A New Derivation of the Gravitational Self Force

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Why another derivation?

Problems with earlier derivations

- many derivations are based on ill-behaved source and/or assumed form for force, or make many assumptions about metric
- no derivation is based on a uniformly accurate, global approximation scheme
- most derivations are vague about worldline (geodesic or not?)
- most derivations abuse global nature of tail integral

Goals of this derivation

- shift emphasis to techniques of PN and singular perturbation theory
- base derivation on global, self-consistent evolution of small body, background metric, and worldline
- clarify previous derivations; make worldline and tail integral more precise
- introduce method that can be applied at any order

Idea of regular expansions

Expand for small ε, keeping coordinates x fixed

- consider a spacetime containing a body of small mass $\sim \varepsilon$ with motion described by a worldline $\gamma(t, \varepsilon)$
- assume $\mathsf{g}_{\mu\nu}(x, \varepsilon) = g_{\mu\nu}(x) + \varepsilon h^{(1)}_{\mu\nu}(x) + \ldots$ and $\mathsf{T}_{\mu\nu}(x, \varepsilon) = \varepsilon \delta T_{\mu\nu}(x) + \ldots$
- the source $\mathsf{T}_{\mu\nu}$ obviously depends on γ
  $\Rightarrow$ to make $\delta T_{\mu\nu}(x)$ independent of ε, expand γ:

$$\gamma(t, \varepsilon) = \gamma^{(0)}(t) + \varepsilon \delta\gamma(t) + \ldots$$

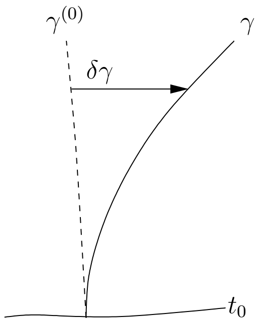
- solve EFE order-by-order:

$$\begin{aligned} G_{\mu\nu}[g](x) &= 0 \\ \delta G_{\mu\nu}[g, h^{(1)}](x) &= \delta T_{\mu\nu}[g, \gamma^{(0)}](x) \\ &\vdots \end{aligned}$$

Expansion of the worldline

Meaning of worldline expansion

- can't add functions at different points in curved space
  $\Rightarrow$ $\gamma^{(0)}$ forced to represent body's worldline
  $\Rightarrow$ corrections are vector fields on $\gamma^{(0)}$
- expansion meaningful only if $\gamma^{(0)}$ tangential to exact γ at some time t_0



Problems with worldline expansion

- deviation between $\gamma^{(0)}$ and γ leads to geodesic-deviation-like terms in equation of motion for $\delta\gamma$
- assume we find γ has an acceleration $a \sim \varepsilon$
  $\Rightarrow$ the error in the worldline grows as $\delta\gamma(t) \sim a(t-t_0)^2$

The metric perturbation

First-order EFE in Lorenz gauge

- impose Lorenz gauge condition $\nabla^\nu \bar{h}^{(1)}_{\mu\nu} = 0$
- linearized EFE becomes wave equation

$$\left(g^\alpha_\mu g^\beta_\nu \nabla_\rho \nabla^\rho + 2R^\alpha{}_\mu{}^\beta{}_\nu\right) h^{(1)}_{\alpha\beta}(x) = -16\pi \delta\overline{T}_{\mu\nu}[g, \gamma^{(0)}](x)$$

First-order metric perturbation

- if $\delta T_{\mu\nu}$ corresponds to point particle, solution to wave equation is

$$\begin{aligned} h^{(1)}_{\mu\nu} &= 4 \int G_{\mu\nu}{}^{\mu'\nu'} \delta\overline{T}_{\mu'\nu'}[\gamma^{(0)}] dV' \\ &= 4 \int_{\gamma^{(0)}} \bar{G}_{\mu\nu\mu'\nu'} u^{\mu'} u^{\nu'} dt, \end{aligned}$$

