A Rigorous Derivation of Electromagnetic Self-Force

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Derivation of Self-Force

Gravitational self-force was derived in S. Gralla and R. Wald, *Class. Quantum Grav.* **25** 205009 (2008) by considering a one-parameter family of solutions to Einstein's equation corresponding to a body (or black hole) whose exterior metric behaves in a way corresponding to having the body "scale to zero size and mass" as $\lambda \to 0$. The assumptions about the exterior field included a "uniformity condition" that the metric is jointly smooth in the variables $\alpha \equiv r$ and $\beta \equiv \lambda/r$. It was necessary to state the assumptions about the one-parameter family in terms of the exterior metric because there is no simple relationship between the exterior field and the “matter source” in general relativity.

Such a relationship exists in electrodynamics if one considers the retarded solution, so the corresponding assumptions about the one-parameter family can be stated in terms of the behavior of the charge/matter source. It is therefore instructive to consider the self-force problem in electrodynamics. This problem is also of considerable interest in its own right.

We should have done this analysis as a “warm-up” problem for the gravitational case. Instead, it was done as an after-thought—and it required considerable effort!

Motion in Electrodynamics

We consider a body in Minkowski spacetime with charge-current J^μ and stress-energy tensor $T^M_{\mu\nu}$. We wish to determine properties of the motion of such a body that are "universal" in the sense that they follow only from Maxwell's equations:

$$\nabla^\nu F_{\mu\nu} = -4\pi J_\nu$$

$$\nabla_{[\mu} F_{\nu\rho]} = 0$$

and conservation of total stress-energy:

$$\nabla^b \left(T^M_{ab} + T^{EM}_{ab} \right) = 0 \ ,$$

where

$$T^{EM}_{ab} \equiv \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \ .$$

It is clear that simple, universal equations can arise only if we consider a limit of zero size of the body. However, if we take the usual point particle limit (zero size at fixed charge and mass), we will encounter the serious problems associated with a singular electromagnetic field and infinite self-energy that have plagued analyses for the past century.

Approaches to Obtaining Equations of Motion

The main approaches used during the past century are:

- Work with true point particles and "regularize" formal expressions to obtain finite results (see, e.g., Dirac). Requires infinite "mass renormalization" and other ad hoc procedures.

- Work with (small) extended bodies and assume "rigidity" to obtain simple equations of motion that do not depend on internal degrees of freedom. Rigidity does not make sense in the context of special (and general) relativity; must consider only models such that the self-consistent, coupled, Maxwell-charged-matter equations admit a well-posed initial value formulation, but this leads to complicated, non-universal equations for finite sized objects.

Our approach: Consider a modified point particle limit, wherein not only the size of the body goes to zero, but its charge and mass also go to zero.

Our Assumptions

We consider a one-parameter family $\{F_{\mu\nu}(\lambda), J^\mu(\lambda), T^M_{\mu\nu}(\lambda)\}$ of solutions to Maxwell's equations and conservation of total stress-energy such that

- $$J^\mu(\lambda, t, x^i) = \lambda^{-2} \tilde{J}^\mu(\lambda, t, [x^i - z^i(t)]/\lambda)$$

  and

  $$T^M_{\mu\nu}(\lambda, t, x^i) = \lambda^{-2} \tilde{T}_{\mu\nu}(\lambda, t, [x^i - z^i(t)]/\lambda)$$

  with $\tilde{J}^\mu$ and $\tilde{T}_{\mu\nu}$ smooth.

- We have $F_{\mu\nu} = F^{\rm ext}_{\mu\nu} + F^{\rm self}_{\mu\nu}$, where $F^{\rm self}_{\mu\nu}$ is the retarded solution of Maxwell's equations with source $J^\mu(\lambda)$ and $F^{\rm ext}_{\mu\nu}$ is a homogeneous solution of Maxwell's equation that is jointly smooth function of λ and the spacetime point.

