Introduction to Self-Force

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Motion of Bodies in General Relativity

General relativity with suitable forms of matter has a well posed initial value formulation. In principle, therefore, to determine the motion of bodies in general relativity—such as binary neutron stars or black holes—one simply needs to provide appropriate initial data (satisfying the constraint equations) on a spacelike slice and then evolve this data via Einstein's equation. It would be highly desireable to obtain simple analytic descriptions of motion. However, it is clear that, in general, the motion of a body of finite size will depend on the details of its composition as well as the details of its internal states of motion. Therefore, one can hope to get a simple description of motion only in some kind of "point particle limit". Such a limit encompasses many cases of physical interest, such as "extreme mass ratio" inspiral. Of particular interest are the "radiation reaction" or "self-force" effects occuring during inspiral (which, of course, are the cause of the inspiral).

Point Particles in General Relativity

Einstein's equation is nonlinear, and a straightforward analysis (see Geroch and Traschen, PRD 36, 1017 (1987)) shows that it does not make any mathematical sense to consider solutions of Einstein's equation with a distributional stress-energy tensor supported on a worldline. Mathematically, the expected behavior of the metric near a "point particle" is too singular to make sense of the nonlinear terms in Einstein's equation, even as distributions. Physically, if one tried to compress a body to make it into a point particle, it should collapse to a black hole.

[By contrast "shells" (i.e., distributional solutions of Einstein's equation with support on a timelike hypersurface) do make mathematical sense. "Strings" are a borderline case.]

Therefore, since point particles do not make sense, it might appear that no simplifications in the description of motion can be achieved.

Point Particles in Linearized Gravity

Solutions, h_{ab}, to the linearized Einstein equation (off of an arbitrary background solution, g_{ab}) with a distributional stress-energy tensor supported on a world-line do make mathematical sense. Therefore, one might begin a treatment of gravitational self-force by considering considering solutions to

$$G^{(1)}_{ab}[h](t,x^i) = 8\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} ,$$

where u^a is the unit tangent (i.e., 4-velocity) of the worldline γ defined by $x^i(t) = z^i(t)$. However, two major difficulties arise in this approach:

- The linearized Bianchi identity implies that the point particle stress-energy must be conserved, which requires that the worldline γ of the particle is a geodesic of the background spacetime. Therefore, there are no solutions for non-geodesic source curves, making it a hopeless to use the linearized Einstein equation to derive corrections to geodesic motion.
- Even if the first problem were solved, solutions to this equation are singular on the worldine of the particle. Therefore, naive attempts to compute corrections to the motion due to h_{ab}—such as demanding that the particle move on a geodesic of $g_{ab} + h_{ab}$—are virtually certain to encounter severe mathematical difficulties,

analogous to the difficulties encountered in treatments of the electromagnetic self-force problem.

Lorenz Gauge Relaxation

The first difficulty has been circumvented by a number of researchers by modifying the linearized Einstein equation as follows: Choose the Lorenz gauge condition, so that the linearized Einstein equation takes the form

$$\nabla^c \nabla_c \tilde{h}_{ab} - 2 R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt}$$

$$\nabla^b \tilde{h}_{ab} = 0$$

where $\tilde{h}_{ab} \equiv h_{ab} - \frac{1}{2} h g_{ab}$ with $h = h_{ab} g^{ab}$. The first equation, by itself, has solutions for any source curve γ; it is only when the Lorenz gauge condition is adjoined that the equations are equivalent to the linearized

Einstein equation and geodesic motion is enforced. Therefore, if one solves the Lorenz-gauge form of the linearized Einstein equation while simply *ignoring* the Lorenz gauge condition that was used to derive this equation, one allows for the possibility non-geodesic motion. Of course, this “gauge relaxation” of the linearized Einstein equation produces an equation inequivalent to the original. However, because deviations from geodesic motion are expected to be small, the Lorenz gauge violation should likewise be small, and it thus has been argued that solutions to the two systems should agree to sufficient accuracy.

