京都大学
Kyoto University

普遍性と創発性から紡ぐ
次世代物理学
フロンティア開拓のための
自立的人材養成

Calculation of the gravitational self-force in Schwarzschild spacetime

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Outline

1. Mode sum scheme
2. Metric perturbation in Lorenz gauge
    - (1+1)-d 10 coupled equations with δ-func. source
    - “divergence dissipating” terms
3. Numerical method
    - grid space (double null, uniform mesh)
    - finite difference scheme (4th order)
    - coupling terms
    - construct the full force modes
4. Results
    - Example of SF
    - Energy balance
    - ISCO shift
5. Summary and Future works

Strategy - mode sum scheme -

Here we consider the gravitational self-force for a bound geodesic in Schwarzschild geometry.

<u>Step 1</u> Solve the MP equations by modes for a given orbit.

- (1+1)-dim. time domain, in Lorenz gauge.
- deal with monopole, dipole separately.

<u>Step 2</u> Construct the full force modes in spherical harmonics.

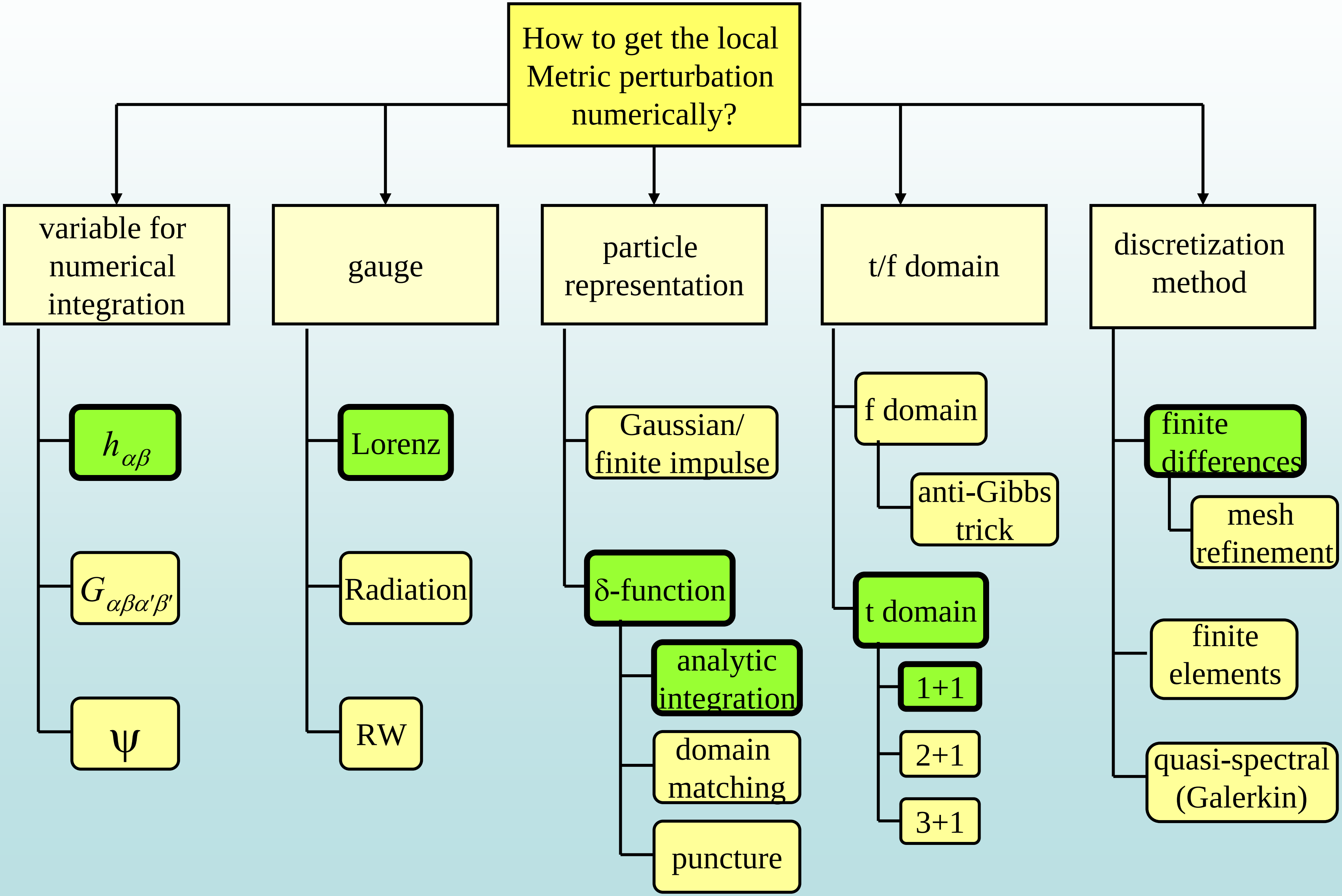
- convert to tensor harmonics to spherical harmonics.

<u>Step 3</u> Apply the mode sum scheme.

$$F^{\alpha}(\tau) = \sum_{l} \left[F_{\text{full}}^{\alpha,l}(x_p) - A^{\alpha} L - B^{\alpha} \right]$$

- subtract the singular part from the full force modes
- derive the fitting formula for high-l tail
- sum over l

Strategy for MP calculation



Metric perturbation in Lorenz gauge

Linearized Einstein equation

$$\delta G^{(1)}[h_{\mu\nu}] = -16\pi T_{\mu\nu}$$

$$T_{\mu\nu} = \mu \int_{-\infty}^{\infty} [-g]^{-1/2} u_\mu u_\nu \delta^{(4)}(x - x_p)$$

($T_{\mu\nu}$ is the stress tensor for geodesic)

Lorenz gauge condition

$$Z_\mu \equiv \bar{h}_{\mu\nu}{}^{;\nu} = 0$$

$$\Downarrow$$

$$\bar{h}_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$$

Divergence dissipating terms

Problem in numerical evolution

The violation of the gauge condition grows up in time-evolution because of the numerical errors.

