Self force and radial fall

(History of fascinations and troubles)
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Plan of the talk

- Newtonian SF – Uniqueness of free fall - Equivalence principle
- Why radial fall (still) ?
- Repulsion and velocity at the horizon : the Schwarzschild controversy 1916-2008 (?)
- State of the art in the SF for radial fall
- Exact 1st order perturbations vis à vis self-force/pragmatic
- Continuity (Conditional) of the RWZ perturbations at the position of the particle
- Approximative correction of the waveforms by radiation reaction through energy balance (with Sofiane Aoudia)

Newtonian self-force (circular)

The Newtonian confusion gauge

$$F_C = F_g \qquad m_1 r_1 = m_2 r_2 \qquad m_2 << m_1$$

$$m_2 \frac{v_2^2}{r_2} = m_2 r_2 \omega_2^2 = \frac{m_1 m_2}{(r_1 + r_2)^2} \qquad \omega_2^2 = \frac{m_1}{r_2 (r_1 + r_2)^2}$$

$$m_2 \to 0$$

$$\omega_2^2 = \frac{m_1}{r_2^3 \left(1 + \frac{r_1}{r_2}\right)^2} = \frac{m_1}{r_2^3 \left(1 + \frac{m_2}{m_1}\right)^2} \approx \frac{m_1}{r_2^3} \left(1 - 2\frac{m_2}{m_1} + \dots\right)$$

r_2 fixed

$$\omega_2^2 = \frac{m_1}{r_2 (r_1 + r_2)(r_1 + r_2)} = \frac{m_1}{r_2^2 (r_1 + r_2)\left(1 + \frac{r_1}{r_2}\right)} = \frac{m_1}{r_2^2 (r_1 + r_2)\left(1 + \frac{m_2}{m_1}\right)} \approx \frac{m_1}{r_2^2 (r_1 + r_2)} \left(1 - \frac{m_2}{m_1} + \dots\right)$$

$r_1 + r_2$ fixed

The finitude of the mass may have ambiguos consequences but it is there…(unless one cancels it in a specific frame, a sort of equivalence principle – for which gravity disappears in a given coordinate system - in a Newtonian form)

Uniqueness of free fall

The Newtonian self acceleration

For circular motion Detweiler S., Poisson E., 2004. PRD, 69, 084019

Origin of coordinate system = centre of mass

Two bodies M, m and the point P have coordinates $\vec{\rho}, \vec{R}, \vec{r}$

In case of a single body the potential at P is : $\Phi_0(\vec{r}) = -\frac{M}{r}$ and the acceleration $\vec{g}_0(\vec{r}) = -\nabla\Phi_0(\vec{r}) = -\frac{M}{r^3}\vec{r}$

In case of two bodies the potential at P is : $\Phi(\vec{r}) = -\frac{M}{|\vec{r}-\vec{\rho}|} - \frac{m}{|\vec{r}-\vec{R}|}$ and for m<<M $\Phi(\vec{r}) = \Phi_0(\vec{r}) + \delta\Phi(\vec{r})$

where $$\boxed{\delta\Phi(\vec{r}) = \underbrace{-\frac{M}{|\vec{r}-\vec{\rho}|} + \frac{M}{r}} - \frac{m}{|\vec{r}-\vec{R}|}}$$

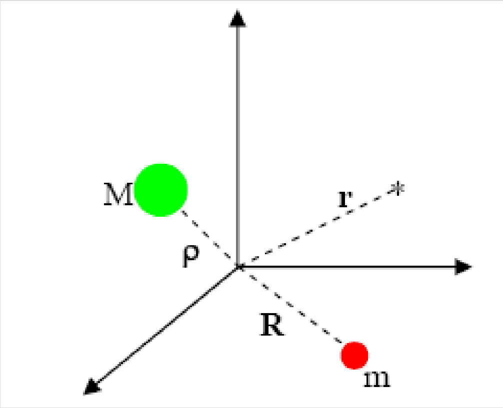
divergent at the particle but isotropic → no force = singular part $\Phi_S(r)$

$\Phi_R(\vec{r}) = -\frac{M}{|\vec{r}-\vec{\rho}|} + \frac{M}{r}$ regular part. Thus $\delta\Phi(\vec{r}) = \Phi_R(\vec{r}) + \Phi_S(\vec{r})$

For the centre of mass definition, we have $\vec{\rho} = -\frac{m}{M}\vec{R}$ Thus

$$\boxed{\Phi_R(\vec{r}) = -\frac{M}{\left|\vec{r} + \frac{m}{M}\vec{R}\right|} + \frac{M}{r} = -\frac{M}{|\vec{r}|\left|1 + \frac{m}{M}\frac{\vec{R}\vec{r}}{r^2}\right|} + \frac{M}{r} \approx -\frac{M}{r}\left(1 - \frac{m}{M}\frac{\vec{R}\vec{r}}{r^2}\right) + \frac{M}{r} = m\frac{\vec{R}\vec{r}}{r^3}}$$

Development of the regular part



and the acceleration is $\vec{g}(r) = -\nabla\Phi_R(\vec{r})$ Since: $\nabla = \frac{\partial}{\partial r}\frac{\vec{r}}{r}$

$$\begin{cases} \vec{g}(r) = \nabla\Phi_R(\vec{r}) = -\left[\frac{m}{r^3}\nabla\left(\vec{R}\vec{r}\right) + m\vec{R}\vec{r}\nabla\left(\frac{1}{r^3}\right)\right] = \\ = -\left[\frac{m}{r^3}\vec{R} - 3m\left(\vec{R}\vec{r}\right)\frac{\vec{r}}{r^5}\right] = m\frac{3\left(\vec{R}\vec{r}\right)\vec{r} - r^2\vec{R}}{r^5} \end{cases}$$

At the particle position:

$$\boxed{\begin{aligned} g_R\left(\vec{R}\right) &= 2m\frac{\vec{R}}{R^3} \\ g_0\left(\vec{R}\right) &= -M\frac{\vec{R}}{R^3} \end{aligned}}$$

Therefore

$$\boxed{g\left(\vec{r}\right) = g_0\left(\vec{r}\right) + g_R\left(\vec{r}\right) = -\frac{M - 2m}{r^3}\vec{r}}$$

The equivalence principle as uniqueness of acceleration doesn’t hold. It should state:

All bodies behave independently of their mass….<u>if we neglect their mass</u>

The (multifaceted) equivalence principle

(approximation)

- I. All bodies fall with the same acceleration independently from the value of their mass (sometimes referred as the uniqueness of accceleration principle). **SEE PREVIOUS SLIDES**

- II. Bodies equally accelerate under inertial or gravitational forces. **SEE NEXT SLIDE**

- III. Bodies equally accelerate independently from the composition of their masses **(UNDER EXPERIMENTAL SCRUTINY : STEP thanks to Kaluza Klein, strings, supersummetries)**

In general relativity, the language style gets more sophisticated:

- IV. At every spacetime point in an arbitrary gravitational field, it is possible to choose a locally inertial coordinate system such that, within a sufficiently small region around the point in question, the laws of nature take the same form as in an unaccelerated coordinate system. The laws of nature concerned might be all laws (strong equivalence principle), or solely those dealing inertial motion (weak equivalence principle version) or all laws but those dealing inertial motion (semi-strong equivalence principle) **(PONT MASSES DON'T EXIST, THUS TIDAL FORCES KILL EP)**

- V. A freely moving particle, of negligeable mass and size, follows a geodesic of spacetime **(GEODESIC g + h^R) .**

The Equivalence principle

(it feels comfortable not to be alone)

…Perhaps they speak of the Principle of Equivalence. If so, it is my turn to have a blank mind, for I have <u>never been able to understand</u> this Principle…

Does it mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world-line. <u>Space-time is either flat or curved</u>….
(the converse is also far reaching)
The Principle of Equivalence performed the essential office of <u>midwife</u> at the birth of general relativity. ..
I suggest that the <u>midwife be now buried</u> with appropriate honours…

J.L. Synge

Synge J.L., 1960. Relativity: the general theory, North-Holland Publishing Co.

Or else see Rohlrich F., 2000. Found. Phys., { 30}, 621.

"**The impasse (or have the relativists fear of free fall?** [..] the problem of the free fall of bodies in the frame of [..] the Schwarzschild solution. More than any other, **this question gathers the optimal conditions of interest, on the technical and epistemological levels**, without inducing nevertheless a focused concern by the experts. Though, is it necessary to emphasise that **it is a first class problem to which classical mechanics has always showed great concern ... from Galileo; which more is the reference model expressing technically the paradigm of the lift in free fall dear to Einstein ? The matter is that the case is the most elementary, most natural, an extremely simple problem ...apparently but which raises extremely delicate questions to which only the less conscious relativists believe to reply with answers** [..] **Exactly the type of naive question that best experts prefer to leave in the shadow, in absence of an answer that has to be patently clear to be an answer.** Without doubts, it is also the reason for which this question induces a very moderate interest among the relativists ..."}

Eisenstaedt J.,1987.
Arch. Hist. Exact Sci., 37, 275.

Why radial fall ? (still)

- Epistemology/Reference (How do we evaluate progress on knowledge?)
- History’s markings (Pisa tower, Cambridge apple tree, Einstein lift)
- No compromise (1 no adiabatic escape* 2 coupling r,t) most delicate issues are present. No cumulation of effects does not mean their not existence).
- Learning case for highly eccentric ? (No claim)
- The final part is plunge (initial velocity ?)
- State of the art (Is it satisfactory ?) Before SF and after SF

* The non-adiabatic gravitational waveforms are aimed by the self-force community since they express

i) the physics closer to the black hole horizon

ii) the most complex trajectories, the most tantalizing theoretical questions.

Repulsion : the Schwarzschild controversy 1916-2008 (?)

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Repulsion : the Schwarzschild controversy 1916-2008 (?)

