Matched Expansion Self-force Calculations

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Overview

- Introduce matched expansions method
- Quasi-local contribution (arXiv:0903.5319, 0906.0005) - Barry
- Distant Past contribution & practical application of matched expansions method (arXiv:0903.0395) - Marc
- Quasinormal Mode expansions for Green function in Schwarzschild - Sam

Self-Force - Formal Expression

- MiSaTaQuWa expression for scalar SF

$$\nabla_\mu \Phi_R = \left(\frac{1}{2}m^2 - \frac{1}{12}(1-6\xi)R\right) q u_\mu + q(g_{\mu\nu} + u_\mu u_\nu)\left(\frac{1}{3}\dot{a}^\nu + \frac{1}{6}R^\nu{}_\lambda u^\lambda\right) + \Phi^{\text{tail}}_\mu$$

- Mainly interested in calculating the tail term - an integral of the derivative of the retarded Green's function over the past world-line of the particle:

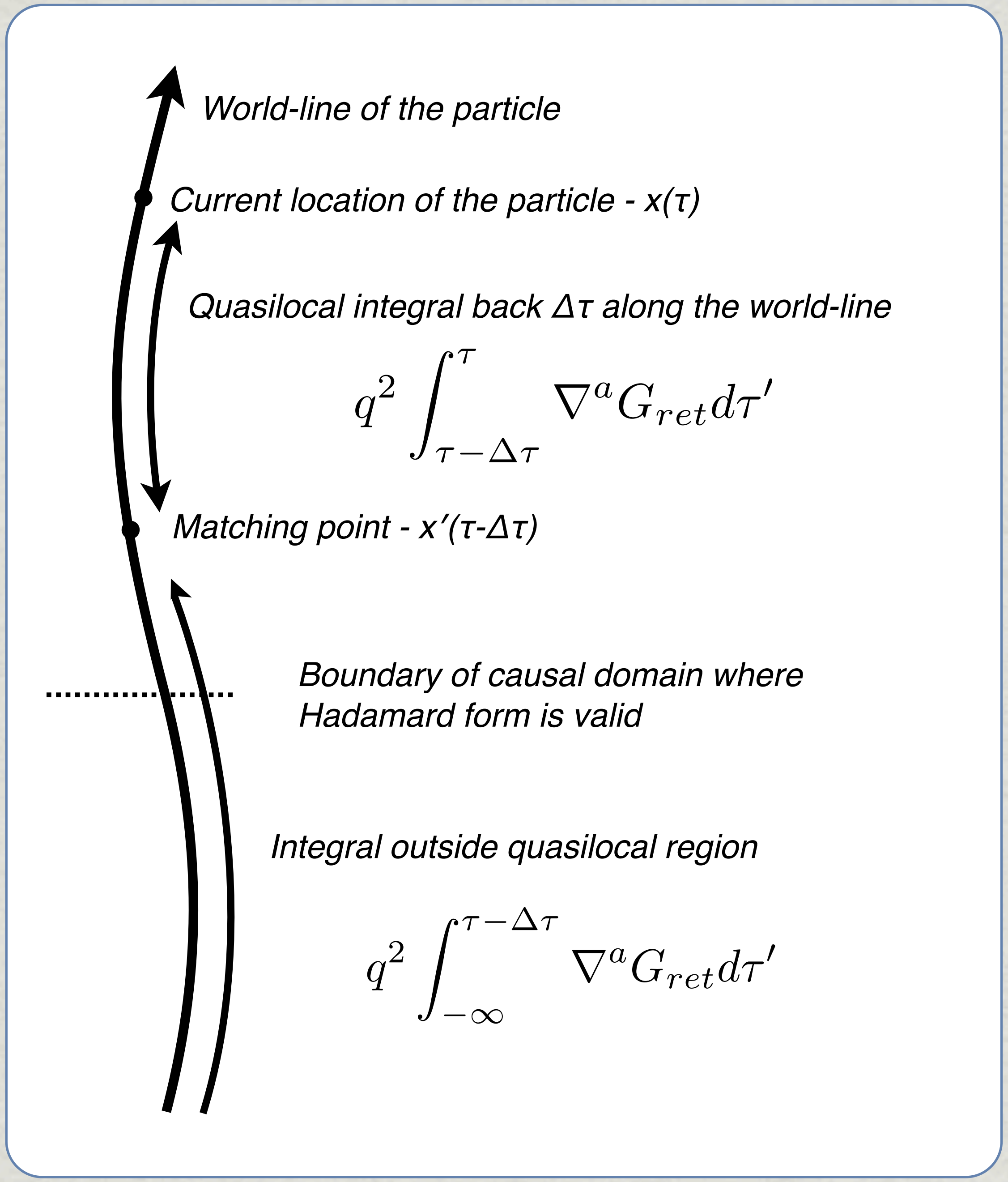
$$\Phi^{\text{tail}}_\mu = q \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau-\epsilon} \nabla_\mu G_{\text{ret}}(z(\tau), z(\tau'))d\tau'$$