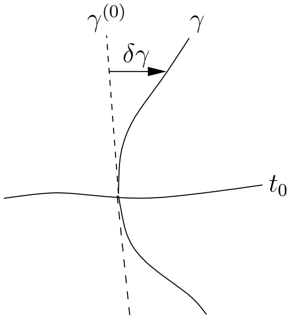
- gauge condition (or Bianchi identity) requires $\gamma^{(0)}$ to be geodesic

The trouble with the tail

The tail integral

- $h_{\mu\nu}$ split into direct piece, due to field propagating on light cone, and tail piece, due to field propagating within light cone
- tail piece given by integral over $\gamma^{(0)}$:

$$h^{\text{tail}}_{\mu\nu}(x) = 4m \int_{-\infty}^{t^-_{ret}} \bar{G}_{\mu\nu\mu'\nu'} u^{\mu'} u^{\nu'} dt'$$



And why it's always wrong

- error grows not only forward in time, but also backward in time
  $\Rightarrow$ the tail integral is never a valid approximation
- tail integral is global, but regular expansion is local
  $\Rightarrow$ chop off at $t = t_0$, replace with contribution from initial data

Idea of singular expansions

Allow expansion coefficients to depend on ε

- the limit $\varepsilon \to 0$ is singular: qualitative change between $\varepsilon = 0$ and $\varepsilon > 0$
- leads to failure/nonuniformity of regular expansion
- solution: work always at $\varepsilon > 0$; use more general expansions

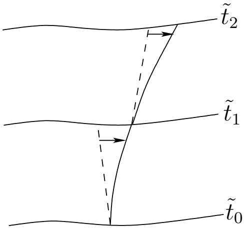
$$\mathsf{g}_{\mu\nu}(x,\varepsilon) = g_{\mu\nu}(x,\varepsilon) + \varepsilon h^{(1)}_{\mu\nu}(x,\varepsilon) + \ldots$$
$$\mathsf{T}_{\mu\nu}(x,\varepsilon) = \varepsilon \delta T(x,\varepsilon) + \ldots$$

- ε-dependence of coefficients prescribed by particular approximation; e.g., might be dependence on a fixed scalar field such as a slow time $\tilde{t} = \varepsilon t$
- typically need to impose extra conditions to solve for each coefficient; e.g., treat slow time as independent variable

Options for uniform approximation

Option 1: patch together regular expansions

- use two-timescale expansion; at each fixed slow-time $\tilde{t}$, regular expansion valid for short time
- uniform approximation formed by letting $\tilde{t}$ vary



Option 2: self-consistent evolution

- let everything vary self-consistently
- in particular, treat γ as fixed structure in the background $g_{\mu\nu}$
- also allow $g_{\mu\nu}$ to depend on ε to account for backreaction

Defining the worldline

Operational definition

- construct one expansion about metric of "external" background $g_{\mu\nu}$ and another about metric of small body
- γ identified as worldline of body if these expansions are identical in a region centered on γ in the spacetime of $g_{\mu\nu}$

Emergence from an exact solution

- surround body by worldtube
- metric can be written as integral over surface of tube
- as body shrinks, boundary data on tube can be expanded about worldline at center of interior of tube
- this curve is identified as worldline by mass dipole of body vanishing on tube

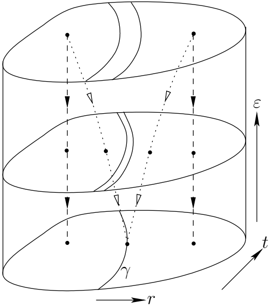
Interior and exterior expansions

Interior expansion: $r \sim \varepsilon$

- choose some "local" coordinates (T, X^a) centered on body
- expand for small ε while holding $\tilde{X}^a = X^a/\varepsilon$ fixed:
  $\mathsf{g}_{\mu\nu} = g^{\text{body}}_{\mu\nu} + \sum \varepsilon^n H^{(n)}(T, \tilde{X}^a)$
- keeps size of body fixed while blowing up all other distances

Exterior expansion: $r \sim 1$

- choose some "global" coordinates x
- expand for small ε holding x, $g_{\mu\nu}$, and γ fixed:
  $\mathsf{g}_{\mu\nu}(x,\varepsilon) = g_{\mu\nu}(x,\varepsilon) + h_{\mu\nu}(x,\gamma,g,\varepsilon)$
- shrinks worldtube around body, keeping other distances fixed

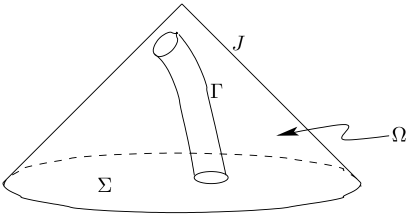


A family of spacetimes. Dashed lines correspond to exterior expansion, dotted lines to interior expansion.

