Self-force with (3+1) codes

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Outline

- Introduction and motivation for prescription
- Test application
- Review of Capra 11 results
- Description of (3+1) codes used
- New window function and effective source
- Representative results
- Lessons + Summary + Future work

Self-force in (3+1)

Why not?

- It's computationally expensive $\rightarrow$ necessarily less accurate
- No one likes delta-function sources on 3D grids.
- All self-force calculations to date have been done either in frequency domain or in (1+1) time domain.

Why?

- Trivial enforcement of backreaction
- Powerful (3+1) codes exist! (as do the brave people who write them!)
- Transition from Schwarzschild to Kerr should be straightforward.

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Example: stationary point charge and conducting plane

What force do you need to exert on the charge to keep it stationary?

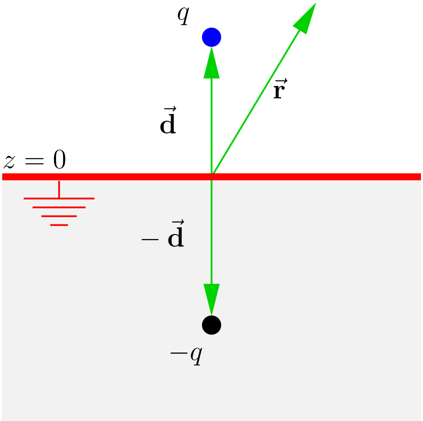
$V = \frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{|\mathbf{r}+\mathbf{d}|}$

$\mathbf{E} = -\vec{\nabla} V = q\frac{\mathbf{r}-\mathbf{d}}{|\mathbf{r}-\mathbf{d}|^3} - q\frac{\mathbf{r}+\mathbf{d}}{|\mathbf{r}+\mathbf{d}|^3}$

Identification: $\mathbf{E}^{\mathsf{S}} = q\frac{\mathbf{r}-\mathbf{d}}{|\mathbf{r}-\mathbf{d}|^3}$

Subtraction: $\mathbf{E}^{\mathsf{R}} = \mathbf{E} - \mathbf{E}^{\mathsf{S}}$

$\mathbf{F} = q\mathbf{E}^{\mathsf{R}}|_{\mathbf{r}=\mathbf{d}} = -\frac{q^2}{4d^2}\hat{\mathbf{z}}$



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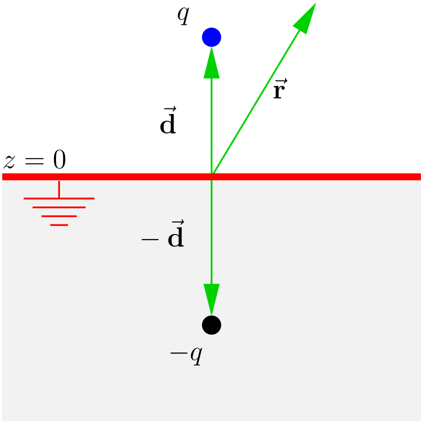


Detweiler-Whiting decomposition

The retarded Green's function can be uniquely split into two parts:

$$G_+(x,x') = G^{\mathsf{R}}(x,x') + G^{\mathsf{S}}(x,x')$$

1. A singular inhomogeneous piece that has nothing to do with the self force.

$$\Box G^{\mathsf{S}}(x,x') = -4\pi\delta_4(x,x')$$

2. A smooth homogeneous piece which completely determines the self force.

$$\Box G^{\mathsf{R}}(x,x') = 0$$

$$F_a^{(\mathsf{self})} = \lim_{x\to z} \nabla_a \psi^{\mathsf{R}} \text{ where } \psi^{\mathsf{R}}(x) = \int G^{\mathsf{R}}(x, z(\tau'))\mathrm{d}\tau'$$

Our prescription: field regularization

- From the DW decomposition, analytically approximate the part of retarded field that doesn't contribute to the self-force.

In a special kind of normal coordinates, called THZ coordinates $\{t, x, y, z\}$, the resulting singular field is locally approximated as:

$$\tilde{\psi}^{\mathsf{S}} \approx q/\rho$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$.

Our prescription: field regularization

- Rearrange field equation.

Let $\psi^{\mathsf{ret}} = \psi^R + W\tilde{\psi}^{\mathsf{S}}$

$$\Box\psi^{\mathsf{ret}} = -4\pi q \int_\gamma \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \mathsf{d}\tau$$

$$\Longrightarrow \Box\psi^{\mathsf{R}} = -\Box(W\tilde{\psi}^{\mathsf{S}}) - 4\pi q \int_\gamma \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \mathsf{d}\tau$$

where W is a suitably chosen "window function".