"Divergence dissipating" terms [Barack & Lousto (2005)]

$$\bar{h}_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} + 2f' t_{(\mu}\tilde{Z}_{\nu)} = -16\pi T_{\mu\nu}$$

$$t_{\mu} = (1, f^{-1}, 0, 0)$$

$$\tilde{Z}_{\mu} = (fZ_r, Z_r, Z_\theta, Z_\varphi)$$

These terms guarantee that the violation of the gauge condition damps in time-evolving the equation.

Reduction to (1+1)-dimension

Expansion of MP in terms of tensor harmonics

$$\bar{h}_{\mu\nu}(x) = \sum_{\ell m} \bar{h}_{lm}^{(i)}(t,r)\mathbf{Y}_{\mu\nu}^{(i),\ell m}(\theta,\phi) \qquad (i = 1,\ldots,10)$$

Field equations in (1+1)-dimension

$$-\partial_t^2 \bar{h}_{lm}^{(i)} + \partial_{r_*}^2 \bar{h}_{lm}^{(i)} + \mathcal{M}_{(j)}^{(i)} \bar{h}_{lm}^{(j)} = S_{lm}^{(i)}(t)\delta(r - r_p(t))$$

(1+1)-dim. 7 (even) + 3 (odd) coupled equations

- Hyperbolic equations
- Delta-function source terms
- (MP is continuous at the particle's location.)

Finite difference scheme

We solve the field equations in double null, uniform grid space by using a finite differential scheme.

Working grid space

Use double null coordinates:

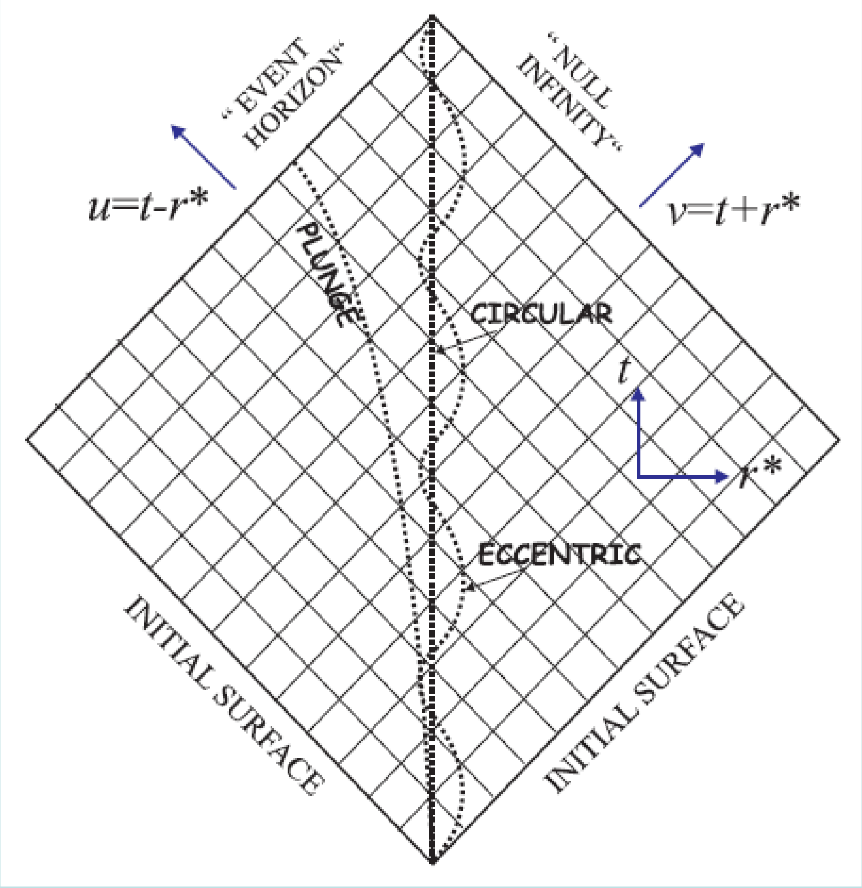
$$v = t + r_*, \quad u = t - r_*$$

Take uniform grid: $\Delta v = \Delta u = h$

Initial condition

Put the initial condition as:

$$\bar{h}_{lm}^{(i)} = 0 \text{ at } v = v_0 \text{ and } u = u_0$$



Finite difference scheme (4th order)

[based on Haas '07]

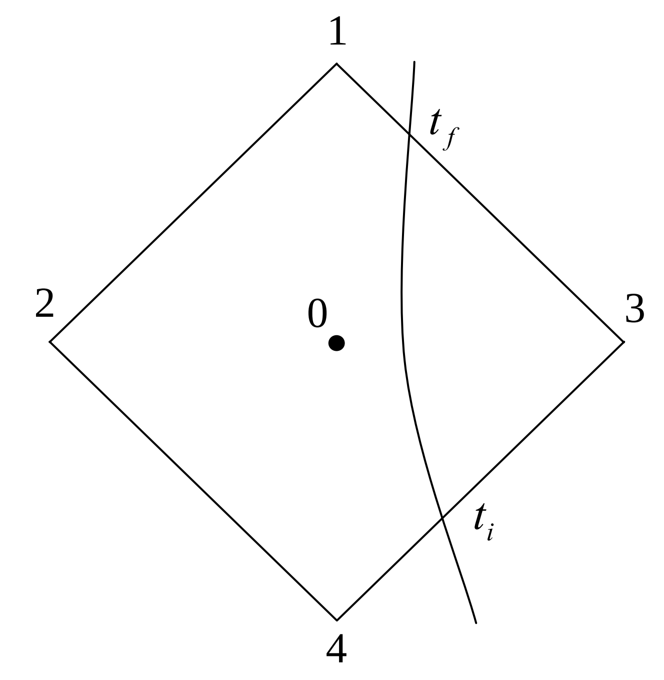
$$\partial_v \partial_u \bar{h}^{(i)}_{lm} + \mathcal{M}^{(i)}_{(j)} \bar{h}^{(j)}_{lm} = S^{(i)}_{lm} \delta(r - r_p(t))$$

principal term — coupling terms — source term

To obtain the finite difference scheme, we integrate them over a cell.

