the formalism

A spherical body of mass M is placed within a tidal environment described by a Newtonian potential $U_{\rm tidal}$.

When $r \ll \mathcal{R}$ the tidal potential can be expressed as a Taylor expansion,

$$U_{\rm tidal} = -\frac{1}{2}\mathcal{E}_{ab}(t)x^a x^b - \frac{1}{6}\mathcal{E}_{abc}(t)x^a x^b x^c + \cdots$$

$$\boxed{\mathcal{E}_{ab} \sim \frac{1}{\mathcal{R}^2} \qquad \mathcal{E}_{abc} \sim \frac{1}{\mathcal{R}^3}}$$

The **tidal moments** $\mathcal{E}_L(t)$ are symmetric-tracefree (STF) tensors; they completely characterize the tidal environment.

Newtonian problem (2)

At first order in the perturbation, the body acquires a quadrupole deformation described by

$$\begin{aligned} Q^{ab} &= \int \rho(x^a x^b - \tfrac{1}{3}\delta_{ab} r^2)\, d^3x = \int \rho x^{\langle a} x^{b\rangle}\, d^3x \\ &= \text{mass quadrupole moment (STF tensor)} \end{aligned}$$

The total potential outside the body is

$$U = \frac{M}{r} + \frac{3}{2} Q_{ab} \frac{\Omega^a \Omega^b}{r^3} - \frac{1}{2} r^2 \underbrace{\mathcal{E}_{ab} \Omega^a \Omega^b}_{\sum_m \mathcal{E}_m Y^{2,m}(\theta,\phi)} + \cdots$$

where $\Omega^a = x^a / r$.

Newtonian problem (3)

Step 2: Model the interior

A computation of Q_{ab}, for any $\mathcal{E}_{ab}$, requires a detailed modeling of the body's interior
(incompressible fluid, degenerate fermi gas, polytrope, . . .).

In general,

$$\boxed{Q_{ab} = -k_2\, R^5 \mathcal{E}_{ab}}$$

$$\begin{aligned} k_2 &= \text{dimensionless constant (Love number)} \\ R &= \text{unperturbed body radius} \end{aligned}$$

The details of the interior determine k_2.

For an incompressible fluid, $k_2 = \frac{1}{2}$.

Newtonian problem (4)

Step 3: Extend to second order

At second order in the perturbation, the body acquires monopole, quadrupole, and hexadecapole deformations (composition of spherical harmonics) described by

$$
\begin{aligned}
Q &= M - k_0' \frac{R^6}{M} \mathcal{E}_{ab} \mathcal{E}^{ab} \\
Q_{ab} &= -k_2 R^5 \mathcal{E}_{ab} + k_2' \frac{R^8}{M} \mathcal{E}_{p\langle a} \mathcal{E}^{p}{}_{b\rangle} \\
Q_{abcd} &= -k_4' \frac{R^{10}}{M} \mathcal{E}_{\langle ab} \mathcal{E}_{cd\rangle}
\end{aligned}
$$

For an incompressible fluid, $k_0' = \frac{5}{16}$, $k_2' = \frac{25}{14}$, $k_4' = \frac{15}{28}$.

Black-hole problem (1)

Step 1: Introduce the formalism (Poisson & Vlasov, 2009)

The unperturbed black hole is a nonrotating, Schwarzschild black hole with metric

$$ds^2 = -\underbrace{(1 - 2M/r)}_{f}\, dv^2 + 2\, dvdr + r^2 \underbrace{\Omega_{AB} d\theta^A d\theta^B}_{d\theta^2 + \sin^2\theta\, d\phi^2}$$

The metric is expressed in light-cone coordinates (v, r, θ^A).

The tidal perturbation preserves the geometrical meaning of the coordinates (Preston & Poisson, 2006):

> v labels incoming light cones; $\theta^A = (\theta, \phi)$ labels generators; r is an affine parameter on generators; the event horizon is described by $r = 2M$.