Formulation of external Einstein equation

External region

- seek solution in vacuum region Ω bounded by spatial surface Σ, timelike tube Γ, and future null surface J
- tube surrounds body; radius R chosen such that $\varepsilon \ll R \ll 1$



Lorenz gauge condition

- in Ω, write total metric as $\mathsf{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$
- impose Lorenz gauge condition on the entire external perturbation $h_{\mu\nu}$, not only on a first-order piece $h^{(1)}_{\mu\nu}$
- vacuum Einstein equation $\mathsf{R}_{\mu\nu} = 0$ becomes

$$\left(g^{\alpha}_{\mu} g^{\beta}_{\nu} \nabla_{\rho}\nabla^{\rho} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\right) h_{\alpha\beta} = 2R_{\mu\nu} + 2\delta^2 R_{\mu\nu}[h] + \ldots$$

Integral representation of external EFE

The integral

- apply Stokes theorem to EFE in Lorenz gauge to get

$$h_{\alpha\beta} = \frac{1}{4\pi}\int_{\partial\Omega}\left(G_{\alpha\beta}{}^{\gamma'\delta'}\nabla_{\mu'}h_{\gamma'\delta'} - h_{\gamma'\delta'}\nabla_{\mu'}G_{\alpha\beta}{}^{\gamma'\delta'}\right)dS^{\mu'} - \frac{1}{2\pi}\int_{\Omega}G_{\alpha\beta}{}^{\gamma'\delta'}\left(R_{\gamma'\delta'} + \delta^2 R_{\gamma'\delta'}[h]\right)dV'$$

- since radius of tube is small, $h_{\mu\nu}$ can be written as functional of worldline γ (and background $g_{\mu\nu}$)

Its pieces

- volume integral contributes only at order ε^2 and r^2
- integral over future null surface vanishes
- initial data surface contributes part of regular homogeneous solution to wave equation

Outline of calculation of self-force

Step 1: general buffer region solution

- construct very general solution to wave equation in region near Γ
- use interior background metric $g_{\mu\nu}^{\text{body}}$ to determine most singular terms in expansion
  —this will be the *only* use of the interior expansion
- impose Lorenz gauge condition to determine acceleration of γ in terms of unknown irreducible pieces of metric perturbation

Step 2: determining unknown pieces of perturbation

- use general buffer region solution as boundary data for integral formulation of EFE
- expand integral representation near surface of tube
- demand that the result agrees with the buffer expansion
  $\Rightarrow$ determines unknown functions in buffer expansion in terms of tail integrals

Buffer region expansion I

Expansion of perturbation for $\varepsilon \ll r \ll 1$

- adopt Fermi coordinates (t, x^a) centered on γ
  —asssume these coordinates are smoothly related to (T, X^a)
- expand metric perturbation for small ε:

$$h_{\mu\nu} = \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \ldots$$

- expand each term for small r (most singular power restricted by interior expansion):

$$h^{(1)}_{\alpha\beta} = \frac{1}{r} h^{(1,-1)}_{\alpha\beta} + h^{(1,0)}_{\alpha\beta} + r h^{(1,1)}_{\alpha\beta} + \ldots$$

$$h^{(2)}_{\alpha\beta} = \frac{1}{r^2} h^{(2,-2)}_{\alpha\beta} + \frac{1}{r} h^{(2,-1)}_{\alpha\beta} + h^{(2,0)}_{\alpha\beta} + \ln r\, h^{(2,0,\ln)}_{\alpha\beta} + \ldots$$