Key Results on $F^{\rm self}_{\mu\nu}$

We have

$$F^{\rm self}_{\mu\nu}(\lambda, t, x^i) = \lambda^{-1} \tilde{F}_{\mu\nu}(\lambda, t, [x^i - z^i(t)]/\lambda) \ ,$$

where $\tilde{F}$ is a smooth function of its arguments.

Define $\alpha = r = \sqrt{\sum [x^i - z^i(t)]^2}$, $\beta = \lambda / r$. We have

$$\lambda F^{\rm self}_{\mu\nu} = \beta^2 \mathcal{F}_{\mu\nu}(t, \alpha, \beta, n^i)$$

where $\mathcal{F}_{\mu\nu}$ is smooth in all of its arguments

Thus, the analog of the "uniformity assumption" of Gralla and Wald holds for $F^{\rm self}_{\mu\nu}$.

"Far Zone" Limit and Unperturbed Motion

Let $\lambda \to 0$ at fixed x^μ. Then $J^\mu(\lambda, t, x^i)$ can be expanded in a *distributional* series. We find that
$J^{(0)\mu} \equiv \lim_{\lambda\to 0} J^\mu(\lambda) = 0$ and

$$J^{(1)\mu} \equiv \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} J^\mu(\lambda) = \mathcal{J}^\mu(t) \frac{\delta(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt}$$

Conservation of J^μ then yields

$$J^{(1)\mu} = q u^\mu \frac{\delta(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \ .$$

Similarly,

$$T^{M,(1)}_{\mu\nu} \equiv \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} T^M_{\mu\nu}(\lambda) = \mathcal{T}^M_{\mu\nu}(t) \frac{\delta(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \ .$$

If we write,

$$T_{\mu\nu}^{EM} = T_{\mu\nu}^{\rm ext} + T_{\mu\nu}^{\rm cross} + T_{\mu\nu}^{\rm self} \ ,$$

then, remarkably, we find

$$T_{\mu\nu}^{{\rm self},(1)} \equiv \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} T_{\mu\nu}^{\rm self}(\lambda) = \mathcal{T}_{\mu\nu}^{\rm self}(t) \frac{\delta(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \ .$$

Conservation of total stress energy then yields

$$\mathcal{T}_{\mu\nu}^{(1)}(t) = m u_\mu u_\nu \frac{\delta(x^i - z^i(t))}{\sqrt{-g}} \frac{d\tau}{dt} \ ,$$

where $\mathcal{T}_{\mu\nu} \equiv T_{\mu\nu}^{M} + \mathcal{T}_{\mu\nu}^{\rm self}$, and

$$m u^\nu \nabla_\nu u_\mu = q u^\nu F_{\mu\nu}^{\rm ext}(\lambda = 0, t, z^i(t)) \ .$$

Thus, to first order in λ, the description of *any* body is is precisely that of a classical point charge/mass moving on a Lorentz force trajectory of the external field. Note that the electromagnetic self-energy of the body contributes to its mass.

“Near Zone” Limit and Perturbed Motion

As $\lambda \to 0$, the body shrinks down to the worldline γ defined by $x^i = z^i(t)$, which satisfies the Lorentz force equation. However, at any $\lambda > 0$, the body is of finite size, so in order to find the “correction” to γ at finite λ, we would need to have a notion of the “center of mass worldline” $\gamma(\lambda)$ of the body to represent its motion. This is highly nontrivial since “electromagnetic self-energy” must be included, but one does not want to include electromagnetic radiation that was emitted in the past. Fortunately, this can be done straightforwardly to the order needed to obtain first-order perturbed motion.