Black-hole problem (2)

The perturbation is charaterized by tidal multipole moments

$$\mathcal{E}_{ab}(v) \sim \mathcal{R}^{-2}, \qquad \mathcal{B}_{ab}(v) \sim \mathcal{R}^{-2}$$
$$\mathcal{E}_{abc}(v) \sim \mathcal{R}^{-3}, \qquad \mathcal{B}_{abc}(v) \sim \mathcal{R}^{-3}$$
$$\mathcal{E}_{abcd}(v) \sim \mathcal{R}^{-4}, \qquad \mathcal{B}_{abcd}(v) \sim \mathcal{R}^{-4}$$

These STF tensors are free functions that are not determined by the vacuum field equations.

They are related to the asymptotic behaviour of the **Weyl tensor** far away from the black hole ($r \gg M$).

It is assumed that the tidal moments vary slowly with time:

$$\dot{\mathcal{E}}_{ab} \sim \mathcal{R}^{-3}, \qquad \ddot{\mathcal{E}}_{ab} \sim \mathcal{R}^{-4}$$

Black-hole problem (3)

The perturbation is built from tidal potentials such as

$$\mathcal{E}^{\mathsf{q}} = \mathcal{E}_{ab}\Omega^a\Omega^b, \qquad \mathcal{E}^{\mathsf{o}} = \mathcal{E}_{abc}\Omega^a\Omega^b\Omega^c, \qquad \mathcal{E}^{\mathsf{h}} = \mathcal{E}_{abcd}\Omega^a\Omega^b\Omega^c\Omega^d$$

$$\mathcal{P}^{\mathsf{m}} = \mathcal{E}_{ab}\mathcal{E}^{ab}, \qquad \mathcal{P}^{\mathsf{q}} = \mathcal{E}_{p\langle a}\mathcal{E}^{p}{}_{b\rangle}\Omega^a\Omega^b, \qquad \mathcal{P}^{\mathsf{h}} = \mathcal{E}_{\langle ab}\mathcal{E}_{cd\rangle}\Omega^a\Omega^b\Omega^c\Omega^d$$

$$\mathcal{Q}^{\mathsf{m}} = \mathcal{B}_{ab}\mathcal{B}^{ab}, \qquad \mathcal{Q}^{\mathsf{q}} = \mathcal{B}_{p\langle a}\mathcal{B}^{p}{}_{b\rangle}\Omega^a\Omega^b, \qquad \mathcal{Q}^{\mathsf{h}} = \mathcal{B}_{\langle ab}\mathcal{B}_{cd\rangle}\Omega^a\Omega^b\Omega^c\Omega^d$$

$$\mathcal{G}^{\mathsf{d}} = \epsilon_{apq}\mathcal{E}^{p}{}_{r}\mathcal{B}^{rq}\Omega^a, \qquad \mathcal{G}^{\mathsf{o}} = \epsilon_{pq\langle a}\mathcal{E}^{p}{}_{b}\mathcal{B}^{q}{}_{c\rangle}\Omega^a\Omega^b\Omega^c$$

where $\boldsymbol{\Omega^a = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]}$.

Black-hole problem (4)

Steps 2 and 3: Solve the field equations (Poisson & Vlasov, 2009)

We integrate the vacuum field equations in the black-hole neighbourhood, imposing regularity conditions at $r = 2M$.