$$\mathcal{D}^A{}_B G_{\text{ret}}{}^B{}_{C'}(x,x') = -4\pi\delta^A{}_{C'}\delta(x,x')$$

$$\mathcal{D}^A{}_B = \delta^A{}_B(\Box - m^2) + P^A{}_B$$

Matched Expansion

- Poisson & Wiseman (Capra 1)
- Anderson & Wiseman (CQG 22 (2005))
- Select point Δτ along the world-line
- Separate tail integral into two regimes:
- 1. Quasilocal region from the recent past (QL)
- 2. Contribution from “distant” past (DP)



Quasilocal Region: Hadamard form

- Provided *x* and *x′* are sufficiently "close" together, the Hadamard Form of the Green function can be used:

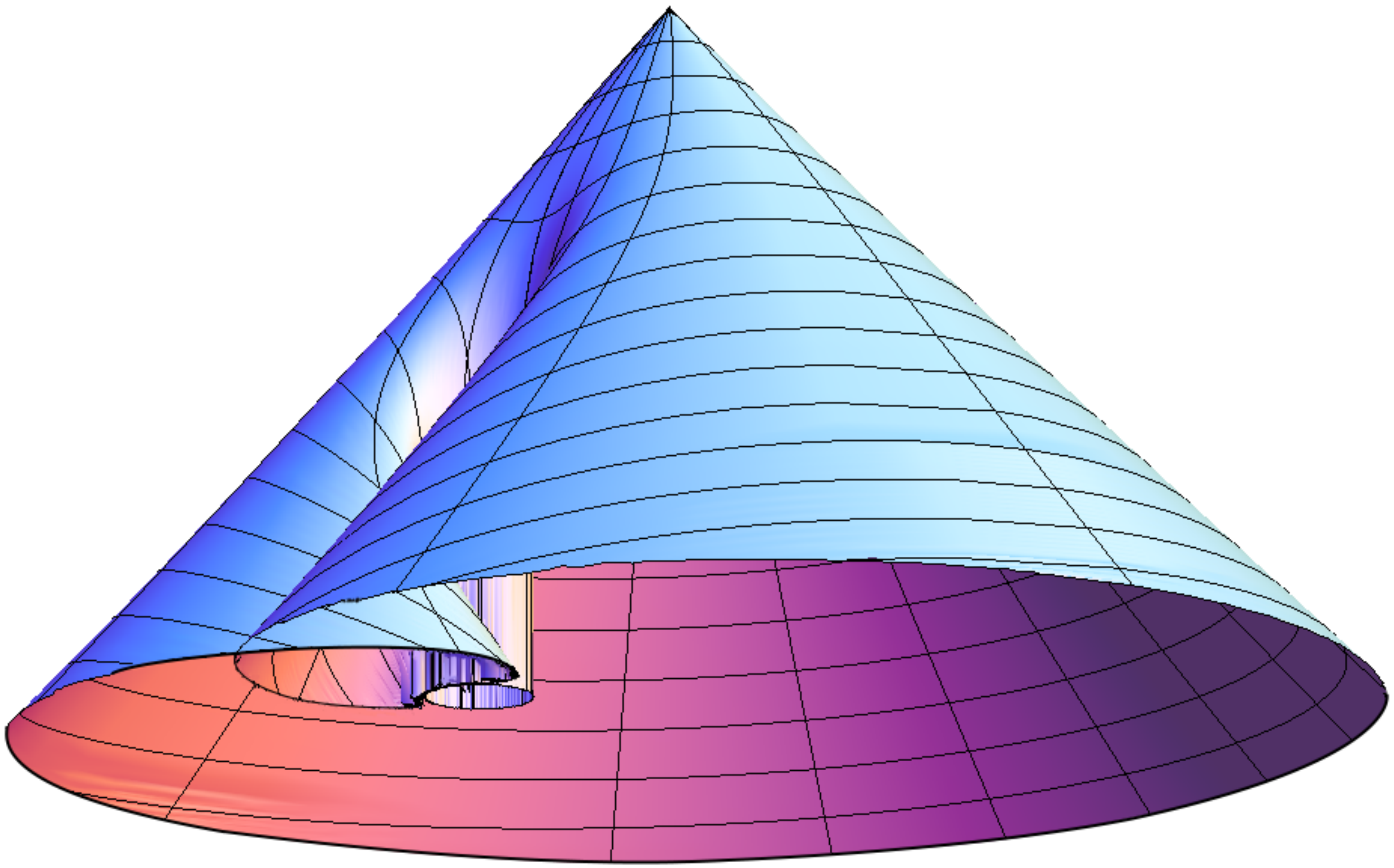
$$G_{\mathrm{ret}}{}^{A}{}_{B'}(x,x') = \theta_{-}(x,x')\left\{U^{A}{}_{B'}(x,x')\delta\left(\sigma(x,x')\right) - V^{A}{}_{B'}(x,x')\theta\left(-\sigma(x,x')\right)\right\}$$

