Self-Force for a Scalar Particle in Kerr spacetime: Circular Equatorial Orbits

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EMRIs

Believe that many astrophysical supermassive black holes will be highly rotating

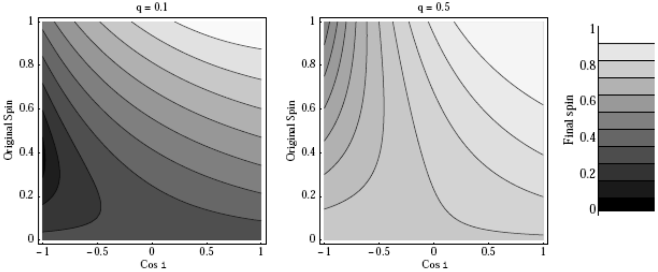


Figure: S. Hughes' and R. Blandford's [2003] estimation of final black hole spin after mergers

State of the art

Field	Trajectory	Author
Scalar	circular	Burko [2000], Detweiler *et al*[2003]
Scalar	radial	Barack & Burko [2000]
Scalar	eccentric	Haas [2007]
Electromagnetic	eccentric	Haas [2008]
Gravitational	radial	Barack & Lousto [2002]
Gravitational	circular	Barack & Sago [2007], Detweiler [2008]
Gravitational	eccentric	Barack & Sago [2009]

- Kerr-Newman: Scalar, static particle. Burko & Liu [2001]

Self force and mode-sum

- In the mode-sum scheme we decompose the full, direct and tail component of the field into spherical harmonics
- Regularization is then done mode-by-mode

$$F_\alpha^{\text{self}} = \sum_{l=0}^{\infty} \left[\lim_{x \to z} F_\alpha^{(\text{full})l}(x) - A_\alpha L - B_\alpha - C_\alpha / L \right] - D_\alpha$$

where $L = l + 1/2$

- Regularization parameters for Kerr derived by Barack & Ori [2002]

Self force for a scalar particle orbiting a Kerr black hole

(circular, equatorial orbits)

Wave equation

The minimally coupled Klein-Gordan equation with source T

$$\Box\Phi \equiv \Phi_{;\alpha}^{\ \alpha} = 4\pi T$$

Point source

$$\begin{aligned} T &= q\int \delta^4(x - x_p(\tau))[-g(x)]^{-1/2} d\tau \\ &= \frac{q}{\rho^2 u^t}\delta(r - r_p)\delta(\phi - \phi_p)\delta(\theta - \theta_p) \end{aligned}$$

Source decomposition

Decomposing the source into spheroidal and frequency modes

$$T = \int \sum_{lm} \hat{T}_{lm}(r) Z_{lm}(\theta, \phi) e^{-i\omega t} \, d\omega$$

and noting that $Z_{lm}e^{-i\omega t}$ are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int f^* g \, d(\cos\theta) d\phi dt$$

we find that for a circular, equatorial orbit

$$\begin{aligned} \hat{T}_{lm} &= \langle Z_{lm} e^{-i\omega t}, T \rangle \\ &= -\frac{q}{\rho^2 u^t} c_{lm} S_{lm}(-a^2 m^2 \Omega^2, 0) \delta(r - r_0) \end{aligned}$$

Seperation of variables

Making the standard seperation of variables ansatz

$$\Phi(t,r,\theta,\phi)=\sum_{lm\omega}R(r)_{lm\omega}S(\theta)_{lm\omega}e^{im\phi}e^{-i\omega t}$$

The homogenius radial equation splits into radial and angular parts

$$\Delta\frac{\partial}{\partial r}\left(\Delta\frac{\partial R}{\partial r}\right)+(a^2m^2-4Mra\omega+(r^2+a^2)^2\omega^2-a^2\omega^2\Delta-\lambda_{lm}\Delta)R=0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)+\left(\lambda_{lm}+a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}\right)S=0$$

- Circular orbits so $\omega = m\Omega_\phi$
- Changing to the tortoise coordinate r_* and defining a new variable $\psi_{lm}(r) = rR_{lm}(r)$ the radial equation becomes

$$\frac{d^2\psi_{lm}}{dr_*^2} + W_{lm}(r)\psi_{lm} = 0$$

where $W_{lm}(r)$ is an effective potential

Physical considerations

Radiation must be ingoing at the horizon and outgoing at infinity

Infinity boundary condition

Asymptotically the field will be of the form

$$\Phi_{lm} \sim e^{-i\omega t} e^{\pm i\kappa r_*} S_{lm}(\theta) e^{im\phi}$$

where $\kappa = [W(r_* \to \infty)]^{1/2}$. The correct choice of sign preceding κ will ensure that the field depends only on the 'asymptotically' outgoing null coordinate $u = t - r_*$

Horizon boundary condition

Things not quite as simple at the horizon due to frame dragging. At the horizon spacetime is dragged with an angular velocity

$$\omega_+ = \frac{a}{2Mr_+}$$

Hence an observer sees the coordinate dependence as

$$\Phi_{lm} \sim e^{-i(\omega - m\omega_+)t} e^{\pm i\kappa_- r_*} S_{lm}(\theta) e^{im\phi}$$

where $\kappa = [W(r_* \to -\infty)]^{1/2}$.