$$
\begin{aligned}
g_{vv} &= -f - r^2 e_1^{\mathsf{q}} \mathcal{E}^{\mathsf{q}} \\
&\quad + \frac{1}{3} r^3 e_2^{\mathsf{q}} \dot{\mathcal{E}}^{\mathsf{q}} - \frac{1}{3} r^3 e_1^{\circ} \mathcal{E}^{\circ} \\
&\quad - \frac{2}{21} r^4 e_3^{\mathsf{q}} \ddot{\mathcal{E}}^{\mathsf{q}} + \frac{1}{6} r^4 e_2^{\circ} \dot{\mathcal{E}}^{\circ} - \frac{1}{12} r^4 e_1^{\mathsf{h}} \mathcal{E}^{\mathsf{h}} + \frac{1}{15} r^4 \left(p_1^{\mathsf{m}} \mathcal{P}^{\mathsf{m}} + q_1^{\mathsf{m}} \mathcal{Q}^{\mathsf{m}} \right) \\
&\quad + \frac{2}{15} r^4 g_1^{\mathsf{d}} \mathcal{G}^{\mathsf{d}} + \frac{2}{7} r^4 \left(p_1^{\mathsf{q}} \mathcal{P}^{\mathsf{q}} + q_1^{\mathsf{q}} \mathcal{Q}^{\mathsf{q}} \right) + \frac{2}{3} r^4 g_1^{\circ} \mathcal{G}^{\circ} \\
&\quad - \frac{1}{3} r^4 \left(p_1^{\mathsf{h}} \mathcal{P}^{\mathsf{h}} + q_1^{\mathsf{h}} \mathcal{Q}^{\mathsf{h}} \right) + O(r^5/\mathcal{R}^5)
\end{aligned}
$$

Black-hole problem (5)

The radial functions are given by $(x = r/2M)$

$$
\begin{aligned}
e_1^{\mathsf{q}} &= f^2 \\
e_2^{\mathsf{q}} &= f\Big[1 + \frac{1}{4x}(5 + 12\log x) - \frac{1}{4x^2}(27 + 12\log x) + \frac{7}{4x^3} + \frac{3}{4x^4}\Big] \\
e_3^{\mathsf{q}} &= 1 + \frac{1}{24x}(89 + 84\log x) + \frac{1}{160x^2}(431 + 996\log x - 1680\,\mathrm{dilog}\,x) \\
&\quad - \frac{1}{10x^3}(315 + 282\log x - 210\,\mathrm{dilog}\,x) \\
&\quad + \frac{1}{120x^4}(4183 + 2322\log x - 1260\,\mathrm{dilog}\,x) - \frac{363}{40x^5} - \frac{809}{480x^6} \\
e_1^{\mathsf{o}} &= f^2\Big(1 - \frac{1}{2x}\Big) \\
e_2^{\mathsf{o}} &= f\Big[1 + \frac{1}{30x}(73 + 60\log x) - \frac{1}{60x^2}(479 + 180\log x) \\
&\quad + \frac{1}{20x^3}(87 + 20\log x) - \frac{3}{20x^4} + \frac{1}{60x^5}\Big] \\
e_1^{\mathsf{h}} &= f^2\Big(f + \frac{3}{14x^2}\Big)
\end{aligned}
$$

Black-hole problem (6)

$$
\begin{aligned}
p_1^{\mathsf{m}} &= f\Big(1 - \frac{19}{15x} + \frac{1}{15x^2} + \frac{1}{15x^3} + \frac{1}{5x^4}\Big) \\
q_1^{\mathsf{m}} &= f\Big(1 - \frac{19}{15x} + \frac{1}{15x^2} + \frac{1}{15x^3} + \frac{1}{5x^4}\Big) \\
g_1^{\mathsf{d}} &= f^2\Big(1 + \frac{2}{15x} - \frac{7}{5x^2} + \frac{1}{15x^3} + \frac{1}{30x^4}\Big) \\
p_1^{\mathsf{q}} &= f^2\Big(-\frac{26}{15x} + \frac{19}{20x^2} + \frac{2}{15x^3} + \frac{1}{15x^4}\Big) \\
q_1^{\mathsf{q}} &= f^2\Big(1 + \frac{4}{15x} - \frac{41}{20x^2} + \frac{2}{15x^3} + \frac{1}{15x^4}\Big) \\
g_1^{\mathsf{o}} &= f^2\Big(1 + \frac{79}{30x} - \frac{53}{20x^2} + \frac{1}{15x^3} + \frac{1}{30x^4}\Big) \\
p_1^{\mathsf{h}} &= f^2\Big(1 - \frac{121}{60x} + \frac{39}{280x^2} + \frac{1}{30x^3} + \frac{1}{60x^4}\Big) \\
q_1^{\mathsf{h}} &= f^2\Big(1 + \frac{29}{60x} - \frac{11}{280x^2} + \frac{1}{30x^3} + \frac{1}{60x^4}\Big)
\end{aligned}
$$