Unrenormalised acceleration 1) $\dfrac{d^2r}{dt^2}$

Renormalised acceleration 2) $\dfrac{d^2R}{dT^2}$

A semi-renormalised acceleration 3) $\dfrac{d^2R}{dt^2}$ $\dfrac{d^2r}{dT^2}$

Despite the mathematical simplicity, confusion and controversy dominated this debate for almost 100 years and invested notorious scientists.
Is there repulsion ? Does a particle reaches the speed of light at the horizon ?

Four types of measurements can be envisaged: local measurement of time dT, non-local measurement of time dt, local measurement of length dR, non-local measurement of length dr. Locality is somewhat a loose definition but hints at those measurements affected by gravity (of the SD black kole), while non-locality hints at measurements not affected by gravity (of the SD black hole)[10]. Therefore, for determining (velocities and) accelerations, four possible combinations do exist:

Coordinates/ Initial conditions/ 2M / Wording / Passion for a gauge

The repulsive gang (Kerr metric excluded): Droste *, Hilbert, Bauer, Page, Eddington, von Laue, de Jans, McVittie, Jaffe and Shapiro, Treder and Fritz, McGruder, Arifov +…

* 1916 independent derivation of the SD metric

The first to introduce the idea of gravitational repulsion was Droste [25, 26]. He defines:

$$dR = \frac{dr}{\sqrt{1 - \frac{2M}{r}}} \tag{8}$$

which, when integrated, Droste calls the δ distance from the horizon. This quantity is derived from the SD metric posing $dt = 0$, delicate operation since the relation between proper and coordinate times varies in space [36], and thus it may be accepted only for a static observer (obviously the notion of static observer raises in itself a series of questions [15].). Droste derives that in radial trajectories, the acceleration in coordinate time is given by (A is a constant of motion, equal to unity for a particle falling with null speed at infinity):

$$\frac{d^2R}{dt^2} = -\frac{M}{r^2}\left[\sqrt{1-\frac{2M}{r}} - \frac{2\left(\frac{dR}{dt}\right)^2}{\sqrt{1-\frac{2M}{r}}}\right] = \frac{M}{r^2}\left(1-2A+\frac{4M}{r}\right)\sqrt{1-\frac{2M}{r}} \tag{9}$$

From eq.(9) two conditions may be derived for the semi-renormalised acceleration, for either of which the repulsion (the acceleration is positive) occurs: for $A = 1$, $r < 4M$ or else $dR/dt > \sqrt{1/2}\sqrt{1-2M/r}$. Instead in his thesis [25], Droste uses the unrenormalised acceleration:

$$\frac{d^2r}{dt^2} = -\frac{M}{r}\left(\frac{r-2M}{r^2} - \frac{3\dot{r}^2}{r-2M}\right) \tag{10}$$

for which repulsion occurs if, for a particle falling from infinity with null initial velocity, $r < 6M$ or else $dr/dt > 1/\sqrt{3}(1-2M/r)$.

In the later and French editions of his book, von Laue [50] writes the radial geodesic in proper time but it is only in 1936 that Drumaux exploits it. The local or proper time given by:

$$dT = \sqrt{1 - \frac{2M}{r}} dt \tag{11}$$

allows him to state that the velocity and the acceleration in proper time, respectively given by (E is the orbital energy defined by $E = dt/dT(1 - 2M/r)$:

$$\frac{dR}{dT} = \frac{1}{\left(1 - \frac{2M}{r}\right)} \frac{dr}{dt} = \frac{1}{E} \sqrt{E^2 - 1 + \frac{2M}{r}} \tag{12}$$

$$\frac{d^2R}{dT^2} = -\frac{M}{r^2} \frac{1}{E^2} \sqrt{1 - \frac{2M}{r}} = -\frac{M}{r^2} \frac{1}{\sqrt{1 - \frac{2M}{r}}} \tag{13}$$

do not show any repulsion. This result is confirmed by Whittaker [52][12]. Nevertheless McVittie, almost thirty years later [42], still reaffirms that the particle is pushed away by the central body; repulsion resurfaces according to Jaffe and Shapiro, in a rather obscure paper [33], Treder [48] and in cooperation with Fritze [49], Arifov [16, 17].

State of the art 2009 a.C.   12 a.S.-F.

"The radial component of the SF is found to point *inward* (i.e., toward the black hole) throughout the entire plunge. This seems to be a universal feature which does not depend on the starting point r_0. Consequently, the work done by the SF on the particle is positive, resulting in that the energy parameter *E increases* throughout the plunge. It is important to stress, however, that this result will be attached to our specific choice of gauge ~as opposed to the energy flux at infinity, which is gauge invariant! "(Barack L., Lousto C., 2002. PRD, 66, 061502)

" the resulting reaction force is repulsive " (Lousto C.O., 2001. CQG, 18, 3989)
"Radiation reaction effects become more important as the particle approaches the maximum of the Zerilli's potential (around *r* max 3.1 *M*. They tend to decelerate the particle with respect to the zeroth order (Schwarzschild) geodesics. This is what one would qualitatively expect *a priori* since the system is losing energy and momentum in the form of gravitational radiation." (Lousto C.O., 2000. PRL, 84, 5251)

Why the (apparent) discrepancy ?
- Proper time / Coordinate time
- Geodesic deviation terms absent in SF paper

6.2 The pragmatic approach

Lousto C.O., 2000. PRL, 84, 5251; 2001, CQG, 18, 3989; Spallicci A., Aoudia S., 2004.CQG, 21, S563.

The straightforward pragmatic approach [125, 126, 128] is is the direct implementation of the geodesic in the full metric (background + perturbations) and it is coupled to a renormalisation by the Riemann-Hurwitz ζ function. Though the application of the ζ function is somewhat artificial and the pragmatic method is somewhat naive, the latter has the merit of a clear identification of the different factors participating in the motion.

Dealing only with time and radial components, two geodesic equations can be written and then combined into a single one, after elimination of the geodesic parameter. Thus, it consists of the computation of the coordinate acceleration, given by, for radial fall, by the sole radial component:

$$\ddot{z}_p = \Gamma^t_{rr}\dot{z}^3_p + \left(2\Gamma^t_{tr} - \Gamma^r_{rr}\right)\dot{z}^2_p + \left(\Gamma^t_{tt} - 2\Gamma^r_{tr}\right)\dot{z}_p - \Gamma^r_{tt} \qquad (46)$$

where $\Gamma^\alpha_{\beta\gamma}$ refers to the full metric and z_p is given by eq. (47). In eq. (46), it is considered:

- the full field $\bar{g}_{\mu\nu}(r,t)$ previously defined;
- the displacement Δz, difference between perturbed (z_p) and unperturbed ($z_u(t)$) positions, and its time coordinate derivatives:

$$z_p = z_u(t) + \Delta z \qquad \dot{z}_p = \dot{z}_u + \Delta\dot{z} \qquad \ddot{z}_p = \ddot{z}_u + \Delta\ddot{z} \qquad (47)$$

- the Taylor development of the field and its spatial derivative:

$$\bar{g}_{\mu\nu}\,|_{z_p} = \bar{g}_{\mu\nu}\,|_{z_u(t)} + \Delta z\bar{g}_{\mu\nu,r}\,|_{r=z_u(t)} \qquad \bar{g}_{\mu\nu,r}\,|_{z_p} = \bar{g}_{\mu\nu,r}\,|_{z_u(t)} + \Delta z\bar{g}_{\mu\nu,rr}\,|_{r=z_u(t)} \qquad (48)$$

$$\frac{[h^{(1)}]^2}{g} \simeq \frac{h^{(2)}}{g} \ll \frac{h^{(1)}}{g} \simeq \frac{\Delta \dot{z}}{\dot{z}_p} \simeq \frac{\Delta z}{z_p} \tag{49}$$

Then the coordinate acceleration correction is given by a development up to 1^{st} order for all quantities:

$$\Delta \ddot{z} = \alpha_1 (g, \dot{z}_u) \Delta z + \alpha_2 (g, \dot{z}_u) \Delta \dot{z} + \alpha_6 (h, \dot{z}_u) \tag{50}$$

which corresponds to the expression in [125, 126][22]. All terms (see Tabs. 1-2) in eq. (50) are of $1/M$ order; the terms $\alpha_{1,2}\Delta$ represent the background field evaluated on the perturbed trajectory (they may alternatively be interpreted as producing the geodesic deviation of two particles initially separated on the radial axis by a Δz distance or two particles at initial differential speed $\Delta \dot{z}$); α_6 represents the perturbed field on the background trajectory. The expression in Tab.2 is in Regge-Wheleer gauge and thus $H_0 = H_2$ and $K = 0$ as in head-on geodesics.

The particle determines in first instance the emission of radiation $h_{\alpha\beta}$, which after backscattering by the black hole potential, interacts with the particle itself resulting into a change in acceleration (α_6 being a h and derivatives depending term). The latter places the particle elsewhere from where it should have been, that is $z_u(t)$. The field is thus to be evaluated at this new position resulting into a further variation in acceleration ($\alpha_1 \Delta z$ and $\alpha_2 \Delta \dot{z}$ being $g_{\alpha\beta}$ and its derivatives depending terms).

Radial geodesic in the perturbed metric $g_{\alpha\beta} + h_{\alpha\beta}$ to be regularised

$$\frac{d^2x^\alpha}{d\lambda^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0 \qquad \Gamma^\alpha_{\beta\gamma}=\frac{1}{2}g^{\alpha\sigma}[g_{\beta\sigma,\gamma}+g_{\gamma\sigma,\beta}-g_{\beta\gamma,\sigma}]$$

$$\ddot{z}_p = \Gamma^t_{rr}\dot{z}_p^3 + \left(2\Gamma^t_{tr}-\Gamma^r_{rr}\right)\dot{z}_p^2 + \left(\Gamma^t_{tt}-2\Gamma^r_{tr}\right)\dot{z}_p - \Gamma^r_{tt}$$

$$z_p = z_u(t)+\Delta z \qquad \dot{z}_p = \dot{z}_u + \Delta\dot{z} \qquad \ddot{z}_p = \ddot{z}_u + \Delta\ddot{z}$$

The field is Taylor developed around the particle

$$\bar{g}_{\mu\nu}\,|_{z_p} = \bar{g}_{\mu\nu}\,|_{z_u(t)} + \Delta z\,\bar{g}_{\mu\nu,r}\,|_{r=z_u(t)} \qquad \bar{g}_{\mu\nu,r}\,|_{z_p} = \bar{g}_{\mu\nu,r}\,|_{z_u(t)} + \Delta z\,\bar{g}_{\mu\nu,rr}\,|_{r=z_u(t)}$$

$$\ddot{z}_p = a_0 + a_1\Delta z + a_2\Delta\dot{z} + a_6$$

Total acceleration

$$a_0 = -\frac{M}{r}\left(\frac{r-2M}{r^2}-\frac{3\dot{r}^2}{r-2M}\right)$$

a_1 is the classic Schwarzschild; a_2, a_3 are not perturbation dependent.