It is convenient to work in Fermi normal coordinates based on the worldline $\gamma(\lambda)$—so $z^i(\lambda, t) = 0$—and it is convenient to do the calculations in the "near zone" limit, defined as follows: Choose a time, t_0 and let $\bar{t} \equiv (t - t_0)/\lambda$, $\bar{x}^i \equiv x^i/\lambda$. The "near zone" limit is $\lambda \to 0$ at fixed $\bar{x}^\mu$ rather than at fixed x^μ. The rescaled fields

$$\bar{g}_{ab} \equiv \lambda^{-2} g_{ab}$$

$$\bar{J}^a \equiv \lambda^3 J^a$$

$$\bar{T}^M_{ab} \equiv T^M_{ab}$$

$$\bar{F}_{ab} \equiv \lambda^{-1} F_{ab}$$

then approach well defined, finite limits as $\lambda \to 0$ at fixed $\bar{x}^\mu$. At $\lambda = 0$, the rescaled fields are stationary.

Center of Mass

Near zone metric (rescaled Minkowski metric in Fermi normal coordinates about $\gamma(\lambda)$):

$$\bar{g}_{00} = -1 - 2\lambda a_i(t_0)\bar{x}^i - \lambda^2[2\dot{a}_i(t_0)\bar{x}^i\bar{t} + (a_i(t_0)\bar{x}^i)^2 + \\ + 2\delta a_i(t_0)x^i] + O(\lambda^3)$$

$$\bar{g}_{i0} = O(\lambda^3)$$

$$\bar{g}_{ij} = \delta_{ij} \ .$$

Define

$$\bar{T}_{\bar{\mu}\bar{\nu}} \equiv \bar{T}^{M}_{\bar{\mu}\bar{\nu}} + \bar{T}^{\text{self}}_{\bar{\mu}\bar{\nu}} \ ,$$

define the zeroth order near-zone mass by

$$m(t_0) \equiv \int \bar{T}^{(0)}_{00} d^3\bar{x}$$

and define the zeroth order near zone center of mass by

$$\bar{X}^i_{\rm CM}(t_0) = \frac{1}{m} \int \bar{T}^{(0)}_{00} \bar{x}^i d^3\bar{x} \ .$$

The perturbed motion is defined by the condition
$\bar{X}^i_{\rm CM} = 0$.

Other Body Parameters

Spin tensor:

$$S^{0j} = -S^{j0} = \int \bar{T}^{(0)00} \bar{x}^j d^3\bar{x} = 0$$

$$S^{ij}(t_0) \equiv 2 \int \bar{T}^{(0)i}{}_0 \bar{x}^j d^3\bar{x}$$

Spin vector:

$$S_i = \frac{1}{2} \epsilon_{ijk} S^{jk}$$

Perturbed mass:

$$\delta m(t_0) \equiv \delta \int_\Sigma \bar{T}_{ab} (\frac{\partial}{\partial \bar{t}})^b d\Sigma^a$$

Charge:

$$q \equiv \int \bar{J}^{(0)0} d^3\bar{x}$$

Perturbed charge:

$$\delta q = \delta \int_\Sigma J^a d\Sigma_a$$

Electromagnetic dipole tensor:

$$Q^{\mu j}(t_0) \equiv \int \bar{J}^{(0)\mu} \bar{x}^j$$

Electric dipole moment:

$$p^i = Q^{0i}$$

Magnetic dipole moment:

$$\mu_i = -\frac{1}{2}\epsilon_{ijk}Q^{jk}$$

Derivation of Motion

Strategy: We write down the equations arising from conservation of total stress-energy and conservation of charge-current at 0th, 1st, and 2nd order in the near-zone expansion. We multiply these relations by various powers of $\bar{x}^i$ and integrate over space to systematically obtain all relationships holding for the body parameters defined above.

- At 0th order, we obtain various relationships, such as the antisymmetry of the spatial components of the spin and electromagnetic dipole tensors.