Numerically we cannot integrate out to the boundaries at $r_* = \pm\infty$

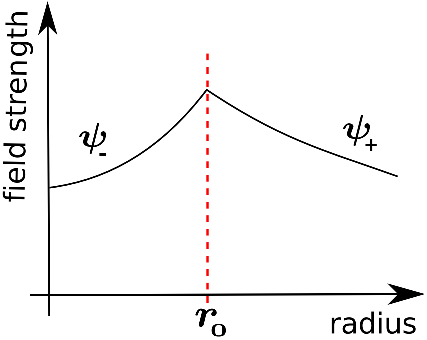
Expansion near boundaries

We expand the radial component in a series those $k > 0$ terms shrink to zero as we approach the relevant boundary

$$\psi_+(r) = e^{+i\kappa_+ r_*} \sum_{k=0}^{\infty} c_k^+ r^{-k}$$

$$\psi_-(r) = e^{\pm i\kappa_- r_*} \sum_{k=0}^{\infty} c_k^- (r - r_+)^k$$

Substitute these back into the radial equation to get recursion relations for series coefficients $c_k^\pm$



Field continuity

Consider the field ψ to be made of two pieces

$$\psi(r_*) = \psi_-(r_*)\Theta(r_{*0} - r_*) + \psi_+(r_*)\Theta(r_* - r_{*0})$$

and substitute into the radial equation

Matching

Comparing coefficients of the delta function and its derivative gives us c_0^+ and c_0^-

Angular piece of decomposition:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)+\left(\lambda_{lm}+a^2\omega^2\cos^2\theta-\frac{m^2}{sin^2\theta}\right)S=0$$

Decomposition into spherical harmonics Y_{jm}

$$S^c_{lm}(\theta)=\sum_{j=0}^{\infty}b^c_j Y_{jm}(\theta)$$

Adopt Dirac-style notation such that

$$Y_{jm}(\theta)\to|jm\rangle,\qquad Y^*_{jm}(\theta)\to\langle jm|,$$

$$\int_0^\pi Y^*_{lm}(\theta)f(\theta)Y_{jm}(\theta)\sin\theta d\theta\to\langle lm|f(\theta)|jm\rangle$$

Substituting series expansion into angular equation and noting that the Y_{jm} satisfy the angular equation with $c = 0$ and $\lambda_{jm} = j(j+1)$ we find

$$\sum_{j=0}^{\infty} b_j^c [c^2 \cos^2\theta - j(j+1)] |jm\rangle = -\lambda_{lm} \sum_{j=0}^{\infty} b_j^c |jm\rangle$$

Now multiply the above by $\langle lm|$.

$$\begin{aligned} \langle lm| \cos^2\theta |jm\rangle &= \frac{1}{3} + \frac{2}{3}\sqrt{\frac{2j+1}{2l+1}} \langle j,2,m,0|l,m\rangle \langle j,2,0,0|l,0\rangle \equiv k^m_{j,l,2} \\ \langle lm|jm\rangle &= \delta_{jl} \end{aligned}$$

Where the numbers $\langle j_1, j_2, m_1, m_2 | jm\rangle$ are Clebsch-Gordan coefficients. The presence of Clebsch-Gordan coefficients tells us that $k^m_{j,l,2} \neq 0$ only for $j \in \{l-2, l-1, l, l+1, l+2\}$. This leads to a simple recursion relation for the series coefficients b_j^c.

Frequency Domain Implementation

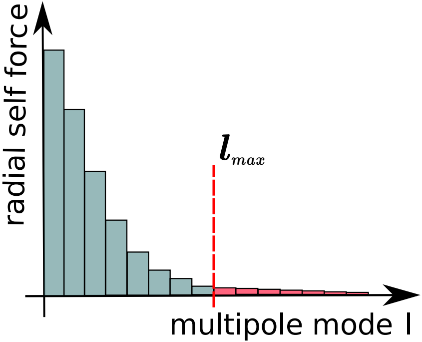
The code

- Written in c
- Makes use of multiple cores
- For a given spin, calculating 55 l modes takes approximately 30mins
- Recently rewritten to run on a cluster

Algorithm

- evolve radial vacuum solutions from the boundaries to the particle
- fix amplitudes by matching
- spheroidal to spherical decomposition
- mode-sum regularization
- estimation of the large-l tail

Estimation of large-l tail



Truncation

Mode-sum method requires summing over all l modes.

$$F^r = \sum_{l=0}^{l_{\text{max}}} F^{rl}_{\text{reg}} + \sum_{l=l_{\text{max}}+1}^{\infty} F^{rl}_{\text{reg}}$$