RW

$$a_1 = \left\{-\frac{2M}{r^2}\left[\frac{3M}{r^2}-\frac{1}{r}+\frac{3(r-M)}{(r-2M)^2}\dot{r}^2\right]\right\}\Delta r \qquad a_2 = \left[\frac{6M}{r(r-2M)}\dot{r}\right]\Delta\dot{r}$$

$$a_6 = \frac{1}{r-2M}\left[\frac{r^2}{2(r-2M)}\dot{H}_0^l - \frac{M}{r-2M}H_1^l - rH_1'^{l}\right] - \frac{3}{2}H_0'^{l}\dot{r}^2 - 3\left(\frac{1}{2}\dot{H}_0^l - \frac{M}{r^2}H_1^l\right)\dot{r}$$

Lousto C.O., 2000. PRL, 84, 5251; 2001, CQG, 18, 3989; Spallicci A., Aoudia S., 2004.CQG, 21, S563.

Table 1 First order radial fall coordinate time geodesic terms

$\Gamma^t_{rr}\dot{z}^3$	$2\Gamma^t_{tr}\dot{z}^2$	$-\Gamma^r_{rr}\dot{z}^2$	$\Gamma^t_{tt}\dot{z}$	$-2\Gamma^r_{tr}\dot{z}$	$-\Gamma^r_{tt}$
	$g^{tt}g_{tt,r}\dot{z}_p^2$	$-\frac{1}{2}g^{rr}g_{rr,r}\dot{z}_p^2$			$\frac{1}{2}g^{rr}g_{tt,r}$
		α_0			
	$g^{tt}_{,r}g_{tt,r}\dot{z}_p^2$	$-\frac{1}{2}g^{rr}_{,r}g_{rr,r}\dot{z}_p^2$			$\frac{1}{2}g^{rr}_{,r}g_{tt,r}$
	$g^{tt}g_{tt,rr}\dot{z}_p^2$	$-\frac{1}{2}g^{rr}g_{rr,rr}\dot{z}_p^2$			$\frac{1}{2}g^{rr}g_{tt,rr}$
		α_1			
	$2g^{tt}g_{tt,r}\dot{z}_p$	$-g^{rr}g_{r,r}\dot{z}_p$			
		α_2			
	$2g^{tt}g_{tt,r}\dot{z}_p$	$-g^{rr}g_{r,r}\dot{z}_p$			
$g^{tt}h_{tr,r}\dot{z}_p^3$	$-h^{tt}g_{tt,r}\dot{z}_p^2$	$\frac{1}{2}h^{rr}g_{rr,r}\dot{z}_p^2$	$\frac{1}{2}g^{tt}h_{tt,t}\dot{z}_p$	$-g^{rr}h_{rr,t}\dot{z}_p$	$-g^{rr}h_{rt,t}$
$-\frac{1}{2}g^{tt}h_{rr,t}\dot{z}_p^3$	$g^{tt}h_{tt,r}\dot{z}_p^2$	$-\frac{1}{2}g^{rr}h_{rr,r}\dot{z}_p^2$	$\frac{1}{2}h^{tr}g_{tt,r}\dot{z}_p$	$h^{rt}g_{tt,r}\dot{z}_p$	$-\frac{1}{2}h^{rr}g_{tt,r}$
$-\frac{1}{2}h^{tr}g_{rr,r}\dot{z}_p^3$					$\frac{1}{2}g^{rr}h_{tt,r}$
		α_6			

Radial geodesic in full spacetime (gauge independent)

The C^0 continuity class of the metric perturbations allows to deal with the divergence with l of the α_6 term [125, 126]. Indeed, the infinite sum over the finite multipole components contributions leads to the problem of dealing infinities in the results. One way of regularising this sum is to subtract to each mode precisely the $l \to \infty$ contribution, since for ever larger l the metric perturbations tend to an asymptotic behaviour. Thus, the subtraction from each mode of the $l \to \infty$ leads to a convergent series. The renormalisation by the Riemann-Hurwitz ζ function was proposed first in [125, 126] and then extended to higher orders in [128]. For $L = l + 0.5$, it can be shown that:

$$\alpha_6 = \sum_{l=0}^{\infty} \alpha_6^l \qquad \alpha_6^l = \alpha_{6\pm}^a L + \alpha_6^b L^0 + \alpha_{6\pm}^c L^{-1} + \alpha_6^d L^{-2} + O(L^{-3}) \quad (51)$$

eq. (51) is casted to have a similar form to the mode-sum expression. The average of $\alpha_{6\pm}^a$ and $\alpha_{6\pm}^c$ vanish at the position of the particle, whereas $\sum_{l=0}^{\infty} \alpha_6^b (l+0.5)^0$ determines the divergence.

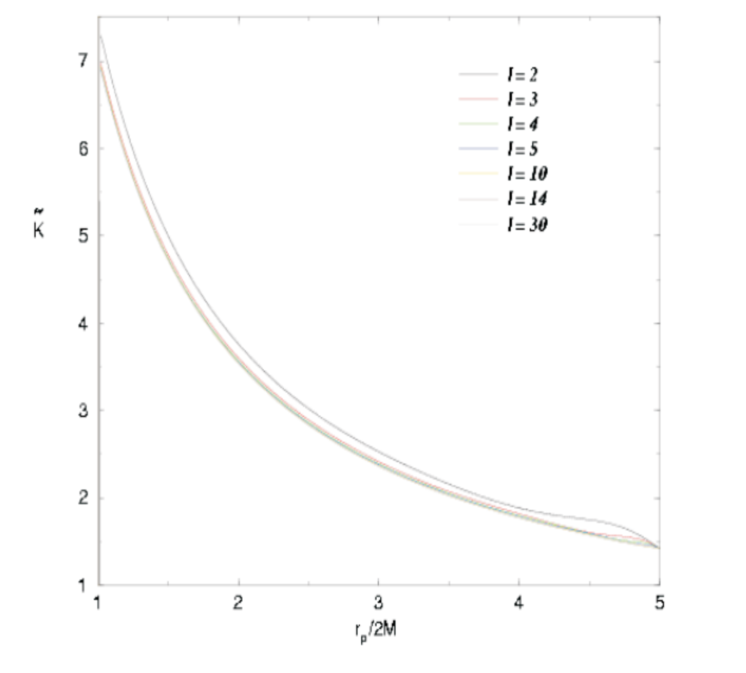
The Riemann ζ function [131] and its generalisation, the Hurwitz ζ function [132], are defined by:

$$\zeta(s) = \sum_{l=1}^{\infty} (l)^{-s} \qquad \zeta(s,a) = \sum_{l=0}^{\infty} (l+a)^{-s} \quad (52)$$

where in our case $a = 0.5$. Two special values of the Hurwitz functions $\zeta(0, 0.5) = 0$ and $\zeta(2, 0.5) = 1/2\pi^2$ nullify the divergent term and determine that the term $\sum_{l=0}^{\infty} \alpha_6^d L^{-2}$ gets a finite value, respectively.

Regularisation (method zeta)

For n>l >3 modes converge to the same behaviour



$$\tilde{h}^{l}_{\alpha\beta} \simeq \tilde{h}^{n}_{\alpha\beta} \simeq \tilde{h}^{\infty}_{\alpha\beta} \quad \left(\tilde{h}^{l}_{\alpha\beta} = \sqrt{(2l+1)} \;\; h^{l}_{\alpha\beta}\right)$$

$$h^{l} \simeq \frac{\tilde{h}^{\infty}}{\sqrt{(2l+1)}}$$

$$h^{l} = \left(h^{l} - \frac{\tilde{h}^{\infty}}{\sqrt{2l+1}}\right) + \frac{\tilde{h}^{\infty}}{\sqrt{2l+1}}$$

$$h = \sum_{l,m}\left[h^{l} Y_{lm}\right] = \sum_{l}\left[\left(h^{ren\ l} + \frac{\tilde{h}^{\infty}}{\sqrt{2l+1}}\right)\sqrt{\frac{2l+1}{4\pi}}\right]$$

$$\sum_{l}\left(\frac{\tilde{h}^{\infty}}{\sqrt{2l+1}}\frac{\sqrt{2l+1}}{\sqrt{4\pi}}\right) = \frac{\tilde{h}^{\infty}}{\sqrt{4\pi}}\sum_{l}\left(l+\frac{1}{2}\right)^{0} = \frac{\tilde{h}^{\infty}}{\sqrt{4\pi}}\,\zeta\left(0,\frac{1}{2}\right) = 0$$

Regularisation by subtraction of the divergent part

It is applicable to RWZ gauge (and any gauge)
Used in quantum mechanics and field theory (vacuum expectation value) + others
Drawback: Based on not evident mathematical properties of the Z function

Objectives

1) Comparison self-force versus pragmatic via 1st order perturbation (GW rigorous derivation)