- each coefficient depends on t and angles θ, ϕ
- possible logarithm arises due to perturbation of characteristics: $t - r \to t - r^*$

Buffer region expansion II

STF decomposition of each term

$$h_{tt}^{(n,m)} = \sum_{\ell\geq 0} \mathscr{A}_L^{(n,m)} \hat{n}^L$$

$$h_{ta}^{(n,m)} = \sum_{\ell\geq 0} \mathscr{B}_L^{(n,m)} \hat{n}_a{}^L + \sum_{\ell\geq 1} \left[\mathscr{C}_{aL-1}^{(n,m)} \hat{n}^{L-1} + \epsilon_{ab}{}^c \mathscr{D}_{cL-1}^{(n,m)} \hat{n}^{bL-1}\right]$$

$$\begin{aligned} h_{ab}^{(n,m)} &= \delta_{ab} \sum_{\ell\geq 0} \mathscr{K}_L^{(n,m)} \hat{n}^L + \sum_{\ell\geq 0} \mathscr{E}_L^{(n,m)} \hat{n}_{ab}{}^L \\ &+ \sum_{\ell\geq 1} \left[\mathscr{F}_{L-1\langle a}^{(n,m)} \hat{n}_{b\rangle}{}^{L-1} + \epsilon^{cd}{}_{(a} \hat{n}_{b)c}{}^{L-1} \mathscr{G}_{dL-1}^{(n,m)}\right] \\ &+ \sum_{\ell\geq 2} \left[\mathscr{H}_{abL-2}^{(n,m)} \hat{n}^{L-2} + \epsilon^{cd}{}_{(a} \mathscr{I}_{b)dL-2}^{(n,m)} \hat{n}_c{}^{L-2}\right] \end{aligned}$$

- angle-dependence in unit vectors n^i, time-dependence in script tensors
- multi-index notation: $\hat{n}^L = n^{\langle i_1} \dots n^{i_\ell \rangle}$

First-order calculation

First-order EFE

- wave equation: $\left(g_\mu^\alpha g_\nu^\beta \nabla_\rho \nabla^\rho + 2R^\alpha{}_\mu{}^\beta{}_\nu\right) h^{(1)}_{\alpha\beta} = O\left(\varepsilon, r^0\right)$
- gauge condition: $\left(g_\mu^\rho g^{\sigma\gamma} - \frac{1}{2} g_\mu^\gamma g^{\rho\sigma}\right) \nabla_\gamma h^{(1)}_{\rho\sigma} = O(\varepsilon)$

Results of expansion

- most singular (m/r) term: $h^{(1,\text{-}1)}_{\alpha\beta} dx^\alpha dx^\beta = 2m(dt^2 + \delta_{ab} dx^a dx^b)$
- $m(t)$ is the ADM mass of the interior background $g^{\text{body}}_{\mu\nu}$
- terms that are not determined by buffer expansion: $\mathscr{A}^{(1,0)}$, $\mathscr{C}^{(1,0)}_a$, $\mathscr{K}^{(1,0)}$, $\mathscr{H}^{(1,0)}_{ab}$, $\mathscr{A}^{(1,1)}_a$, $\mathscr{B}^{(1,1)}$, $\mathscr{C}^{(1,1)}_{ab}$, $\mathscr{D}^{(1,1)}_a$, $\mathscr{K}^{(1,1)}_a$, $\mathscr{H}^{(1,1)}_{abc}$, $\mathscr{F}^{(1,1)}_a$, $\mathscr{I}^{(1,1)}_{ab}$
- evolution equations determined by gauge condition:

$$\partial_t m = O\left(\varepsilon^2\right) \text{ and } a^i = O(\varepsilon)$$

Go to STF decomposition

Second-order calculation

Second-order EFE

- wave equation: $\left(g^\alpha_\mu g^\beta_\nu \nabla_\rho \nabla^\rho + 2R^\alpha{}_\mu{}^\beta{}_\nu\right) h^{(2)}_{\alpha\beta} = 2\delta^2 R_{\mu\nu}\left[h^{(1)}\right] + O(\varepsilon)$
- gauge condition: $\left(g^\rho_\mu g^{\sigma\gamma} - \frac{1}{2} g^\gamma_\mu g^{\rho\sigma}\right) \nabla_\gamma \left(h^{(1)}_{\rho\sigma} + \varepsilon h^{(2)}_{\rho\sigma}\right) = O\left(\varepsilon^2\right)$