$$\int_{cell} dv du \partial_v \partial_u \bar{h}^{(i)} \rightarrow \bar{h}^{(i)}_1 - \bar{h}^{(i)}_2 - \bar{h}^{(i)}_3 + \bar{h}^{(i)}_4$$

$$\int_{cell} dv du S^{(i)}_{lm}(t) \delta(r - r_p(t)) \rightarrow \int_{t_i}^{t_f} dt S^{(i)}_{lm}(t)$$



Finite difference scheme

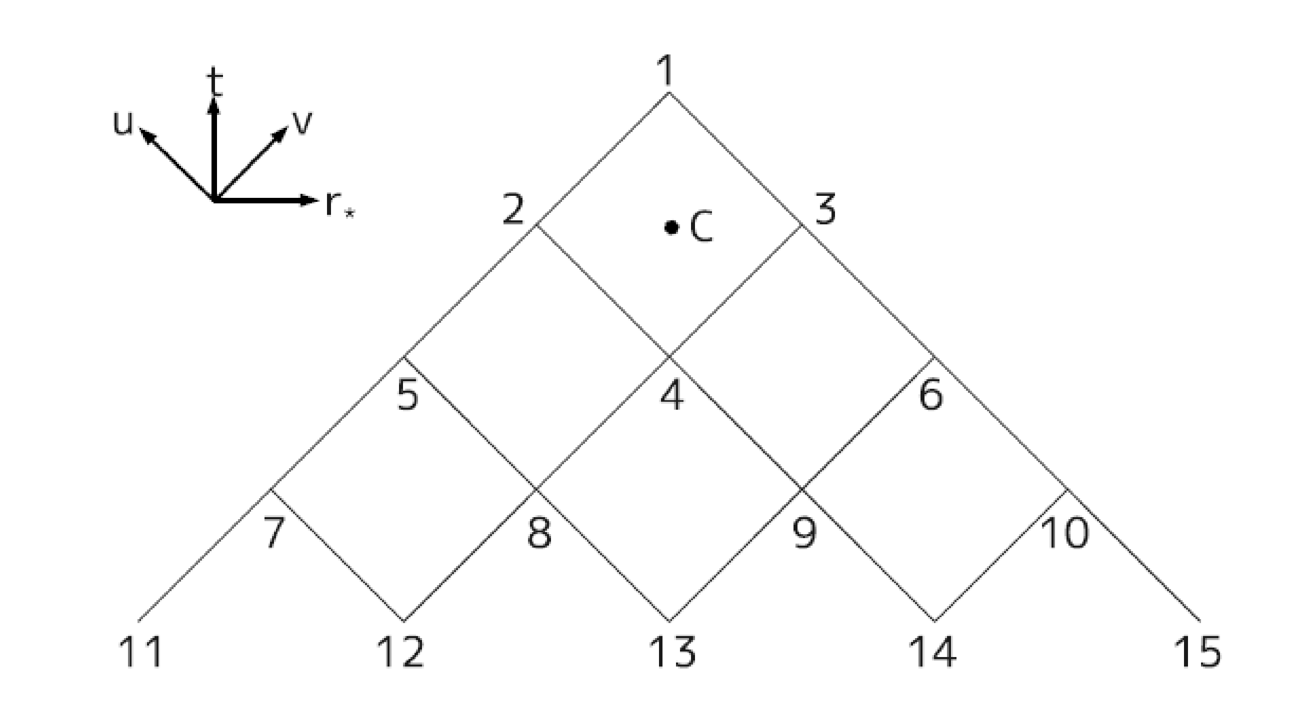
$$\bar{h}^{(i)}_1 = \bar{h}^{(i)}_2 + \bar{h}^{(i)}_3 - \bar{h}^{(i)}_4 - \int_{cell} dv du M^{(i)}_{(j)} \bar{h}^{(j)} + \int_{t_i}^{t_f} dt S^{(i)}(t)$$

Coupling terms (for vacuum case)

$$\partial_v \partial_u \bar{h}^{(i)}_{lm} + \mathcal{M}^{(i)}_{(j)} \bar{h}^{(j)}_{lm} = S^{(i)}_{lm} \delta(r - r_p(t))$$

coupling terms

$$\psi \equiv V(r) \bar{h}^{(i)}_{lm} \qquad \partial_v \psi \qquad \partial_{r*} \psi$$