2) Visibility in coordinate time

$$u^\gamma \nabla_\gamma (u^\beta \nabla_\beta Z^\alpha) = -R_{\beta\gamma\delta}{}^\alpha u^\beta Z^\gamma u^\delta - (g^{\alpha\beta} + u^\alpha u^\beta)(\nabla_\delta h^{\text{tail}}_{\beta\gamma} - \frac{1}{2}\nabla_\beta h^{\text{tail}}_{\gamma\delta}) u^\gamma u^\delta \tag{57}$$

$$F^\alpha_{self} = m\frac{Du^\alpha}{d\tau} = \frac{d^2x^\alpha}{d\tau^2} + {}^0\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$$

$$\frac{d^2t}{d\tau^2} = F^t_{self} - m\,{}^0\Gamma^t_{\beta\gamma}u^\beta u^\gamma = -mg^{t\beta}\left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\gamma\delta\,;\beta}\right)u^\gamma u^\delta - mk^t$$

* = tail or regular

$$\frac{d^2r}{d\tau^2} = F^r_{self} = -m\,{}^0\Gamma^r_{\beta\gamma}u^\beta u^\gamma - mg^{r\beta}\left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\gamma\delta\,;\beta}\right)u^\gamma u^\delta - mk^r$$

$$k^\alpha = \left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\delta\gamma\,;\beta}\right)u^\alpha u^\beta u^\gamma u^\delta$$

Since:

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} \qquad\qquad \frac{d^2r}{d\tau^2} = \frac{dr}{dt}\frac{d^2t}{d\tau^2} + \frac{d^2r}{dt^2}\left(\frac{dt}{d\tau}\right)^2$$

after some computation:

$$m\frac{d^2r}{dt^2} = m\left[{}^0\Gamma^t_{\beta\gamma}v^\beta v^\gamma_+ g^{t\beta}\left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\gamma\delta\,;\beta}\right)v^\gamma v^\delta\right]\dot{z}_u(t)$$

$$-m{}^0\Gamma^r_{\beta\gamma}v^\beta v^\gamma_- m g^{r\beta}\left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\gamma\delta\,;\beta}\right)v^\gamma v^\delta \qquad (64)$$

The term k^α disappears when the self-force is expressed in coordinate time. Furthermore, a tedious computation shows that eq. (65) 'expresses the self-force in coordinate time and it is nothing else than the α_6 term of eq. (50) apart regularisations by mode-sum or Riemann-Hurwitz ζ function:

$$\alpha_6 \leftrightarrow g^{t\beta}\left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\gamma\delta\,;\beta}\right)v^\gamma v^\delta \dot{z}_u(t) - g^{r\beta}\left(h^*_{\beta\gamma\,;\delta} - \frac{1}{2}h^*_{\gamma\delta\,;\beta}\right)v^\gamma v^\delta \qquad (65)$$

The recasting of the Riemann tensor term and of the left-hand side of eq. (57) into coordinate time should show the equivalence to the $\alpha_{1,2}$ terms in eq. (50).

τ

$$u^\gamma \nabla_\gamma (u^\beta \nabla_\beta Z^\alpha) = -R_{\beta\gamma\delta}{}^\alpha u^\beta Z^\gamma u^\delta - (g^{\alpha\beta} + u^\alpha u^\beta)(\nabla_\delta h^{\mathrm{tail}}_{\beta\gamma} - \frac{1}{2}\nabla_\beta h^{\mathrm{tail}}_{\gamma\delta}) u^\gamma u^\delta \tag{57}$$

t

$\alpha_1 \Delta z$ $\alpha_2 \Delta \dot{z}$

α_6

Lousto C.O., 2000. PRL, 84, 5251

Barack L., Lousto C., 2002. PRD, 66, 061502)

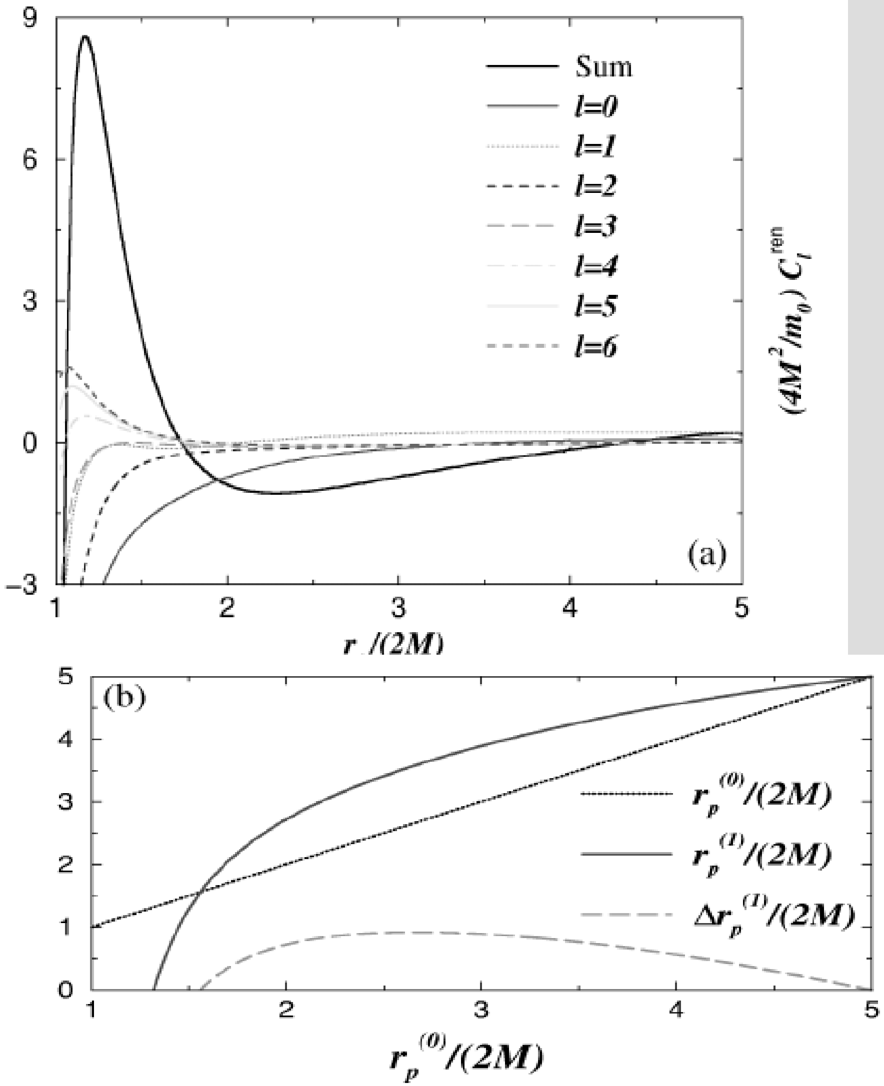
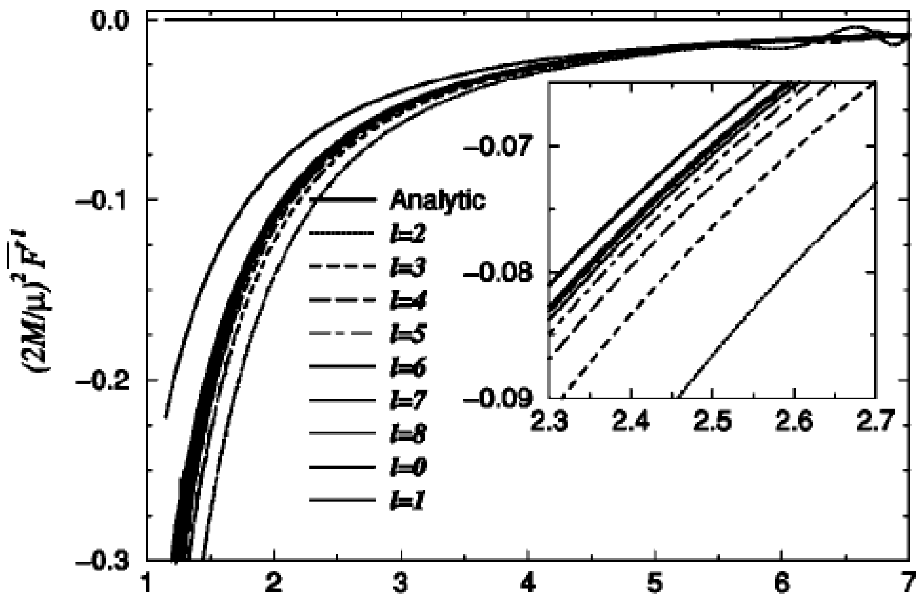


FIG. 2. The renormalized radiative piece of the reaction on $\ddot{r}_p$. We show the regularized sum over $\ell \leq 10$ multipole contributions to C, as defined in Eq. (5). Below, the first-order trajectory $r_p^{(1)}$, for $m_0 = 0.1M$, is compared to the zeroth order one, $r_p^{(0)}$.



(a)

$$F^{rl}_{\rm reg} = -\frac{15}{16}\mu^2 \frac{E^2}{r_p^2}\left(E^2 + \frac{4M}{r_p} - 1\right)L^{-2} + O(L^{-4}),$$

$$F^{l}_{t\,\rm reg} = -\frac{15}{16}\mu^2 E \frac{d}{d\tau}\left(\frac{\dot{r}_p^2}{r_p}\right)L^{-2} + O(L^{-4}),$$

Why the (apparent) discrepancy ?
- Proper time / Coordinate time
- Geodesic deviation terms absent in SF paper

Where are we (Orléans) ?

Determination of regularisation parameters via mode-sum and confirmation of existing results (Aoudia thesis)

Development of a time domain numerical code for solving Regge-Wheeler-Zerilli equation. Waveforms at infinity like LP-MP

Comparison between pragmatic and self-force expressions (of relevance also in the frame suggested by the GW "self-consistent" prescription). Role of geodesic deviations.

Extension of the Riemann-Hurwitz Z function regularisation for higher orders, Class. Quantum. Grav., 21, S563.

We got stuck in the computation of the perturbations at the position of the particle and suspected for a while the perturbations not being C° ↓

Approximative determination of the corrections in the waveforms due to radiation reaction via energy balance for masses in radial fall.