- At 1st order, we obtain other relationships including

$$\frac{d}{dt_0}m = 0 \ , \quad \frac{d}{dt_0}S_{ij} = -Q^{\mu}{}_{[i}F^{\text{ext}}_{j]\mu} \ , \quad ma_i = qF^{\text{ext}}_{0i} \ .$$

- At 2nd order, we obtain

$$\begin{aligned} m\delta a_i &= -(\delta m)a_i + (\delta q)F^{\text{ext}}_{0i} + q\delta F^{\text{ext}}_{0i} + \frac{2}{3}q^2\dot{a}_i + \\ &\quad + \frac{1}{2}Q^{\mu\nu}\partial_i F^{\text{ext}}_{\mu\nu} + \frac{d}{dt_0}\left(a^j S_{ji} + 2Q^j{}_{[i}F^{\text{ext}}_{0]j}\right) \\ \frac{d}{dt_0}\delta m &= \frac{1}{2}Q^{\mu\nu}\partial_0 F^{\text{ext}}_{\nu\mu} - \frac{\partial}{\partial t_0}\left(Q^{\mu 0}F^{\text{ext}}_{0\mu}\right) \end{aligned}$$

Note that there is no evolution equation for $Q^{\mu\nu}$.

Perturbed Equations of Motion in Covariant Form

Define

$$\delta\hat{m} \equiv \delta m - u_b u^c Q^{bd} F^{\rm ext}_{cd} \ .$$

Then, we have

$$\delta[\hat{m} a_a] = \delta[q F^{\rm ext}_{ab} u^b] + \left(g_a{}^b + u_a u^b\right) \left\{ \frac{2}{3} q^2 \frac{D}{d\tau} a_b + \frac{1}{2} Q^{cd} \nabla_b F^{\rm ext}_{cd} + \frac{D}{d\tau} \left(a^c S_{cb} + 2 u^d Q^c{}_{[b} F^{\rm ext}_{d]c} \right) \right\}$$

$$\frac{D}{d\tau} S_{ab} = -2 \left(g^a{}_c + u^a u_c\right) \left(g^b{}_d + u^b u_d\right) Q^e{}_{[c} F^{\rm ext}_{d]e} - 2 a^c S_{c[a} u_{b]}$$

$$\frac{D}{d\tau} \delta\hat{m} = -\frac{1}{2} Q^{ab} \frac{D}{d\tau} F^{\rm ext}_{ab} - 4 Q_a{}^b F^{\rm ext}_{bc} a^{[c} u^{a]}$$

Self-Consistent Motion

One can rewrite the perturbed equations of motion in terms of a deviation vector describing the perturbation of the worldline from Lorentz force motion. However, even if the deviation from Lorentz force motion is *locally* small, at late times the deviation from a single, fixed Lorentz force trajectory will, in general, become large. Would like to invent a *self consistent perturbative equation* that corrects the Lorentz force trajectory "as one goes along" so as to give a good, global-in-time, description of the motion. Candidate equations: "Remove the δ's", and "reduce order" on the right side, i.e., replace a by

$$a_a = \frac{q}{m} F^{\text{ext}}_{ab} u^b$$

and replace $da^a/d\tau$ by

$$\begin{aligned} \frac{da_b}{d\tau} &= u^c \nabla_c \left(\frac{q}{m} F^{\text{ext}}_{bd} u^d \right) \\ &= u^d u^c \nabla_c \left(\frac{q}{m} F^{\text{ext}}_{bd} \right) + \left(\frac{q}{m} F^{\text{ext}}_{bd} \right) \frac{q}{m} F^{\text{ext}\,d}{}_e u^e \end{aligned}$$

The resulting equations should give an accurate and completely satisfactory description of the motion of a sufficiently small blob of charge.

Non-Relativistic Form of Final Equations of Motion

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + \frac{2}{3} q^2 \frac{d\vec{a}}{dt} + p_i \vec{\nabla} E^i + \mu_i \vec{\nabla} B^i$$
$$+ \frac{d}{dt}\left(\vec{s} \times \vec{a} + \vec{\mu} \times \vec{E} + \vec{p} \times \vec{B}\right) \ ,$$

where it is understood that $\vec{a}$ and $d\vec{a}/dt$ on the right side of this equation are to be eliminated by reduction of order using the Lorentz force equation.