Terms defined by interior background

- most singular (ε^2/r^2) coefficient:

$$h^{(2,-2)}_{tt} = -2m^2 + 2M_i n^i$$
$$h^{(2,-2)}_{ta} = 2\epsilon_{aij} n^i S^j$$
$$h^{(2,-2)}_{ab} = \delta_{ab}\left(\tfrac{8}{3}m^2 + 2M_i n^i\right) - 7m^2 \hat{n}_{ab}$$

- M^i is mass dipole of $g^{\text{body}}_{\mu\nu}$, S^i is ADM angular momentum of $g^{\text{body}}_{\mu\nu}$

Go to STF decomposition

Second-order results I

Mass evolution

- at order ε^2/r, "mass" of body becomes ambiguous due to ambiguity in local definition of gravitational energy
  $\Rightarrow \mathscr{A}^{(2,\text{-}1)} = \mathscr{K}^{(2,\text{-}1)}$ is arbitrary—it will never be determined
- EFE gives result for time-evolution of combination $2m + \mathscr{K}^{(2,\text{-}1)}$

$$\partial_t(2m + \mathscr{K}^{(2,\text{-}1)}) = \tfrac{1}{3}m\partial_t(2\mathscr{A}^{(1,0)} + 5\mathscr{K}^{(1,0)}),$$

- m might be slowly evolving, $\mathscr{K}^{(2,\text{-}1)}$ both slowly and rapidly evolving
- determining separate equations for m and $\mathscr{K}^{(2,\text{-}1)}$ requires more specific knowledge of $\mathscr{A}^{(1,0)}$ and $\mathscr{K}^{(1,0)}$

Go to STF decomposition

Second-order results II

Equation of motion

- EFE determines equation of motion for mass dipole:

$$\partial_t^2 M_a + R_{a0i0}M^i = -m a_a^{(1)} + \tfrac{1}{2}\epsilon_a{}^{jk}R_{0ijk}S^i - \tfrac{1}{2}m\mathscr{A}_a^{(1,1)} + m\partial_t\mathscr{C}_a^{(1,0)}$$

- if expansion were regular, then $a^i \equiv 0$, and equation of motion would include deviation term $R_{a0i0}M^i$
- we define γ by M^i vanishing for all time $\Rightarrow$ this is guaranteed by unique choice of acceleration:

$$a_a^{(1)} = \frac{1}{2m}\epsilon_a{}^{jk}R_{0ijk}S^i - \tfrac{1}{2}\mathscr{A}_a^{(1,1)} + \partial_t\mathscr{C}_a^{(1,0)}$$

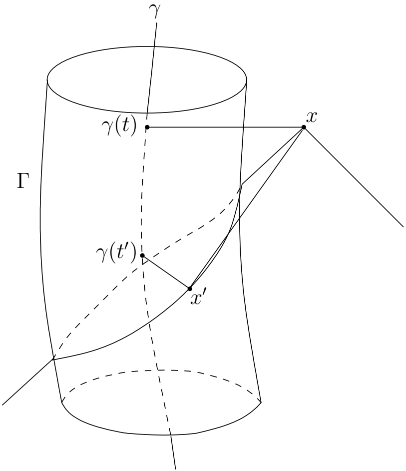
- self-force determined in terms of unknowns $\mathscr{A}_a^{(1,1)}$ and $\mathscr{C}_a^{(1,0)}$
- in principle, the job is done: given solution to wave equation at given time, force is uniquely determined by irreducible pieces of the field