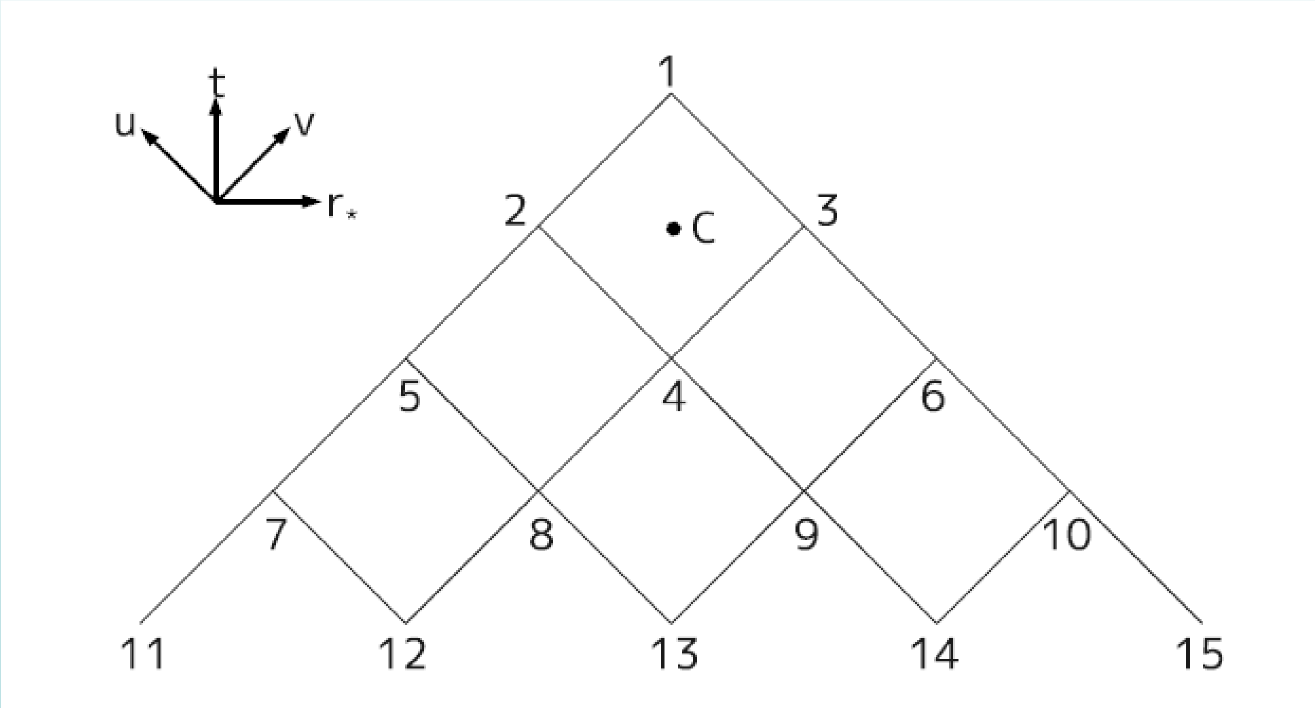


Coupling terms (for vacuum case)

$$\iint \psi du dv = \left(\frac{h}{6}\right)^2 \left[3\psi_1 + 15(\psi_2+\psi_3+\psi_4) - 6(\psi_5+\psi_6) + \frac{3}{2}(\psi_7-\psi_8-\psi_9+\psi_{10})\right] + O(h^6),$$

$$\iint \partial_v \psi du dv = \frac{h}{24}\left[9(\psi_1-\psi_2) + 19(\psi_3-\psi_4) - 5(\psi_6-\psi_9) + (\psi_{10}-\psi_{14})\right] + O(h^6),$$

$$\iint \partial_{r_*} \psi du dv = \frac{h}{24}\left[28(\psi_3-\psi_2) - 5(\psi_6-\psi_5) + (\psi_{10}-\psi_7) + 5(\psi_9-\psi_8) - (\psi_{14}-\psi_{12})\right] + O(h^6).$$

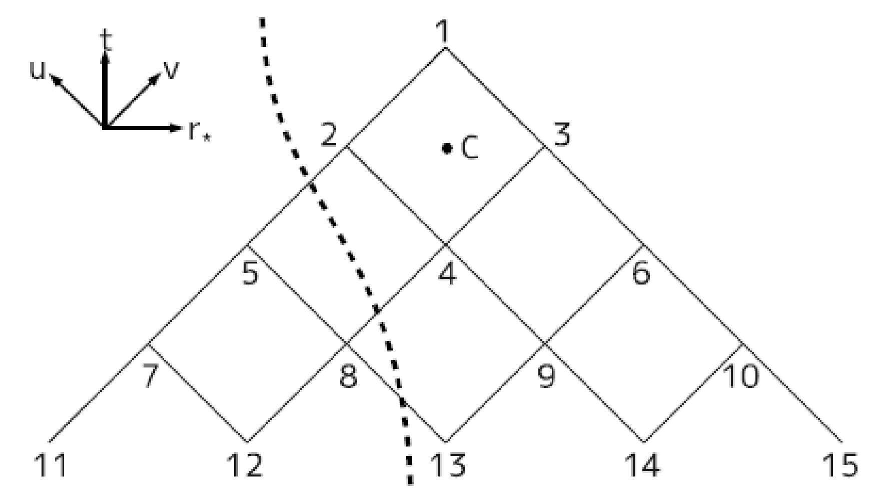


Coupling terms (close to the particle)

Expand the integrands around the particle position:

$$\psi(u,v) = \sum_{a+b=0}^{4} \frac{c_{ab}^{\pm}}{a!b!}(u-u_p)^a(v-v_p)^b + O(h^5)$$

To obtain the coefficients,
 we use 15 grid points and 15 jump conditions at the particle position.



$$\left[\bar{h}_{lm}^{(i)}\right] = 0,$$

$$\left[\partial_v \bar{h}_{lm}^{(i)}\right] = -\left|\frac{dv_p}{d\tau}\right|^{-1} \frac{2E}{f_p^2} S_{lm}^{(i)},$$

$$\left[\partial_u \bar{h}_{lm}^{(i)}\right] = \left|\frac{du_p}{d\tau}\right|^{-1} \frac{2E}{f_p^2} S_{lm}^{(i)},$$

$$\vdots$$

Construct of full force modes

Metric perturbation in tensor harmonics

$l \geq 2$: Use finite difference scheme in time domain

$l = 0,1$: Use Fourier technique in frequency domain

Our TD code does not work stably for these modes because some potentials functions turn negative for some r outside the horizon.

Construction of full force modes

$$[F^{\alpha l}_{\rm full}(x_0)]_{\pm} = \frac{\mu^2}{r_0^2} \sum_{m=-l}^{l} Y^{lm}(\theta_0, \varphi_0) \mathcal{F}^{\alpha}_{lm}[\bar{h}^{(i)}_{lm}, \bar{h}^{(i)}_{l\pm 1,m}, \bar{h}^{(i)}_{l\pm 2,m}, \bar{h}^{(i)}_{l\pm 3,m}]$$

Linear combination of the MP functions and their derivatives, including seven tensor harmonics modes.