In radial fall, it has been shown by two different heuristic arguments [125, 127] that the metric perturbations are of C^0 continuity class at the location of the particle. One argument [125] is based on the integration over r of the Hamiltonian constraint, which is the tt component of the Einstein equations (eq.[C7a] in [71]) and the resulting leading order of $\partial_r \Psi$; the other [127] on the structure of selected even perturbations equations. In the Appendix, a more stringent demonstration on the C^0 continuity is given in terms of the jump conditions that the wavefunctions and derivatives have to satisfy to guarantee continuity of the perturbations. Anyhow, the connection coefficients and metric perturbation derivatives have a finite jump and they can be computed as the average of their values at $z_u \pm \epsilon$ with $\epsilon \to 0$.

The C^0 continuity class for the perturbations at the position of the particle in the Regge-Wheeler gauge is analysed herein. The purpose is to identify the conditions on the wavefunction Ψ that determine such continuity. After visual inspection of eq. (25), containing a derivative of the Dirac delta distribution, it is evinced that the wavefunction Ψ is of C^{-1} continuity class (Ψ may contain functions of higher continuity class) and thus can be written as:

$$\Psi(t,r) = \Psi^{+}(t,r)\ \Theta_1 + \Psi^{-}(t,r)\ \Theta_2 \tag{68}$$

while its derivatives are given by:

$$\Psi_{,r} = \Psi^{+}_{,r}\Theta_1 + \Psi^{-}_{,r}\Theta_2 + \left(\Psi^{+} - \Psi^{-}\right)\delta \tag{69}$$

$$\Psi_{,rr} = \Psi^{+}_{,rr}\Theta_1 + \Psi^{-}_{,rr}\Theta_2 + 2\left(\Psi^{+}_{,r} - \Psi^{-}_{,r}\right)\delta + \left(\Psi^{+} - \Psi^{-}\right)\delta' \tag{70}$$

$$\Psi_{,t} = \Psi^{+}_{,t}\Theta_1 + \Psi^{-}_{,t}\Theta_2 + \left(\Psi^{+} - \Psi^{-}\right)\dot{z}_u\delta \tag{71}$$

$$\Psi_{,rt} = \Psi^{+}_{,rt}\Theta_1 + \Psi^{-}_{,rt}\Theta_2 + \left(\Psi^{+}_{,r} - \Psi^{-}_{,r}\right)\dot{z}_u\delta + \left(\Psi^{+}_{,t} - \Psi^{-}_{,t}\right)\delta$$

$$- \left(\Psi^{+} - \Psi^{-}\right)\dot{z}_{u,r}\delta - \left(\Psi^{+} - \Psi^{-}\right)\dot{z}_u\delta' \tag{72}$$

where $\Theta_1 = \Theta\left[r - z_u(t)\right]$, and $\Theta_2 = \Theta\left[z_u(t) - r\right]$ are two Heaviside step distributions; δ and δ' stand for $\delta[r - z_u(t)]$ and $\delta'[r - z_u(t)]$ respectively.

The perturbation functions K, H_2, H_1 are given by [125]:

$$K = \frac{6M^2 + 3M\lambda r + \lambda(\lambda+1)r^2}{r^2(\lambda r + 3M)}\Psi + \left(1 - \frac{2M}{r}\right)\Psi_{,r} - \frac{\kappa\, U^0 (r-2M)^2}{(\lambda+1)(\lambda r + 3M) r}\delta \tag{73}$$

$$H_2 = -\frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2(\lambda+1)r^3}{r^2(\lambda r + 3M)^2}\Psi + \frac{3M^2 - \lambda M r + \lambda r^2}{r(\lambda r + 3M)}\Psi_{,r} + (r-2M)\Psi_{,rr}$$

$$+ \frac{\kappa U^0 (r-2M)[\lambda^2 r^2 + 2\lambda M r - 3Mr + 3M^2]}{r(\lambda+1)(\lambda r + 3M)^2}\delta - \frac{\kappa U^0 (r-2M)^2}{(\lambda+1)(\lambda r + 3M)}\delta' \tag{74}$$

$$H_1 = r\Psi_{,tr} + \frac{\lambda r^2 - 3M\lambda r - 3M^2}{(r-2M)(\lambda r + 3M)}\Psi_t - \frac{\kappa\, U^0\, \dot{z}_u\, (\lambda r + M)}{(\lambda+1)(\lambda r + 3M)}\delta + \frac{\kappa\, U^0\, \dot{z}_u\, r(r-2M)}{(\lambda+1)(\lambda r + 3M)}\delta' \tag{75}$$

It is wished that the discontinuities of Ψ and its derivatives are such that they are canceled when combined in K, H_2 and H_1. To this end, the perturbations, expressed as functions of Ψ and its derivatives, may be written as:

$$K = f_1(r)\Psi + f_2(r)\Psi_{,r} + f_3(r)\delta[r - z_u(t)] \tag{76}$$

$$H_2 = f_4(r)\Psi + f_5(r)\Psi_{,r} + f_6(r)\Psi_{,rr} + f_7(r)\delta[r - z_u(t)] + f_8(r)\delta'[r - z_u(t)] \tag{77}$$

$$H_1 = f_9(r)\Psi_{,rt} + f_{10}(r)\Psi_{,t} + f_{11}(r)\delta[r - z_u(t)] + f_{12}(r)\delta'[r - z_u(t)] \tag{78}$$

where the definitions of the f functions are drawn by visual inspections of eq.s (73, 74, 75). After replacing Ψ and its derivatives in eq.s (76, 77, 78), continuity requires that the coefficients of Θ_1 must be equal to the coefficients of Θ_2, while the coefficients of δ and δ' must vanish separately. After some tedious computing and making use of one of the Dirac delta distribution properties: $f(r)\delta'[r - z_u(t)] = f(z_u(t))\delta'[r - z_u(t)] - f'(z_u(t))\delta[r - z_u(t)]$, at the position of the particle, it is obtained for K:

$$\Psi^+_{,r} - \Psi^-_{,r} = \frac{f_1 f_3}{f_2^2} \qquad \Psi^+ - \Psi^- = -\frac{f_3}{f_2} \tag{79}$$

while for H_2:

$$\Psi^+_{,r} - \Psi^-_{,r} = \frac{1}{f_6}\left(\frac{f_5 f_3}{f_2} - f_7 + f_{8,r} - \frac{f_{6,r} f_3}{f_2}\right) \qquad \Psi^+ - \Psi^- = -\frac{f_8}{f_6} \tag{80}$$

and finally for H_1:

$$\Psi^+ - \Psi^- = \frac{f_{12}}{\dot{z}_u f_9} \tag{81}$$

The jump conditions set by eq.s (79, 80, 81) are equivalent. The other jump conditions to be satisfied (coming from H_2 and H_1) are:

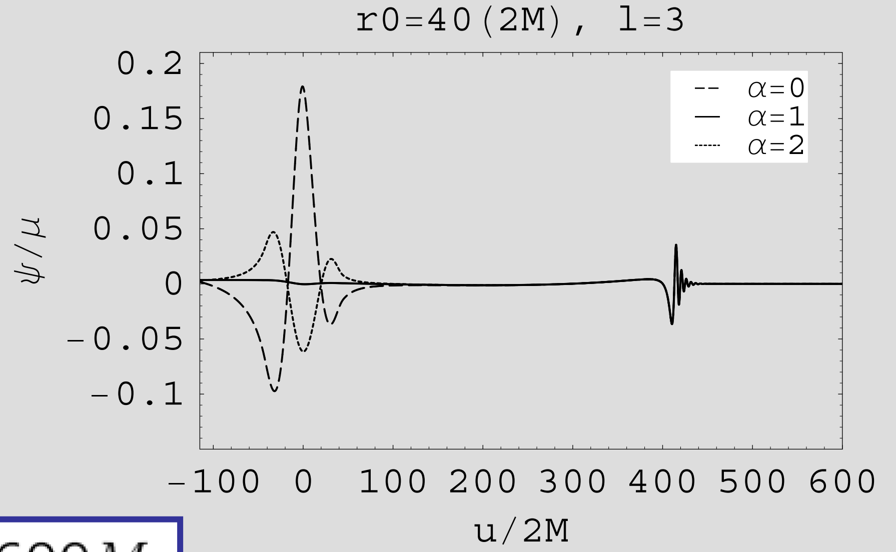
$$\Psi^+_{,rr} - \Psi^-_{,rr} = \frac{f_4\left(\Psi^+ - \Psi^-\right) - f_7\left(\Psi^+_{,r} - \Psi^-_{,r}\right)}{f_6} \tag{82}$$

$$\Psi^+_{,t} - \Psi^-_{,t} = \frac{\left(f_9 \dot{z}_{u,r} - f_{9,r}\dot{z}_u + f_{10}\dot{z}_u\right)\left(\Psi^+ - \Psi^-\right) - f_{11} + f_{12,r}}{f_9} \tag{83}$$

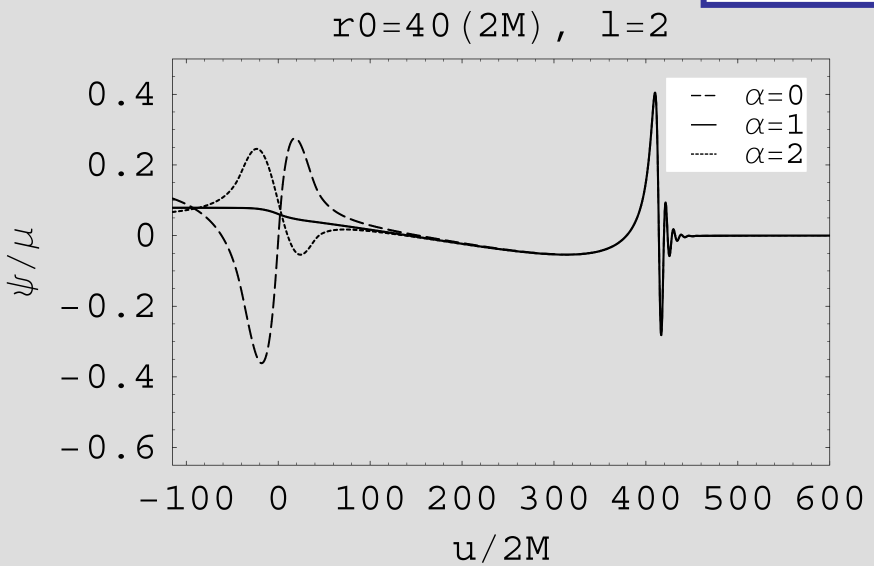
$$\Psi^+_{,rt} - \Psi^-_{,rt} = -\frac{f_{10}\left(\Psi^+_{,t} - \Psi^-_{,t}\right)}{f_9} \tag{84}$$