- Only part with *V(x,x′)* contributes to the QL self-force

$$f^{a}_{\mathrm{QL}} = -q^2 \int_{\tau-\Delta\tau}^{\tau} \nabla^{a} V(x,x')d\tau'$$

- The problem is now to calculate *V(x,x′).*

Normal Neighborhood



Hadamard Series

- Express V(x,x′) as an expansion in powers of σ

$$V^{AB'}(x,x') = \sum_{r=0}^{\infty} V_r{}^{AB'}(x,x')\sigma^r(x,x')$$

⬇

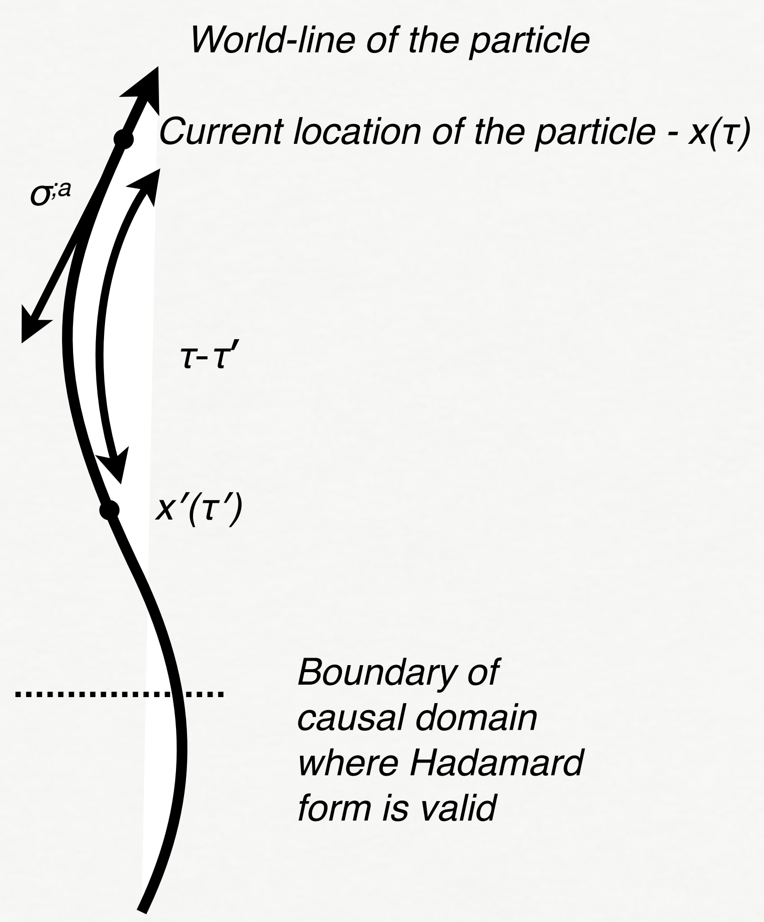
$$\sigma^{;\alpha'}(\Delta^{-1/2}V_0^{AB'})_{;\alpha'} + \Delta^{-1/2}V_0^{AB'} + \frac{1}{2}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}g^{AC'}) = 0$$

$$\sigma^{;\alpha'}(\Delta^{-1/2}V_r^{AB'})_{;\alpha'} + (r+1)\,\Delta^{-1/2}V_r^{AB'} + \frac{1}{2r}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}V_{r-1}^{AC'} = 0$$