Mode sum scheme

Reguralize and sum over l :

$$F^{\alpha}(\tau) = \sum_{l} F^{\alpha}_{\text{reg}}(x_p)$$

$$F^{\alpha,l}_{\text{reg}}(x_p) \equiv F^{\alpha,l}_{\text{full}}(x_p) - A^{\alpha} L - B^{\alpha}$$

$$A^t_{\pm} = \mp \frac{\mu^2 u^r}{r^2 f V}, \quad A^r_{\pm} = \mp \frac{\mu^2 \mathcal{E}}{r^2 V}, \quad A^{\varphi}_{\pm} = 0,$$

$$B^t = \frac{\mu^2 \mathcal{E} u^r}{\pi r^2 f V^{3/2}} \left[-\hat{K}(w) + 2(1-V)\hat{E}(w) \right]$$

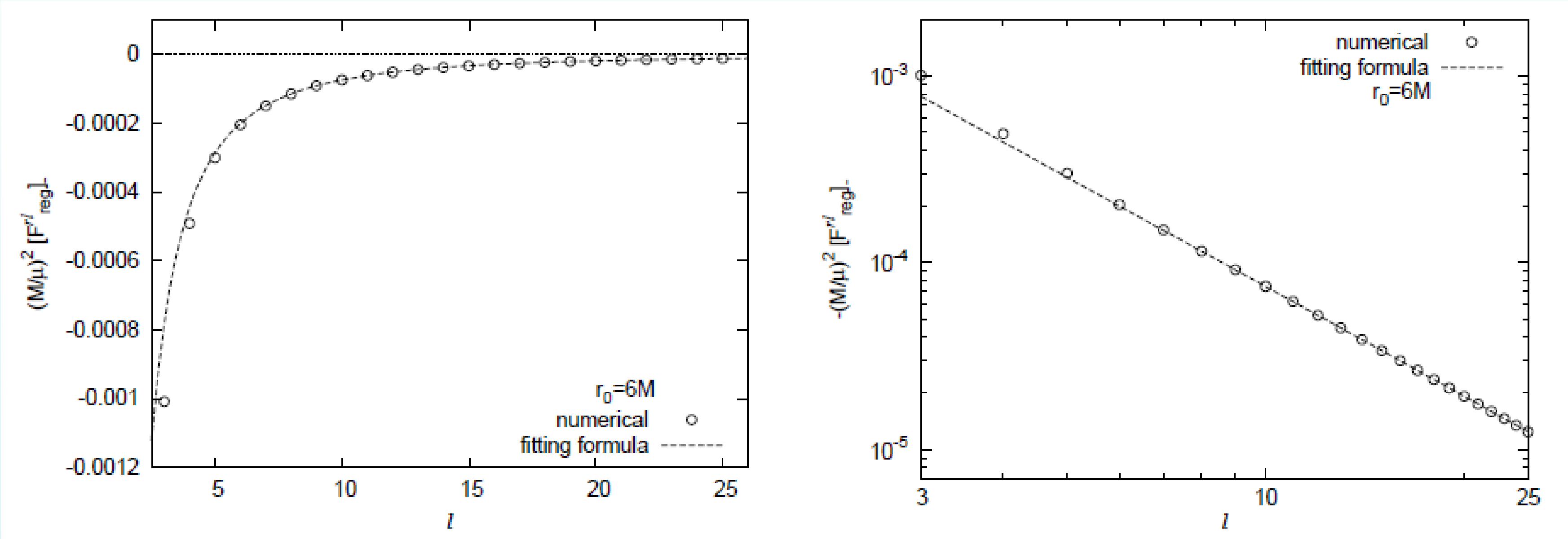
$$B^r = \frac{\mu^2}{\pi r^2 V^{3/2}} \left[\left((u^r)^2 - 2\mathcal{E}^2 \right) \hat{K}(w) + \left(\mathcal{E}^2 + (u^r)^2 (1-2V) \right) \hat{E}(w) \right]$$

$$w = \frac{\mathcal{L}^2}{r^2 + \mathcal{L}^2}, \quad f = 1 - \frac{2M}{r}, \quad V = 1 + \frac{\mathcal{L}^2}{r^2}$$

Large l-tail behaviour

For large l, the regularized full force modes behave like:

$$F^{\alpha l}_{\rm reg} = \frac{D^{\alpha}_{2}}{L^{2}} + \frac{D^{\alpha}_{4}}{L^{4}} + \cdots$$



Strong test for our regularization scheme!

Summation over multipole modes

To obtain the SF, take summation over l after regularization.

$$F^{\alpha} = \sum_{l=0}^{\infty} F_{\text{reg}}^{\alpha l} = \sum_{l=0}^{l_{\max}} F_{\text{reg}}^{\alpha l} + \sum_{l=l_{\max}+1}^{\infty} F_{\text{reg}}^{\alpha l}$$

Calculate finite number of modes, $0 \le l \le l_{\max}$ $(l_{\max} = 15)$

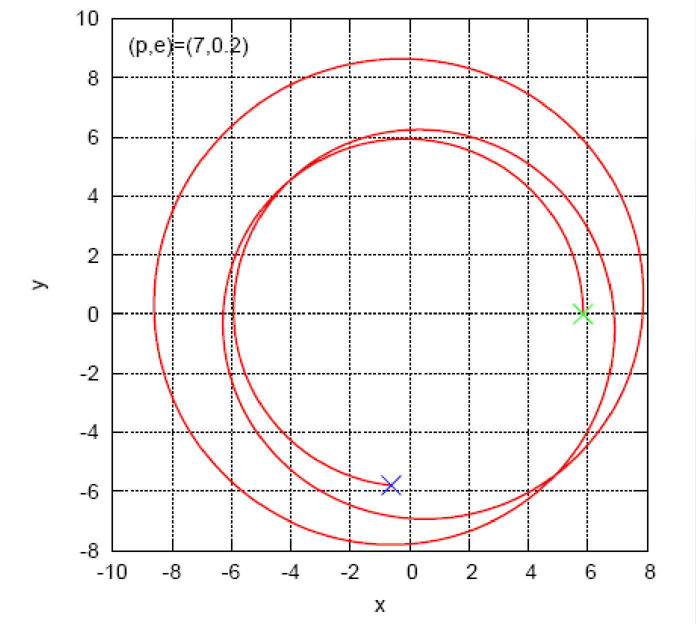
Derive a fitting formula for truncated part.