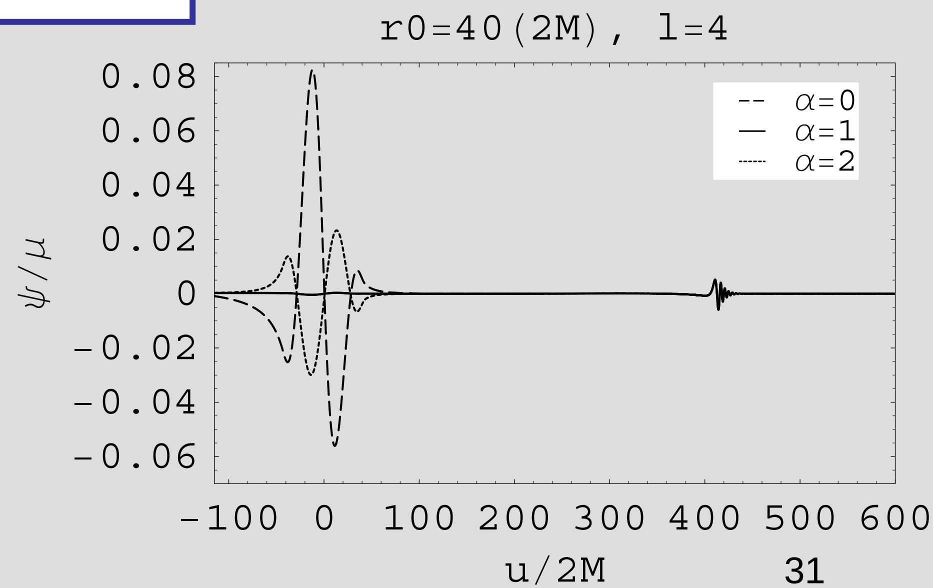
Résolution numérique de l'équation de Zerilli

Pour d'importantes séparations initiales, les trois phases de radiations se séparent distinctement.



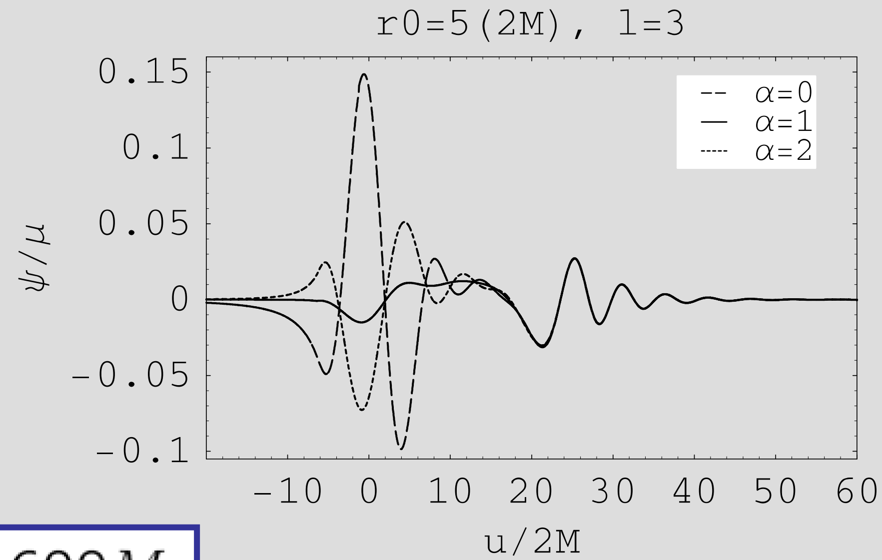
$$r_{obs} = 1600M$$



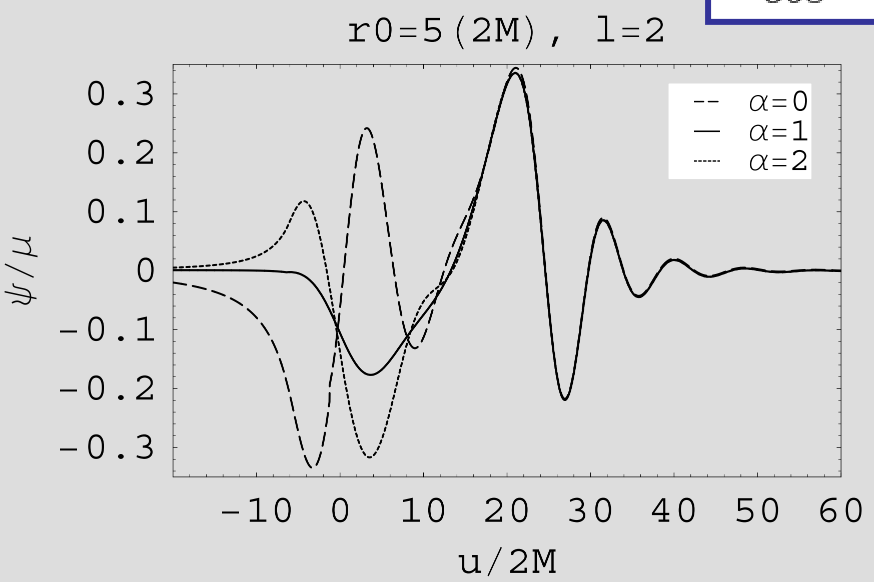


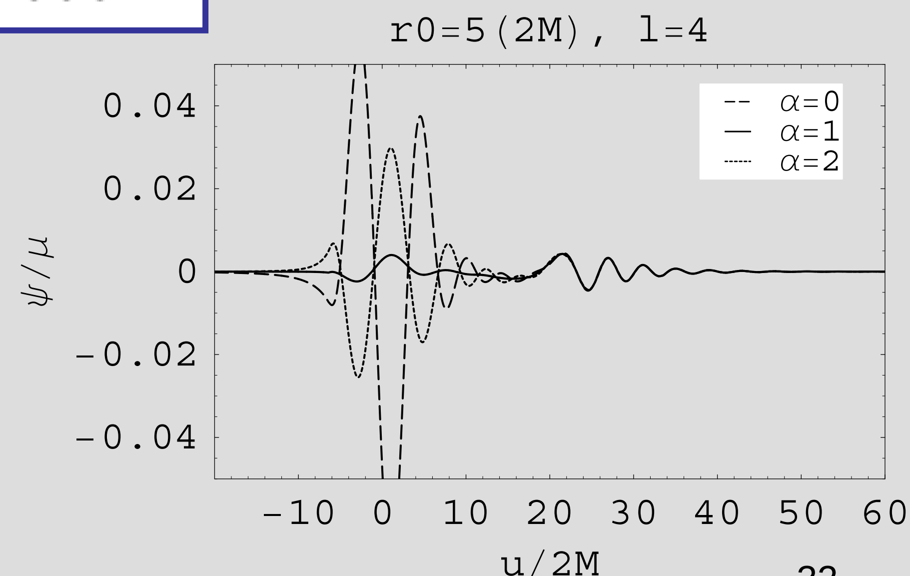
Résolution numérique de l'équation de Zerilli

Pour de moyennes séparations initiales, les trois phases de radiations commencent à ne plus se discerner.