Our prescription: field regularization

- Simultaneously integrate the following equations:

$$\Box\psi^{\mathsf{R}} = S_{\mathsf{eff}}(x, z(\tau))$$

$$\frac{\mathsf{d}^2 z^a}{\mathsf{d}\tau^2} = \frac{q}{m}(g^{ab} + u^a u^b)\nabla_b \psi^{\mathsf{R}}.$$

Some comments

- $\tilde{\psi}^S$ is not the exact D-W singular field. Therefore, S_{eff} is only of finite differentiability at the location of the charge.
- The window function W is an arbitrary construct, except for the following conditions:
  1. $W \to 1$ sufficiently fast as one approaches the particle,
  2. $\nabla_a W \to 0$ sufficiently fast as one approaches the particle, and
  3. $W \approx 0$ outside a compact region R surrounding the particle.

These conditions guarantee that
(a) $\nabla_a \psi^{\mathsf{R}}|_{\text{point charge}}$ gives the self-force.
(b) ψ^{R} gives fluxes in the wavezone.

Advantages of this approach

- No delta functions.
  - Overcomes (3+1) difficulties with delta functions on a grid.
  - Allows the use of numerical relativity infrastructure.
- ψ^{R} gives the retarded field in the wavezone.
  - Important for wave generation
- The self-force is easy to compute from ψ^{R}.
  - It's just the gradient of ψ^{R}.
  - No mode sum!

Test application: scalar charge in a circular orbit of Schwarzschild

- In Schwarzschild coordinates, F_a has only two unknown independent components.

$$F_t = ?, \;\; F_r = ?, \;\; F_\theta = 0, \;\; F_\phi = -\frac{1}{\Omega} F_t$$

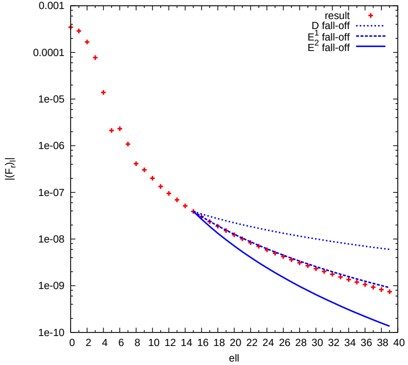
- The fluxes through the event horizon and infinity are related to F_t:

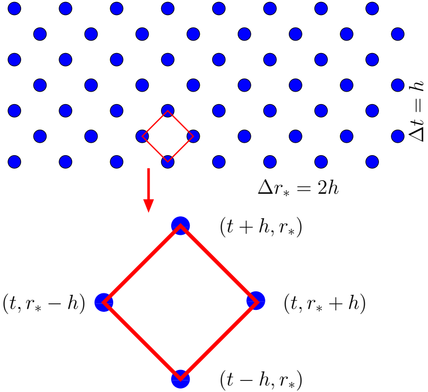
$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{r=2M} + \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{r=\infty} = -\sqrt{1-\frac{3M}{R}} F_t .$$

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{r=R} = R^2 \sqrt{\frac{R}{R-2M}} \oint_R \left[\frac{2M}{R} \dot{\psi}^2 + \left(1 - \frac{2M}{R}\right) \dot{\psi} \partial_r \psi \right] \mathrm{d}\Omega$$

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right|_{r=2M} = -4M^2 \oint \dot{\psi}^2 \; \mathrm{d}\Omega$$

Capra 11: Tests on (1+1) code

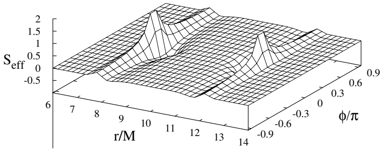


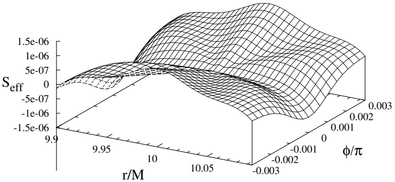


We achieved self-force accuracies to within $\sim 1\%$ and matched the retarded field in the wavezone to within relative error of $\sim 10^{-6}$.

Effective source

Equatorial profile of the effective source with $W = \exp\left[-\frac{(r-R)^N}{\sigma^N}\right]$, and $N = 8$, $\sigma = 2M$.