Go to STF decomposition

Expansion of boundary integral near the worldtube

Procedure

- use buffer-region expansion as data on worldtube
- expand fields on worldtube about $\gamma(t')$
- expand fields at x about $\gamma(t)$
- if $\gamma(t')$ is close to $\gamma(t)$, expand fields at former about latter
- perform STF decomposition of result
- add contribution from initial data surface
- demand consistency with the buffer region expansion



Evolution equations in terms of a tail integral

The tail

- the tail has contributions from γ and Σ:

$$h^{\text{tail}}_{\alpha\beta} = \int_0^{t^-} 4m\big(G_{\alpha\beta\alpha'\beta'} - \tfrac{1}{2}g_{\alpha\beta}G^{\delta}{}_{\delta\alpha'\beta'}\big)u^{\alpha'}u^{\beta'}dt' + h^{(1,0)}_{\Sigma\alpha\beta}$$

$$h^{\text{tail}}_{\alpha\beta\gamma} = \int_0^{t^-} 4m\nabla_{\gamma}\big(G_{\alpha\beta\alpha'\beta'} - \tfrac{1}{2}g_{\alpha\beta}G^{\delta}{}_{\delta\alpha'\beta'}\big)u^{\alpha'}u^{\beta'}dt' + h^{(1,1)}_{\Sigma\alpha\beta\gamma}$$

Evolution equations

- $\partial_t(2m + \mathscr{K}^{(2,\text{-}1)}) = \frac{1}{9}m\left(5g^{\alpha\beta} + 11u^{\alpha}u^{\beta}\right)h^{\text{tail}}_{\alpha\beta\gamma}u^{\gamma}$
- $a^{(1)}_{\alpha} = -\frac{1}{2}\left(g_{\alpha}{}^{\delta} + u_{\alpha}u^{\delta}\right)\left(2h^{\text{tail}}_{\delta\beta\gamma} - h^{\text{tail}}_{\beta\gamma\delta}\right)u^{\beta}u^{\gamma} + \frac{1}{2m}R_{\alpha\beta\gamma\delta}u^{\beta}S^{\gamma\delta}$ where $S^{\gamma\delta} = e^{\gamma}_{c}e^{\delta}_{d}\epsilon^{cde}S_e$

Summary

Point of view

- self-force problem is a singular perturbation problem
- to derive self-consistent results, use singular perturbation theory
- in particular, use inner and outer expansions (well known) and treat worldline and background as fixed (not usually considered)

Techniques

- get force using buffer region expansion
- relate force to past history via integral representation

Prospects

- self-consistent approach offers prospect of global, uniform solution
- method can be used at any order

Buffer region expansion

STF decomposition

$$h^{(n,m)}_{tt} = \sum_{\ell\geq 0} \mathscr{A}^{(n,m)}_{L} \hat{n}^{L}$$

$$h^{(n,m)}_{ta} = \sum_{\ell\geq 0} \mathscr{B}^{(n,m)}_{L} \hat{n}_{a}{}^{L} + \sum_{\ell\geq 1} \left[\mathscr{C}^{(n,m)}_{aL-1} \hat{n}^{L-1} + \epsilon_{ab}{}^{c} \mathscr{D}^{(n,m)}_{cL-1} \hat{n}^{bL-1}\right]$$

$$h^{(n,m)}_{ab} = \delta_{ab} \sum_{\ell\geq 0} \mathscr{K}^{(n,m)}_{L} \hat{n}^{L} + \sum_{\ell\geq 0} \mathscr{E}^{(n,m)}_{L} \hat{n}_{ab}{}^{L} + \sum_{\ell\geq 1} \left[\mathscr{F}^{(n,m)}_{L-1\langle a} \hat{n}_{b\rangle}{}^{L-1} + \epsilon^{cd}{}_{(a} \hat{n}_{b)c}{}^{L-1} \mathscr{G}^{(n,m)}_{dL-1}\right] + \sum_{\ell\geq 2} \left[\mathscr{H}^{(n,m)}_{abL-2} \hat{n}^{L-2} + \epsilon^{cd}{}_{(a} \mathscr{I}^{(n,m)}_{b)dL-2} \hat{n}_{c}{}^{L-2}\right]$$