$$r_{obs} = 1600M$$

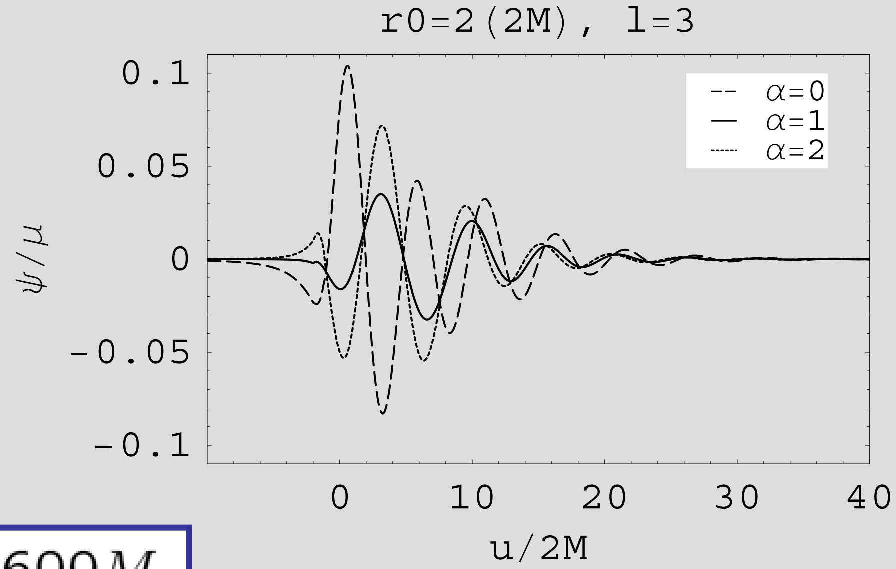


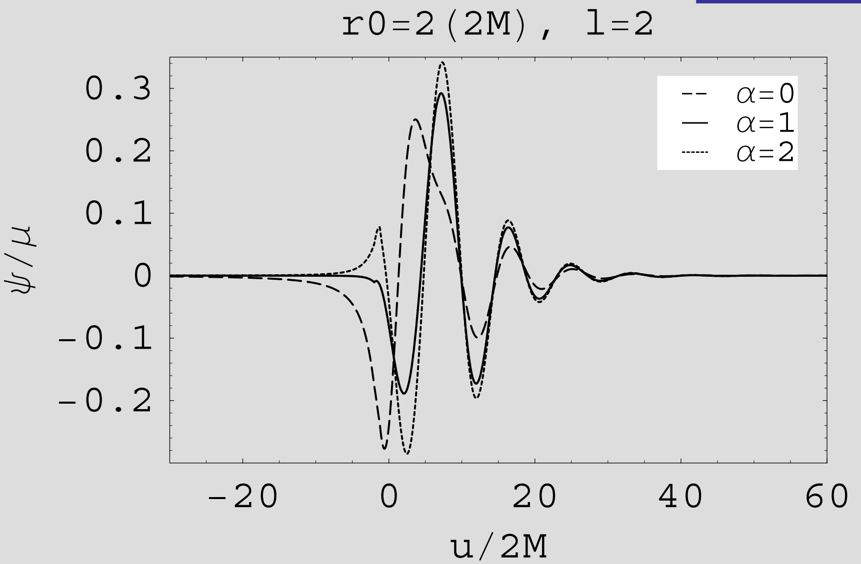


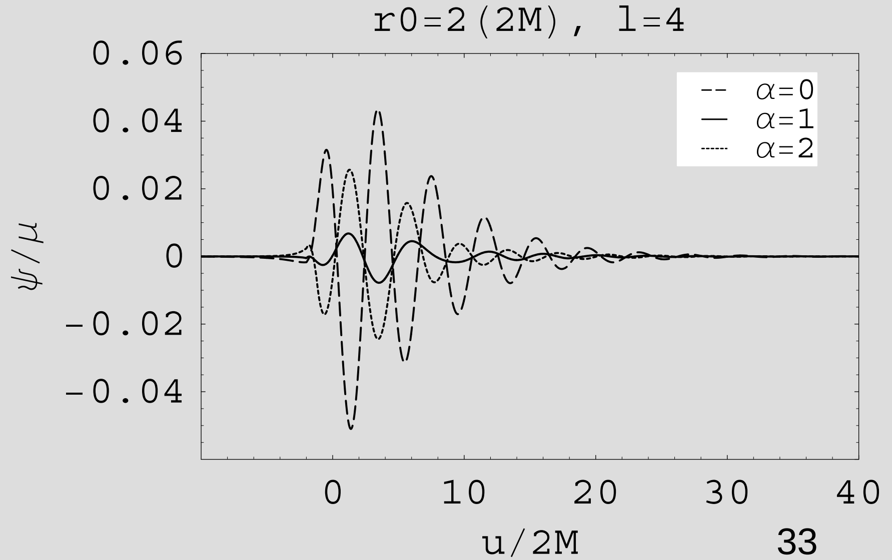
Résolution numérique de l’équation de Zerilli

Pour de faibles séparations initiales, les trois phases de radiations se confondent.

$$r_{obs} = 1600M$$







Energy balance (l=2) (Approximate since instantaneous energy balance is used)

Flux at infinity

$$\frac{dE_\infty}{du} = \frac{1}{64\pi}\frac{(l+2)!}{(l-2)!}\left(\frac{\partial\psi}{\partial u}\right)^2$$

Radiated energy at infinity retarded time u

$$E_\infty(u) = \frac{1}{64\pi}\frac{(l+2)!}{(l-2)!}\int_0^u \left(\frac{\partial\psi}{\partial u}\right)^2 du$$

Energy absorbed by the BH at retarded time u

$$E_H^{tot} = 0.25\frac{\mu}{M}$$

$$E_H(u) = \frac{E_\infty(u)}{E_\infty^{tot}} E_H^{tot}$$

$$E_\infty^{tot} = \frac{1}{64\pi}\frac{(l+2)!}{(l-2)!}\int_0^\infty \left(\frac{\partial\psi}{\partial u}\right)^2 du$$