Hadamard Expansions

In order to solve the (relaxed) linearized Einstein equation near the worldline of the particle, we would like to have a short distance expansion for the (retarded) Green's function for a general wave equation

$$g^{ab}\nabla_a\nabla_b\phi + A^a\nabla_a\phi + V\phi = 0$$

Such an expansion was provided by Hadamard in the 1920's. It is easiest to explain in the Riemannian case, where one is solving a generalized Laplace equation, and the Green's function is unique up to smooth solutions. In 4-dimensions, in Euclidean space with $A^a = 0$ and $V = 0$, the Green's function with source at x' is simply

$$G(x,x') = \frac{1}{\sigma(x,x')}$$

where $\sigma(x,x')$ denotes the squared geodesic distance between x and x'. This suggests that we seek a solution to the generalized Laplace equation of the form

$$G(x,x') = \frac{U(x,x')}{\sigma(x,x')} + V(x,x')\ln\sigma(x,x') + W(x,x')$$

where V and W are, in turn, expanded as

$$V(x,x') = \sum_{j=0}^{\infty} v_j(x,x')\sigma^j \ , \quad W(x,x') = \sum_{j=0}^{\infty} w_j(x,x')\sigma^j$$

If one substitutes these expansions into the generalized

Laplace equation and formally sets the coefficient of each power of σ to zero, one gets an equation that can be uniquely solved for U—the solution is the square root of the van Vleck-Morette determinant—and one gets "recursion relations" for the v_j and w_j, which uniquely determine them—except for w_0, which can be chosen arbitrarily. In the analytic case, one can then show that the series have a finite radius of convergence (which, however, clearly must be smaller than the radius of the normal neighborhood in which σ is defined!), and that the above expansion provides a Green's function.

A similar construction works in the Lorentzian case (and can be obtained from the Riemannian result by insertion of suitable $i\epsilon$'s in σ). The corresponding Hadamard expansion for the retarded Green's function is

$$G_+(x,x') = U(x,x')\delta(\sigma)\Theta(t,t') + V(x,x')\Theta(-\sigma)\Theta(t,t')$$

where V again is given by a series whose coefficients v_j are uniquely determined by recursion relations. The following points should be noted

- For a self-adjoint equation, $V(x,x') = V(x',x)$, and (where defined!) V is a smooth solution of the homogeneous wave equation in each variable.
- For a globally hyperbolic spacetime, $G_+(x,x')$ is globally well defined. By contrast the Hadamard expansion of $G_+(x,x')$ can be valid at best within a

convex normal neighborhood. One occasionally sees in the literature Hadamard formulae that are purported to be valid when multiple geodesics connect x and x'. I do not believe that there is any mathematical justification for these formulae.

- It is rigorously true that, globally, $G_+(x, x')$ is singular if and only if there is a future directed null geodesic from x' to x (whether or not this geodesic enters the chronological future of x').

Hadamard Expansion for a Point Particle Source

Using the above Hadamard expansion for the retarded Green's function, we find that the solution to the relaxed linearized Einstein equation with a point particle source is

$$h_{\alpha\beta} = \frac{2M}{r}\delta_{\alpha\beta} - 8Ma_{(\alpha}u_{\beta)}(1 - a_i x^i) + h^{\text{tail}}_{\alpha\beta} + M\mathcal{R}_{\alpha\beta} + O(r^2)$$

where

$$h^{\text{tail}}_{\alpha\beta} \equiv M\int_{-\infty}^{\tau^-}\left(G_{+\alpha\beta\alpha'\beta'} - \frac{1}{2}g_{\alpha\beta}G_{+\ \ \gamma\alpha'\beta'}^{\ \gamma}\right)u^{\alpha'}u^{\beta'}d\tau'$$

The symbol τ^- means that this integration is to be cut short of $\tau' = \tau$ to avoid the singular behavior of the Green's function there; this instruction is equivalent to using only the "tail" (i.e., interior of the light cone) portion of the Green's function, i.e., the portion arising from $V(x, x')$ where V is defined.

Equations of Motion Including Self-Force

With the above type of formula for $h_{\alpha\beta}$ as a starting point, the equations of motion of a point particle—accurate enough to take account of self-force corrections—have been obtained by the following 3 approaches:

- One can proceed in parallel with the derivations of Dirac and DeWitt and Brehme for the electromagnetic case and derive the motion from conservation of total stress-energy (Mino, Sasaki, and Tanaka, PRD 55, 3457 (1997)). This requires an (ad hoc) regularization of the "effective stress energy" associated to $h_{\alpha\beta}$.