Description of (3+1) Codes

- Multiblock code (with P. Diener (LSU)) - gr-qc/0602104
  - high-order finite differencing code
  - able to calculate quasinormal modes of Kerr to high accuracy
  - uses Kerr-Schild coordinates
  - six blocks; outer boundary at $r = 400M$.
- SGRID code (with W. Tichy (FAU)) - gr-qc/0609087
  - pseudospectral code
  - able to evolve a single black hole
  - uses Kerr-Schild coordinates
  - four domains; outer boundary at $r = 210M$

New window function

Consider the transition function f depending on 4 parameters: $\{r_0, w, q, s\}$.

$$f(r|r_0,w,q,s) = \begin{cases} 0, & r \le r_0 \\ \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{s}{\pi}\left(\tan\left(\frac{\pi}{2w}(r-r_0)\right) - \frac{q^2}{\tan\left(\frac{\pi}{2w}(r-r_0)\right)}\right)\right], & r_0 < r < r_0 + w \\ 1, & r \ge r_0 + w. \end{cases}$$

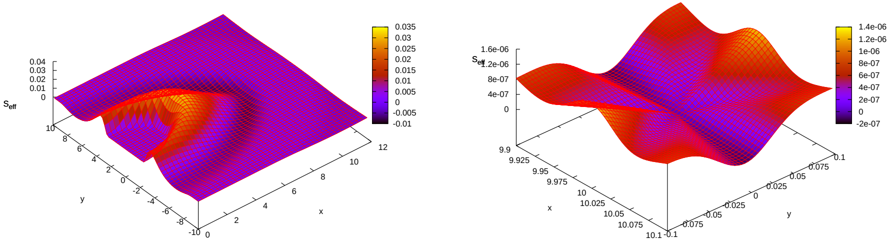
With this, we construct the window

$$W(r) = \begin{cases} f(r|(R-\delta_1-w_1), w_1, q_1, s_1) & r \le R \\ 1 - f(r|(R+\delta_2), w_2, q_2, s_2) & r > R \end{cases}$$

which depends on 8 parameters: 2 sets of $\{\delta, w, q, s\}$.

New effective source

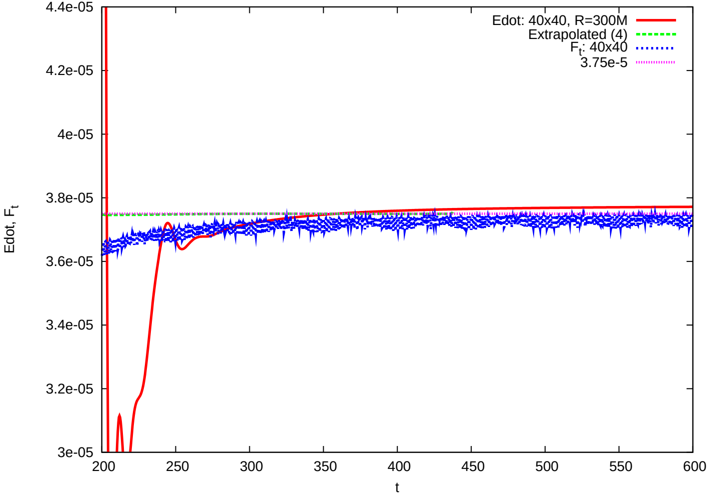
Our new effective source has less structure and is of lower amplitude away from the location of the charge.



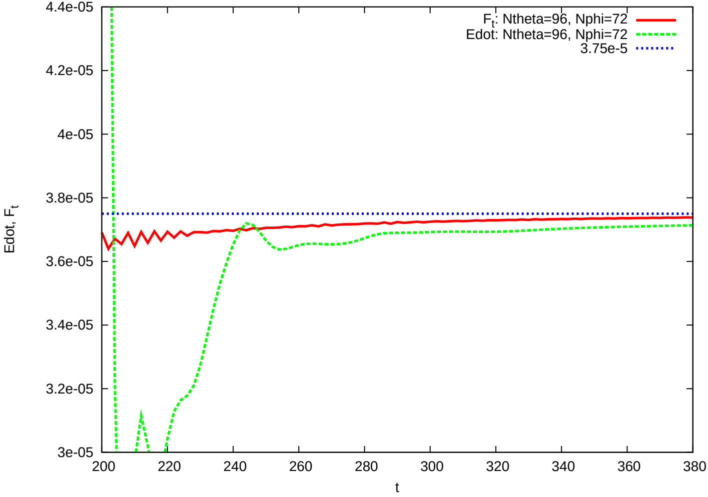
Parameters:

$\{\delta_1 = \delta_2 = 0; q_1 = 0.6, q_2 = 1.2; s_1 = 3.6, s_2 = 1.9; w_1 = 7.9, w_2 = 20\}.$