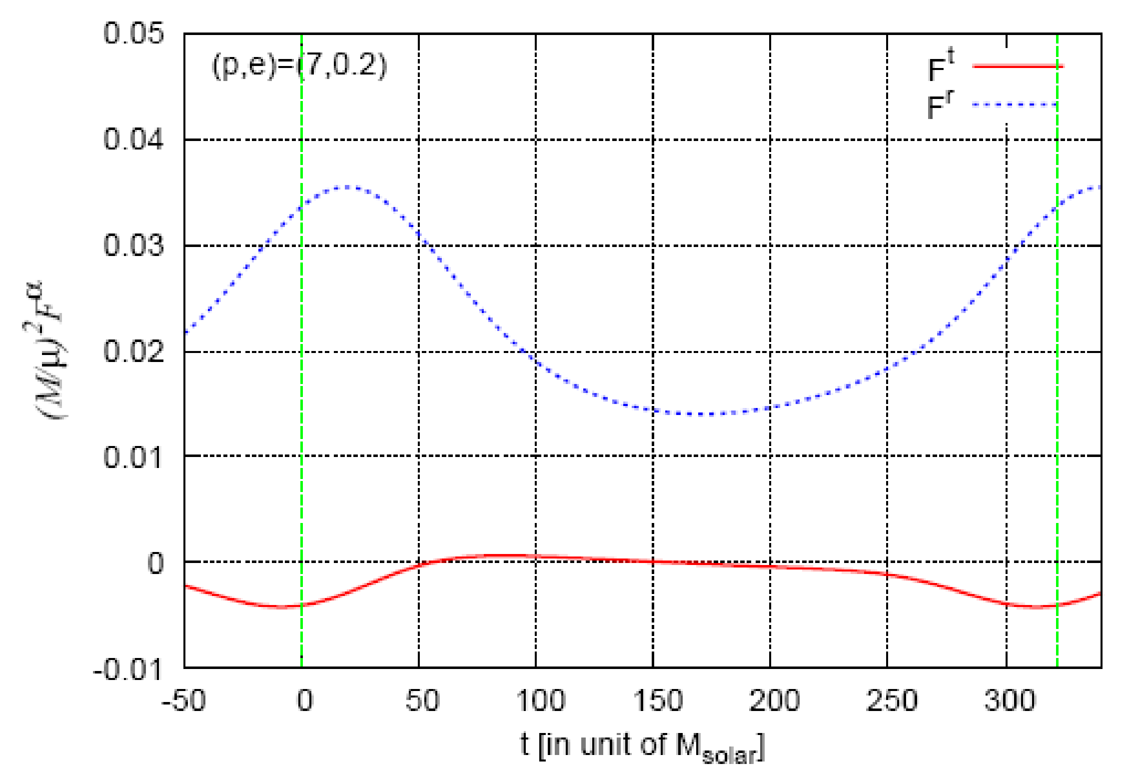
$$F_{\text{reg}}^{\alpha l} = \frac{D_2^{\alpha}}{L^2} + \frac{D_4^{\alpha}}{L^4} + \cdots$$

(Here we use six data points, to derive the fitting parameters.)

Example (moderate eccentric orbit)