- One can derive equations of motion from some simple axioms (Quinn and Wald, PRD 56, 3381 (1997)), specifically that: (i) the difference in "gravitational force" between different curves of the same acceleration (in possibly different spacetimes) is given by the (angle average of) the difference in $-\Gamma^{\mu}{}_{\alpha\beta}u^{\alpha}u^{\beta}$ where $\Gamma^{\mu}{}_{\alpha\beta}$ is the Christoffel symbol associated with $h_{\alpha\beta}$ and (ii) the gravitational self-force vanishes for a uniformly accelerating worldline in Minkowski spacetime.

- One can derive equations of motion via matched asymptotic expansions (Mino, Sasaki, and Tanaka; Poisson, Liv. Rev. Rel. 7, 6 (2004)). The idea here is to postulate a suitable metric form (namely, Schwarzschild plus small perturbations) near the "particle", and then "match" this "near zone" expression to the "far zone" formula for $h_{\alpha\beta}$. Equations of motion then arise from the matching.

The MiSaTaQuWa Equations

All three approaches have led to the following system of equations:

$$\nabla^c \nabla_c \tilde{h}_{ab} - 2R^c{}_{ab}{}^d \tilde{h}_{cd} = -16\pi M u_a(t) u_b(t) \frac{\delta^{(3)}(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt}$$

$$u^b \nabla_b u^a = -(g^{ab} + u^a u^b)(\nabla_d h^{\text{tail}}_{bc} - \frac{1}{2}\nabla_b h^{\text{tail}}_{cd}) u^c u^d$$

where it is understood that the retarded solution to the equation for $\tilde{h}_{ab}$ is to be chosen. Note that the equation of motion for u^a corresponds to the geodesic equation in the metric $g_{ab} + h^{\text{tail}}_{ab}$. However, h^{tail}_{ab} is not a (homogeneous) solution to the (relaxed) linearized Einstein equation, nor is it smooth.

The Detweiler-Whiting Reformulation

The symmetric Green's function is defined by $G_{\rm sym} = (G_+ + G_-)/2$ where G_- is the advanced Green's function. The Hadamard expansion of $G_{\rm sym}$ is

$$G_{\rm sym}(x,x') = \frac{1}{2}[U(x,x')\delta(\sigma) + V(x,x')\Theta(-\sigma)]$$

Now V is a homogeneous solution (where defined!). Detweiler and Whiting define a new Green's function by

$$G_{\rm DW}(x,x') = \frac{1}{2}[U(x,x')\delta(\sigma) + V(x,x')\Theta(\sigma)]$$

The Detweiler-Whiting Green's function has the odd property of having no support in the interior of the future or past light cones. Detweiler and Whiting show that the

MiSaTaQuWa equation is equivalent to geodesic motion in the metric $g_{ab} + h^R_{ab}$ where h^R_{ab} is the homogeneous solution of the (relaxed) linearized Einstein obtained from applying $G_+ - G_{\rm DW}$ to the worldline source.

How Should Gravitational Self-Force be Derived?

A precise formula for gravitational self-force can hold only in a limit where the size, R, of the body goes to zero. However, to avoid difficulties associated with the non-existence of point particles, it is essential that one let M go to zero as well. This suggests that we consider a one-parameter family of solutions to Einstein's equation, $g_{ab}(\lambda)$, for which the body scales to zero size and mass in an asymptotically self-similar way as $\lambda \to 0$. In the limit as $\lambda \to 0$ (where the body shrinks down to a worldline γ and "disappears"), geodesic motion is obtained. Last year, Gralla and I have proved that, in the Lorenz gauge, to first order in λ, the deviation Z^i from γ satisfies

$$\frac{d^2 Z^i}{dt^2} = \frac{1}{2M} S^{kl} R_{kl0}{}^{i} - R_{0j0}{}^{i} Z^j - M \left(h^{\mathrm{tail}\,i}{}_{0,0} - \frac{1}{2} h^{\mathrm{tail},i}_{00} \right)$$

The MiSaTaQuWa equations arise as "self-consistent perturbative equations" associated with this perturbative result.