Dipole and acceleration (1)

In post-Newtonian theory, the equations of motion of a tidally-deformed body are

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{g} - \boldsymbol{d} + \text{post-Newtonian terms}$$

$$g_a = \partial_a U_{\text{ext}}(t, \boldsymbol{r}) \qquad d_a = \frac{1}{2M} Q^{bc} \mathcal{E}_{abc} + \cdots$$

The Newtonian potential in the body's comoving frame is

$$U = U_{\text{body}} + \underbrace{d_a x^a}_{\text{tidal dipole}} + U_{\text{tidal}}$$

Dipole and acceleration (2)

With $Q_{ab} \sim R^5 \mathcal{E}_{ab} \sim R^5/\mathcal{R}^2$ and $\mathcal{E}_{abc} \sim 1/\mathcal{R}^3$, the tidal dipole moment d_a scales as

$$d_a \sim \frac{1}{M}(R/\mathcal{R})^5$$

This measures the body's failure to move on a geodesic of the external spacetime.

Dipole and acceleration (3)

The black-hole metric contains a dipole term,

$$g_{vv} = -f + \cdots + \frac{2}{15} r^4 g_1^{\mathsf{d}} \mathcal{G}^{\mathsf{d}} + \cdots$$

The linear piece has the form of $d_a x^a$, with

$$d_a \propto M^3 \epsilon_{apq} \mathcal{E}^p{}_r \mathcal{B}^{rq}$$

This scales as

$$\boxed{d_a \sim \frac{1}{M} (M/\mathcal{R})^4}$$

instead of the previous $M^{-1}(M/\mathcal{R})^5$.

Interpretation?

Deformed horizon (1)

The horizon's intrinsic metric is

$$
\begin{aligned}
\gamma_{AB} \quad = \quad & r^2\Omega_{AB} - \frac{1}{6}r^4\bigl(\mathcal{E}^{\mathsf{q}}_{AB} + \mathcal{B}^{\mathsf{q}}_{AB}\bigr) - \frac{1}{60}r^5\bigl(\mathcal{E}^{\circ}_{AB} + \mathcal{B}^{\circ}_{AB}\bigr) \\
& - \frac{1}{840}r^6\bigl(\mathcal{E}^{\mathsf{h}}_{AB} + \mathcal{B}^{\mathsf{h}}_{AB}\bigr) + \frac{1}{90}r^6\Omega_{AB}\bigl(\mathcal{P}^{\mathsf{m}} + \mathcal{Q}^{\mathsf{m}}\bigr) - \frac{2}{45}r^6\Omega_{AB}\,\mathcal{G}^{\mathsf{d}} \\
& - \frac{1}{21}r^6\Omega_{AB}\bigl(\mathcal{P}^{\mathsf{q}} + \mathcal{Q}^{\mathsf{q}}\bigr) + \frac{1}{18}r^6\Omega_{AB}\,\mathcal{G}^{\circ} + \frac{1}{72}r^6\Omega_{AB}\bigl(\mathcal{P}^{\mathsf{h}} + \mathcal{Q}^{\mathsf{h}}\bigr) \\
& + \frac{1}{84}r^6\mathcal{P}^{\mathsf{h}}_{AB} - \frac{5}{126}r^6\mathcal{Q}^{\mathsf{h}}_{AB} + \frac{13}{252}r^6\mathcal{H}^{\mathsf{h}}_{AB} + O(r^7/\mathcal{R}^5)
\end{aligned}
$$

where $r = 2M$ and $\Omega_{AB} =$ diag$[1, \sin^2\theta]$.