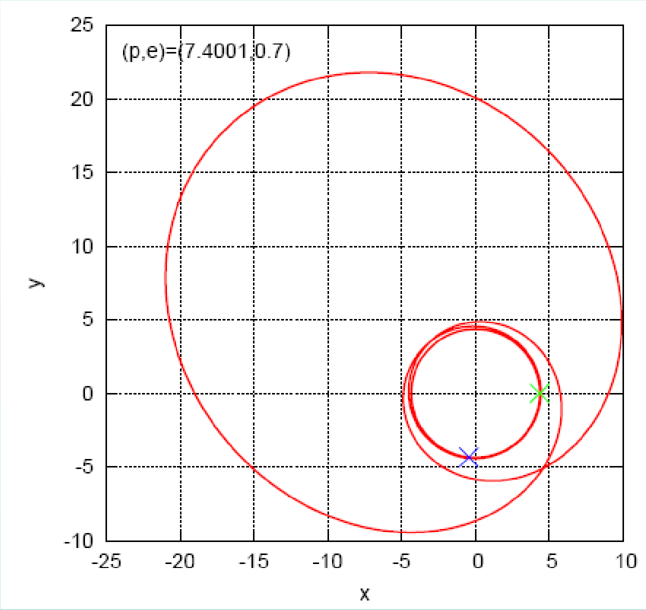
$(p, e) = (7, 0.2)$ case

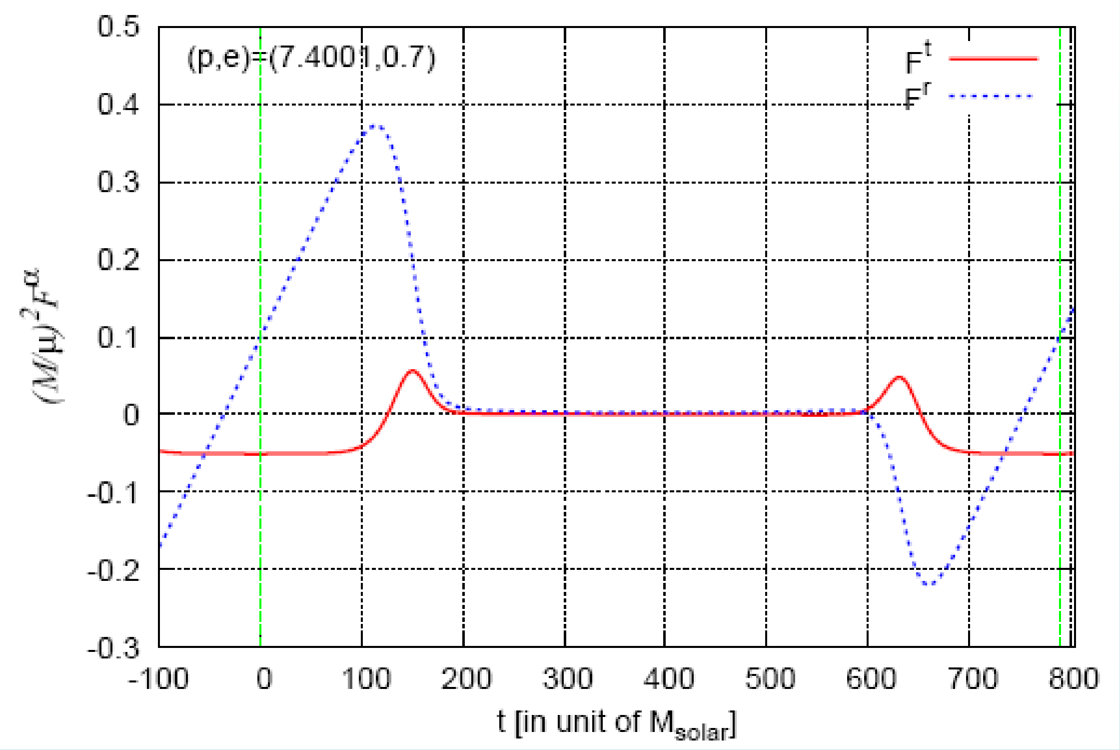




Example (zoom-whirl case)

$(p,e) = (7.4001, 0.7)$ case





Energy balance

t-component of local SF is related to the energy loss of the particle.

Energy balance formula (energy loss of particle) = (radiated energy)

$$\left\langle F_t / u_0^t \right\rangle = \left\langle \dot{E}_{\infty} \right\rangle + \left\langle \dot{E}_{\mathrm{EH}} \right\rangle \equiv \left\langle \dot{E}_{\mathrm{total}} \right\rangle$$

(p, e)=(7, 0.2) case

Energy loss	Radiated energy	rel. err.
$\left\langle F_t / u_0^t \right\rangle = -4.8976 \times 10^{-4}$	$-\left\langle \dot{E}_{total} \right\rangle = -4.89738 \times 10^{-4}$	0.02%
	$-\left\langle \dot{E}_{\infty} \right\rangle = -4.88438 \times 10^{-4}$ $-\left\langle \dot{E}_{BH} \right\rangle = -0.01300 \times 10^{-4}$	

[Estimated error = O(10^{-4})] [Estimated error = O(10^{-5})]

$$\frac{\left\langle \dot{E}_{EH} \right\rangle}{\left\langle \dot{E}_{\infty} \right\rangle} \approx 2 \times 10^{-3}$$

consistent with Martel's ('04)

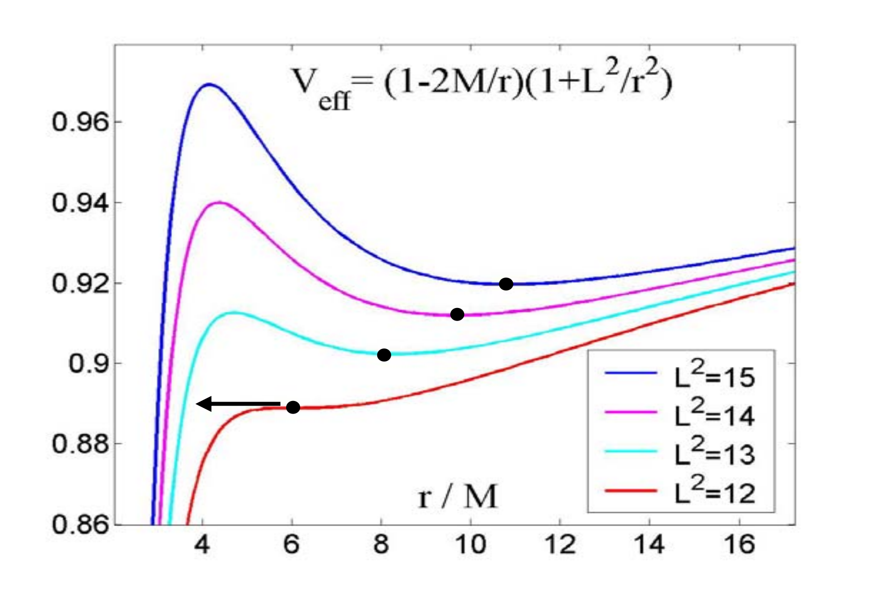
ISCO (test particle case)

Radial geodesic equation

$$\ddot{r}_p = -\frac{1}{2}\frac{\partial V(r)}{\partial r}\bigg|_{r=r_p} \; ; \quad V(r) = \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right)$$

For $L_z^2 > 12M^2$, a stable circular orbit exists.

$L_z^2 = 12M^2$ is the minimal value so that a stable circular orbit can exist.



Inner-most stable circular orbit (ISCO)

Consider the small oscillation around the minimal point.

$$\ddot{r}_p = -\omega_r^2 r_p \; ; \quad \omega_r^2 = \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\bigg|_{r=r_p}$$

ISCO is at the radius so that $\omega_r^2 = 0$.

For a test particle, $r_{isco}=6M$.

ISCO (including the conservative SF)

Conservative and dissipative pieces

$$F^r_{\text{cons/diss}}(t) \equiv \tfrac{1}{2}\left[F^r(t) \pm F^r(-t)\right], \quad F^t_{\text{cons/diss}}(t) \equiv \tfrac{1}{2}\left[F^t(t) \mp F^t(-t)\right]$$

Equation of radial motion with conservative SF

$$\ddot{r}_p = F_{\text{eff}} + F^r_{\text{cons}} \;; \quad F_{\text{eff}}(r_p; L) \equiv -\frac{1}{2}\frac{\partial V(r)}{\partial r}\bigg|_{r=r_p}$$

$$\dot{E} = -F_t^{\text{cons}} \,, \quad \dot{L} = -F_\varphi^{\text{cons}}$$

Corrected ISCO

Consider the small oscillation around a circular orbit.

$$\ddot{r}_p = -\omega_r^2 r_p \;; \quad \omega_r^2 = \frac{\partial}{\partial r_p}\left[F_{\text{eff}}(r_p; L(r_p)) + F_r^{\text{cons}}\right]$$

Again, we define ISCO radius so that $\omega_r^2 = 0$.