Estimation

Fit the F^r for $l < l_{\text{max}}$ using

$$F^{rl}_{\text{reg}} \simeq \sum_{n=1}^{N} \frac{D^r_{2n}}{L^{2n}}$$

Code validation: high-l form

Contribution to the self force per mode is known to fall of as $1/L^2$

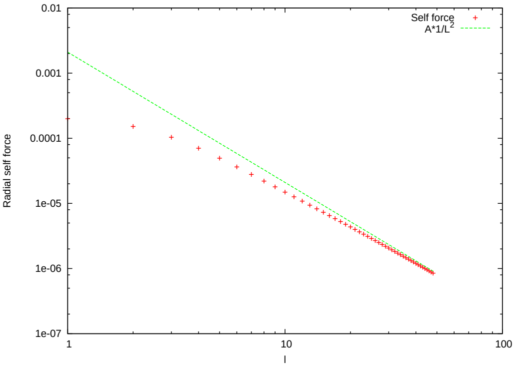


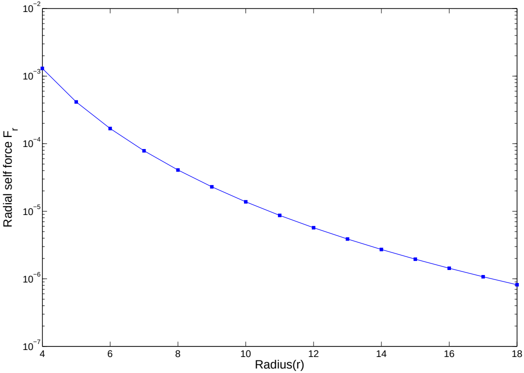
Figure: a=0.5 r=5

Code validation: energy flux

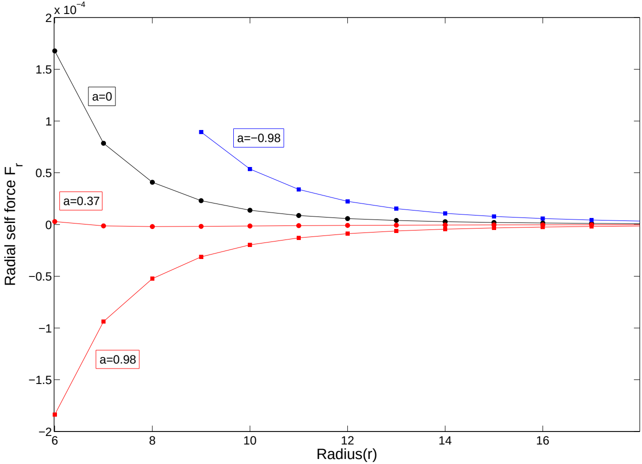
- The time component of the self force $F_t = -\mu(\dot{E}_\infty + \dot{E}_+)$
- Can calculate both sides of the above equation from the code.
- Find perfect agreement to 14 s.f.
- F_t^l converges exponentially, so only low modes contribute significantly. Hence only low modes are tested.

Code validation: zero spin

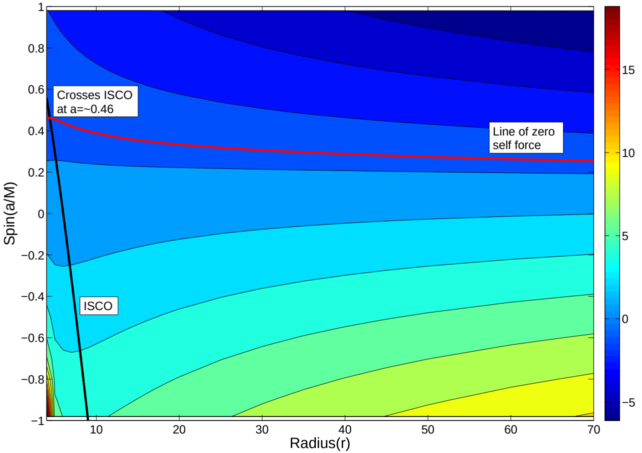
Reproduces known Schwarzschild results of Diaz-Rivera, Messaritaki, Whiting, and Detweiler [2004]



Results


x 10^−4
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
6
8
10
12
14
16
Radius(r)
Radial self force F_r
a=0
a=−0.98
a=0.37
a=0.98

Results - $r^5 F_r$



Future work

- Immediate: understand consequences of sign change in the force
- Near future: effect of spin on the Inner Most Stable Circular Orbit (ISCO)
- Next few months: extension to eccentric equatorial orbits ($\omega = n\Omega_r + m\Omega_\phi$)
- Long term plan: extension to generic orbits in Kerr and gravitational case

Summary

- Mode-sum implementation for Kerr
- First self force calculation in Kerr for a realistic orbit
- Interesting new effects in Kerr

Any questions?

Supplementary slides

Eccentric orbits

- Method of extended homogenius solutions put information here that might be useful for answering questions