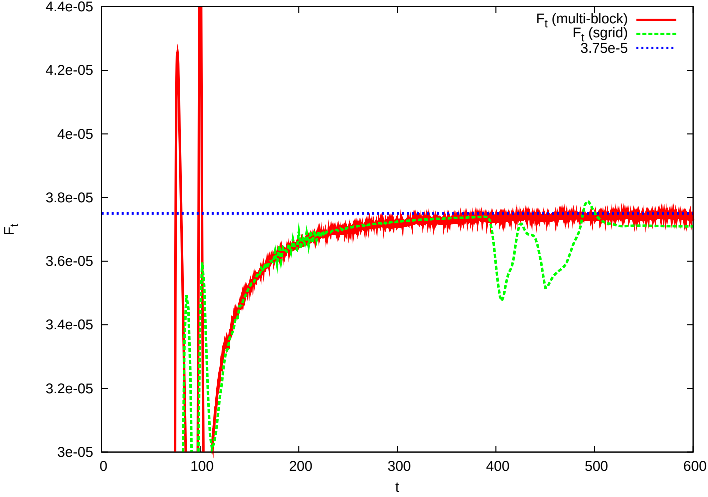
(3+1) Results: F_t and the energy flux (multiblock)



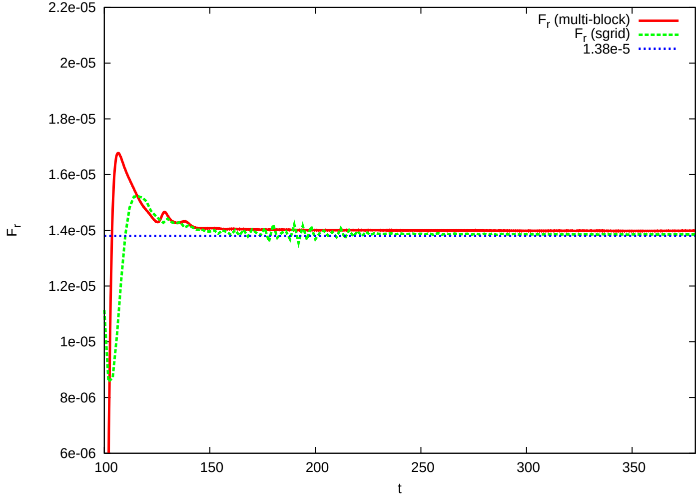
(3+1) Results: F_t and the energy flux (SGRID)



(3+1) Results: Comparing F_t results



(3+1) Results: Comparing F_r results



Summary of (3+1) results

	Code	(3+1)	Frequency-domain	error
F_t	mb	$\sim 3.7 \times 10^{-5}$	3.750227×10^{-5}	$\sim$ 1%
F_t	sg	3.739×10^{-5}	3.750227×10^{-5}	0.2%
$\dot{E}$ as F_t	mb	3.76×10^{-5}	3.750227×10^{-5}	0.6%
$\dot{E}$ as F_t	mbe*	3.7502×10^{-5}	3.750227×10^{-5}	0.007%
$\dot{E}$ as F_t	sg	3.749×10^{-5}	3.750227×10^{-5}	0.02%
F_r	mb	1.387×10^{-5}	1.378448×10^{-5}	0.6%
F_r	sg	1.3852×10^{-5}	1.378448×10^{-5}	0.5%

Summary of (3+1) self-force results for a circular orbit at $R = 10M$.

Lessons

- Finite differentiability of the source reduces the convergence of NR codes. (No surprise!)
- It also impacts the uncertainty in the computed self-force.
- Choosing a good window function is important.
- The effect of imperfect BCs is exacerbated in a self-force problem.
- The expertise of numerical relativity can be beneficial to the self-force community (and vice-versa) in the context of EMRI modeling.

Summary

- We now possess a method that allows a (3+1) approach to the self-force programme. This method
  1. provides a smeared out representation for the delta-function source;
  2. permits direct access to the self-force, and thus providing a natural way to update particle trajectories;
  3. simultaneously gives the retarded field and fluxes in the wavezone.
- We have tested our method on the test case of a scalar charge moving in a circular orbit of a Schwarzschild black hole by
  1. developing (1+1) code for the wave equation;
  2. and successfully collaborating with numerical relativists with existing (3+1) codes.
- We have performed the very first (3+1) calculation of a self-force, though it's not very accurate (yet).

For the next Capra

- Improvements to the effective source.
- Effective scalar source for general geodesics of Schwarzschild (and Kerr?).
- Self-consistent scalar point charge dynamics in Schwarzschild (and Kerr?)