Correction of ISCO frequency

For a slightly eccentric orbit:

$$r_p(t) = r_0\left[1 + e\cos(\Omega_r t)\right]$$

$$F^r = F_0^r + e\left[F^r_{1,\mathrm{cons}}\cos(\Omega_r t) + F^r_{1,\mathrm{diss}}\sin(\Omega_r t)\right]$$

$$F_t = F_t^0 + e\Omega_r\left[F_t^{1,\mathrm{diss}}\cos(\Omega_r t) + F_t^{1,\mathrm{cons}}\sin(\Omega_r t)\right]$$

Conservative correction of ISCO (switch off the dissipation)

$$\omega_r^2 = 0$$

$$\Delta r_{isco} = (18M/\mu)\left[3MF_0^r - 6MF^r_{1,\mathrm{cons}} - F_t^{1,\mathrm{cons}}\right]_{r_0\to 6M} \approx -3.269\mu$$

$$\Delta\Omega_\phi/\Omega_\phi = -\frac{\Delta r_{isco}}{4M} - \frac{27M}{2\mu}F_0^r\Big|_{r_0\to 6M} \approx 0.4870(\mu/M)$$

Comparison to dissipative effect

If we consider the dissipative effect, the transition from inspiral to plunge occurs around the ISCO. [Ori-Thorne '00]

Bandwidth during transition

$$\frac{\Delta\Omega_{\rm diss}}{\Omega_{\rm isco}} \cong 4.3874\left(\frac{\mu}{M}\right)^{2/5}$$

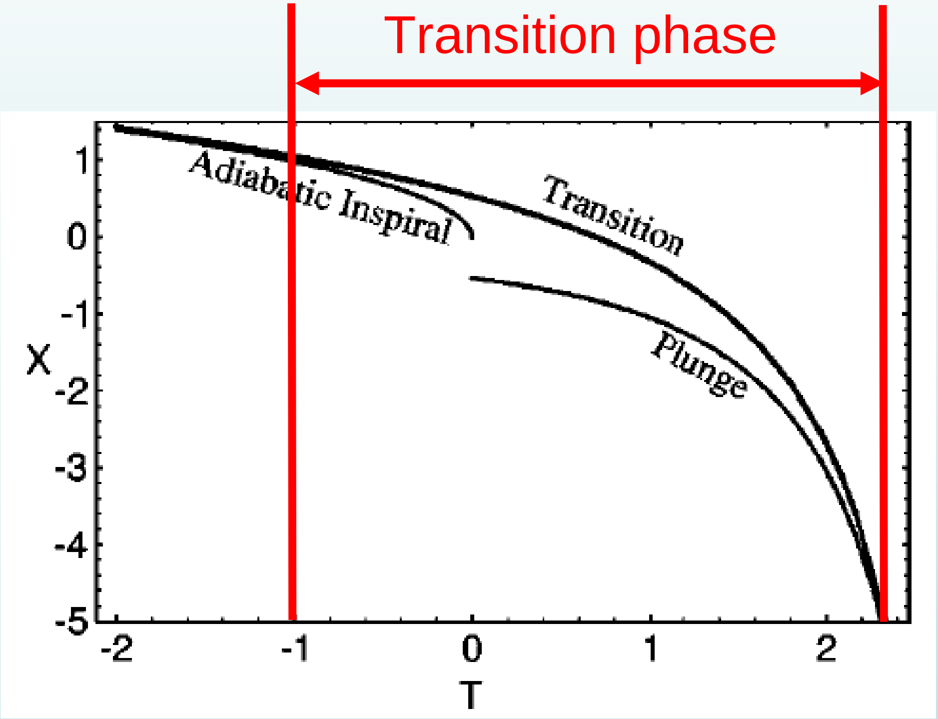
Conservative vs. Dissipative

$$\frac{\Delta\Omega_{\rm diss}}{\Delta\Omega_{\rm isco}} \approx 9\left(\frac{\mu}{M}\right)^{-3/5}$$

$$\approx 35830 \ \text{for}\ \eta = 10^{-6}$$

$$\approx 9000 \ \text{for}\ \eta = 10^{-5}$$

$$\approx 2261 \ \text{for}\ \eta = 10^{-4}$$



Summary

This work

Schwarzschild background case:

- Time-domain code to solve the Lorenz gauge MP
- Calculate the SF for circular and eccentric orbits
- Check the energy balance
- Evaluate the conservative correction of ISCO

Future works

Extension to Kerr geometry:

- Develop a time domain scheme for Kerr case
- Implement of "m-mode sum" scheme

Waveform including the SF effect:

- Include the SF effect to orbits
- 2nd order perturbation

Extension to Kerr case

Differences between scalar and gravitational cases

Gauge problem

It does not appear in our Lorenz gauge strategy.

Non-radiative modes

In multipole expansion, $l=0,1$ modes bother us.
But, in 'm-mode sum' scheme, it does not appear.

Complication of equations

The numerical scheme will be basically same as scalar case because the principal terms are decoupled in grav. case.

In principle, it is not difficult to extend the scalar code to grav. case.

Next plan

First step

reproduce Barack-Golbourn ('07) with some modifications:

From Schwarzschild to Kerr

To develop the finite difference scheme in Kerr
Comparison to FD results (done by Neils and Leor)

Replace the puncture function

To implement 'm-mode sum' scheme [Barack,Golbourn,NS '08]
To check the convergence of m-modes