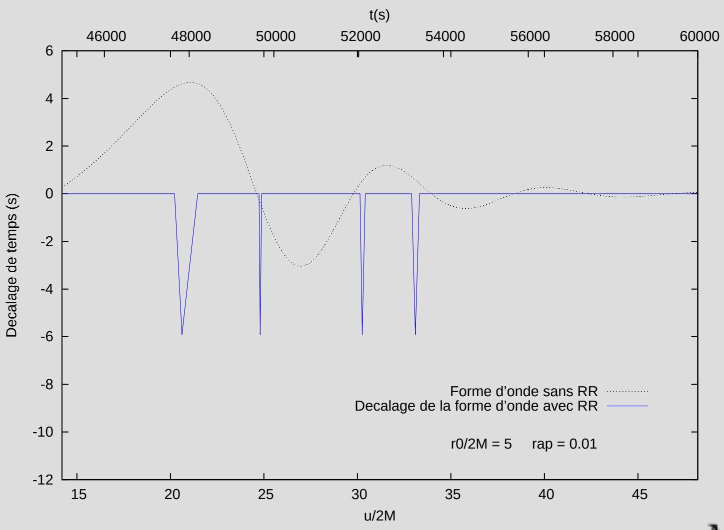
New energy of the particle

$$E_n(u) = E - E_\infty(u) - E_H(u)$$

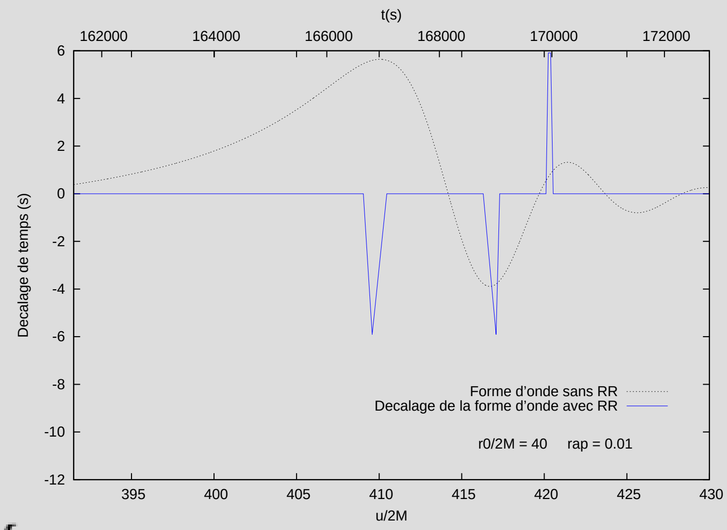
$$E = \sqrt{1 - 2M/r_0}$$

Test case for initial null velocity at starting point: limited effects

Waveforms corrections



$\frac{\mu}{M} = 0.01$



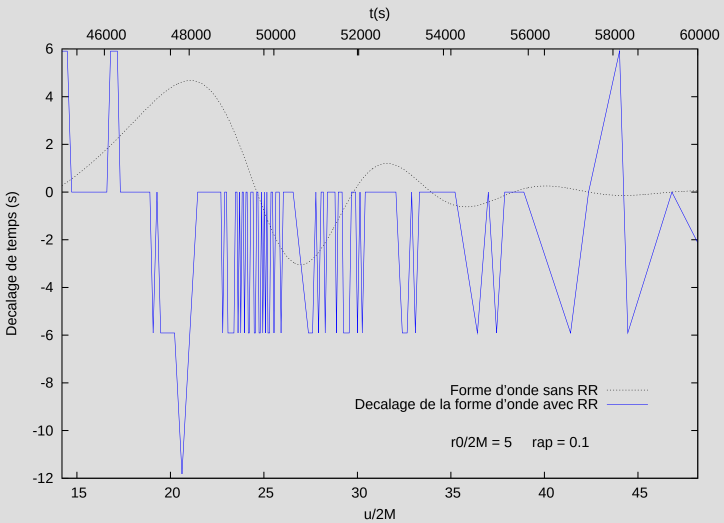
$l = 2$

$r_0 = 5(2M)$

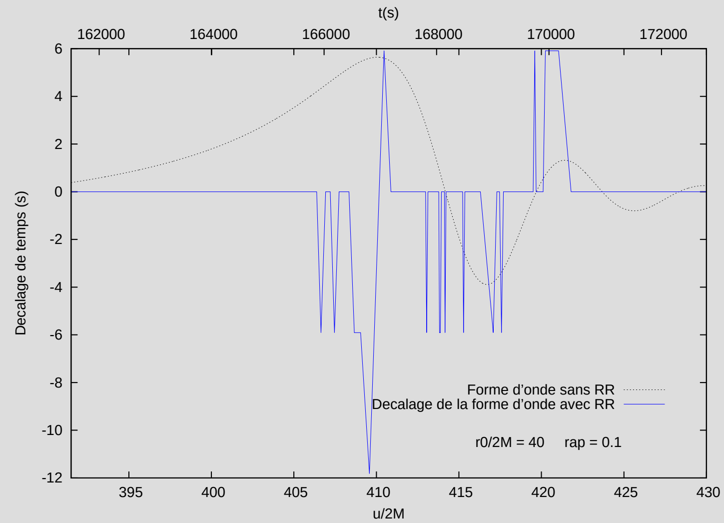
$M = 3 10^6 M_\odot$

$r_0 = 40(2M)$

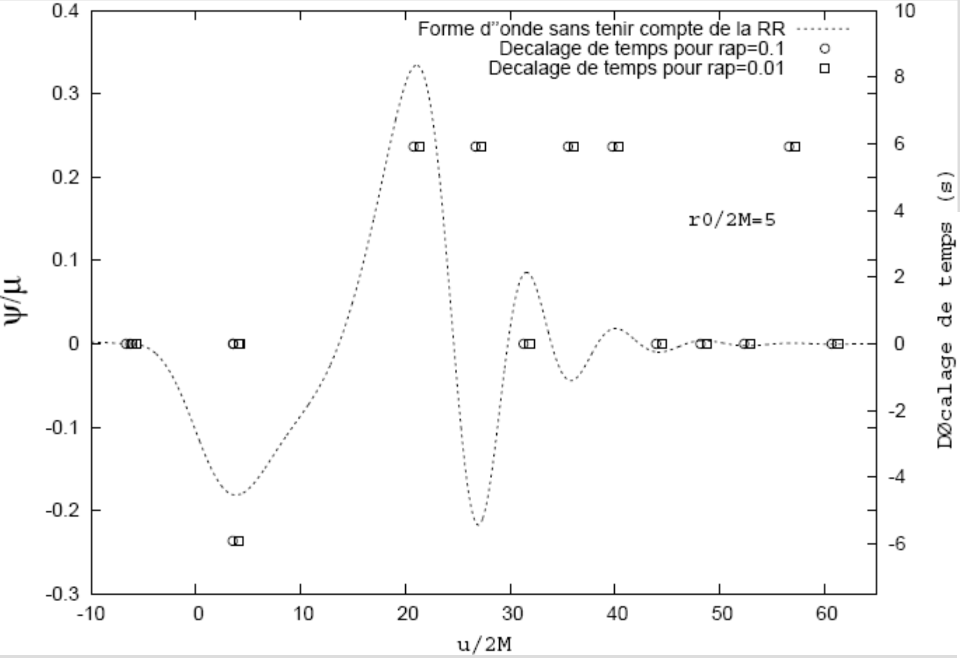
$\alpha = 1$

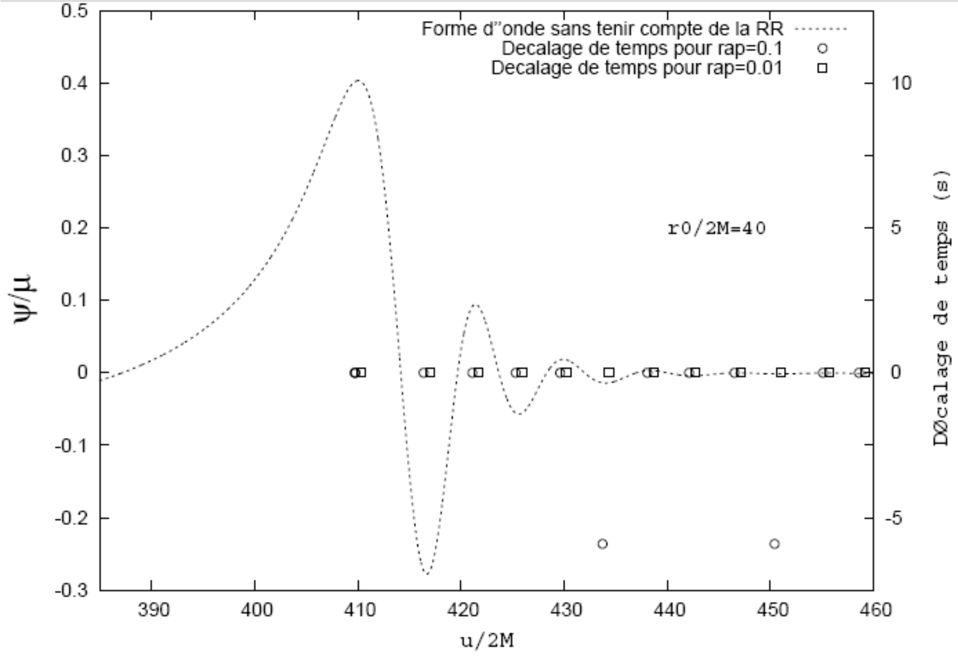


$\frac{\mu}{M} = 0.1$



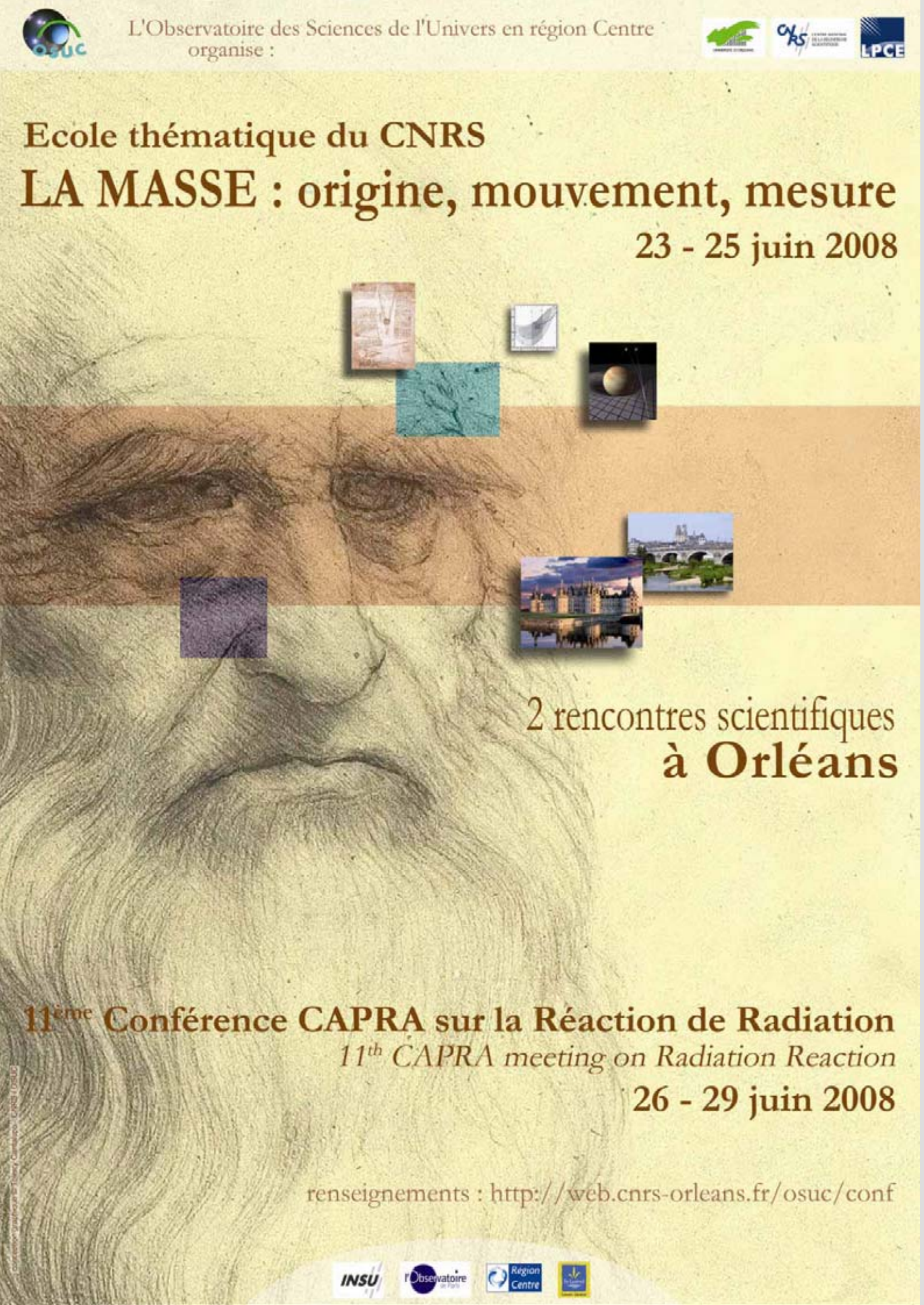
+ Corrected waveforms (0,01 m^2c^2/M for all l; 0,25 m^2c^2/M for l=2)





$$t = \frac{G}{c^3} 2M\left[\frac{u}{2M}\right] \approx 3\times 10^2 \left[\frac{u}{2M}\right]$$

For SgrA



Springer
coming up in 2009

$$\zeta\left(s,\frac{1}{2}\right)=\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)^{-s}=2^{s}\sum_{k=0}^{\infty}(2k+1)^{-s}$$

$$=2^{s}\left[\sum_{k=0}^{\infty}(k)^{-s}-\sum_{k=0}^{\infty}(2k)^{-s}\right]$$

$$=2^{s}\left[\sum_{k=0}^{\infty}(k)^{-s}-2^{-s}\sum_{k=0}^{\infty}(k)^{-s}\right]$$

$$=(2^{s}-1)\,\zeta(s)$$

$$\zeta\left(0,\frac{1}{2}\right)=0$$

Extension de la régularisation par la fonction zêta

$$L = l + \frac{1}{2}$$

Calcul du comportement en L des dérivées supérieures des ψ à la position de la particule

$\overline{\Psi} \propto L^{-2.5}$

$\overline{\Psi}_{,r} \propto L^{-2.5}$

$\overline{\Psi}_{,rr} \propto L^{-0.5}$

$\overline{\Psi}_{,rrr} \propto L^{-0.5}$

$\overline{\Psi}_{,t} \propto L^{-2.5}$

$\overline{\Psi}_{,tr} \propto L^{-0.5}$

$\overline{\Psi}_{,trr} \propto L^{-0.5}$

$$\alpha_i = \sum_l \alpha_i^l$$

$$\alpha_6^l = \alpha_6^a L^0 + \alpha_6^b L^{-2}$$

$$\alpha_7^l = \alpha_7^a L^2 + \alpha_7^b L^0 + \alpha_7^c L^{-2}$$

$$\alpha_8^l = \alpha_8^a L^0 + \alpha_8^b L^{-2}$$

$$\alpha_6 = \alpha_6^a\ \zeta(0,0.5) + \alpha_6^b\ \zeta(2,0.5)$$

$$\alpha_7^l = \alpha_7^a\ \zeta(-2,0.5) + \alpha_7^b\ \zeta(0,0.5) + \alpha_7^c\ \zeta(2,0.5)$$

$$\alpha_8^l = \alpha_8^a\ \zeta(0,0.5) + \alpha_8^b\ \zeta(2,0.5)$$

$\zeta(-2{,}0.5) = 0 \quad \zeta(0{,}0.5) = 0$

$\zeta(2{,}0.5) = \frac{1}{2}\pi^2$

$$\Delta\ddot{r}_p = \alpha_1 \Delta r_p + \alpha_2 \Delta\dot{r}_p + \alpha_3 \Delta r_p^2 + \alpha_4 \Delta\dot{r}_p^2 + \alpha_5 \Delta r_p \Delta\dot{r}_p + \alpha_6 + \alpha_7 \Delta r_p + \alpha_8 \Delta\dot{r}_p$$

Spallicci & Aoudia (2004).Class. Quantum. Grav., 21, S563.