Deformed horizon (2)

The **expansion** Θ and **shear** σ_{AB} of the horizon's null generators are determined by

$$\dot{\gamma}_{AB} = \Theta \gamma_{AB} + 2\sigma_{AB}, \qquad \gamma^{AB}\sigma_{AB} = 0$$

and we find that $\Theta = O(M^6/\mathcal{R}^6)$.

The **surface gravity** is

$$\begin{aligned} \kappa \;=\; & \kappa_0 \Big[1 + \frac{16}{3} M^3 \dot{\mathcal{E}}_{ab} \Omega^a \Omega^b + \frac{32}{9} M^4 \ddot{\mathcal{E}}_{ab} \Omega^a \Omega^b + \frac{8}{9} M^4 \dot{\mathcal{E}}_{abc} \Omega^a \Omega^b \Omega^c \\ & - \frac{16}{225} M^4 (\mathcal{E}_{ab}\mathcal{E}^{ab} + \mathcal{B}_{ab}\mathcal{B}^{ab}) + O(M^5/\mathcal{R}^5) \Big] \end{aligned}$$

where $\kappa_0 = (4M)^{-1}$.

Deformed horizon (3)

The change in black-hole area $\mathcal{A}$ is governed by

$$\dot{\Theta} = \kappa\Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB}$$

Integration over a cross-section of the horizon yields

$$\kappa_0 \dot{\mathcal{A}} - \ddot{\mathcal{A}} = 8\pi\mathcal{F} + O(M^9/\mathcal{R}^9)$$

with

$$\begin{aligned} \mathcal{F}(v) &= \frac{1}{8\pi}\int \sigma_{AB}\sigma^{AB}\sqrt{\gamma}\, d^2\theta \\ &= \frac{16}{45}M^6\Big(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab} + \dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}\Big) \\ &\quad + \frac{16}{4725}M^8\Big(\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc} + \frac{16}{9}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}\Big) \end{aligned}$$

Deformed horizon (4)

Integration yields

$$\frac{\kappa_0}{8\pi}\dot{\mathcal{A}} = \mathcal{F} + \frac{d}{dv}\left[\frac{\mathcal{F}}{\kappa_0} + \frac{\dot{\mathcal{F}}}{\kappa_0^2}\right] + O(M^9/\mathcal{R}^9)$$

The flux $\mathcal{F}$ is positive-definite, and represents the rate at which the black-hole absorbs mass as a result of tidal processes.

This is the **tidal heating of black holes**,

$$\begin{aligned}\dot{M} &= \frac{16}{45}M^6\Big(\dot{\mathcal{E}}_{ab}\dot{\mathcal{E}}^{ab} + \dot{\mathcal{B}}_{ab}\dot{\mathcal{B}}^{ab}\Big) \\ &\quad + \frac{16}{4725}M^8\Big(\dot{\mathcal{E}}_{abc}\dot{\mathcal{E}}^{abc} + \frac{16}{9}\dot{\mathcal{B}}_{abc}\dot{\mathcal{B}}^{abc}\Big)\end{aligned}$$

Conclusions

- The metric of a tidally-deformed black hole was worked out to order $(r/\mathcal{R})^4$ and expressed in geometrically meaningful coordinates (v, r, θ^A).
- The metric is written in terms of (electric-type and magnetic-type) tidal multipole moments that are operationally defined in terms of the asymptotic behaviour of the Weyl tensor far from the black hole.
- The tidal moments characterize the tidal environment; they must be determined by matching the black-hole metric to a global metric that contains the black hole and the external bodies.
- The tidal dynamics of a black hole produce tidal heating.