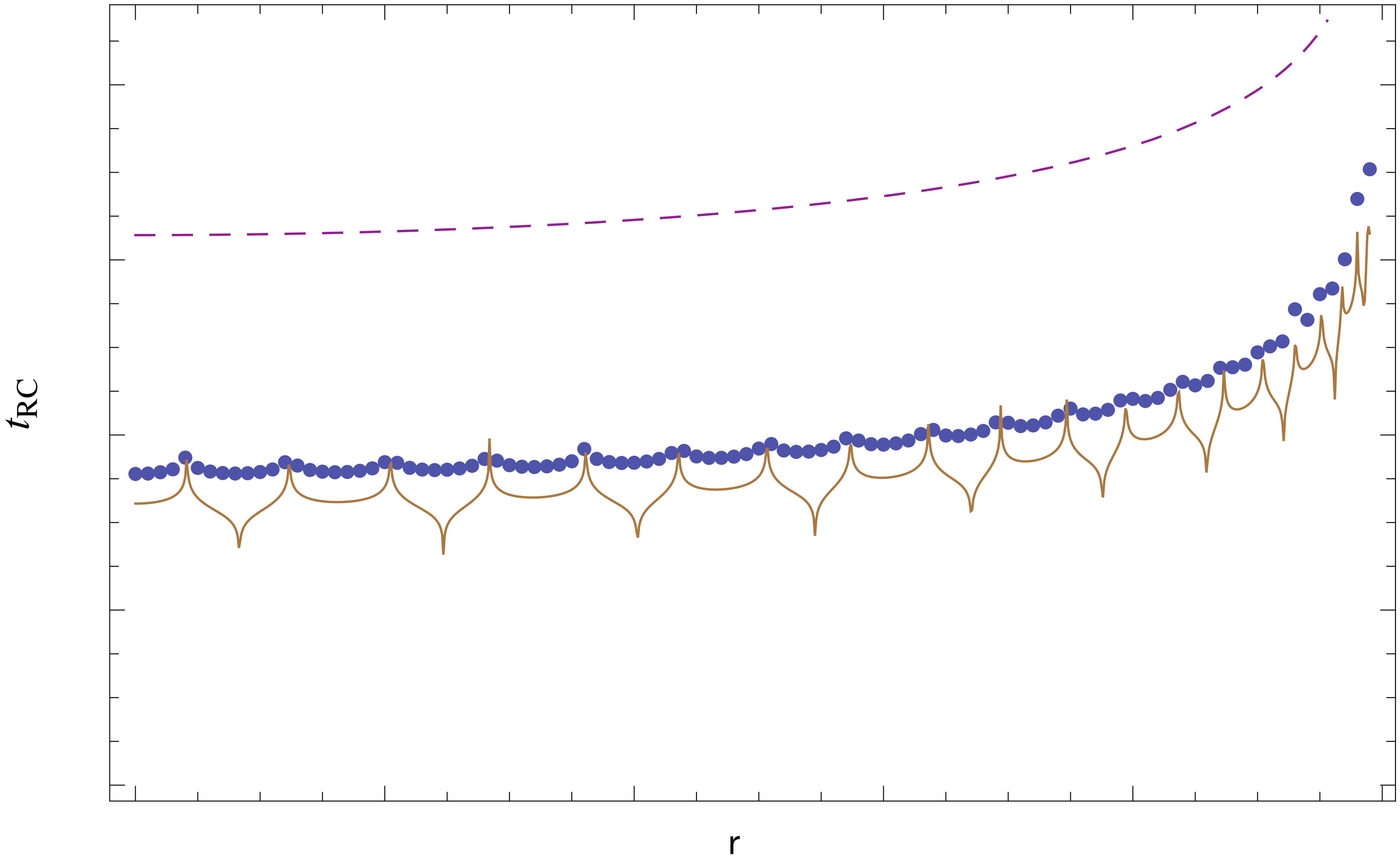
Calculating *V(x,x′)* - *Series*

- *V(x,x′)* expanded as a covariant Taylor series.

$$V^{AB'}(x,x') = \sum_{p=0}^{\infty} \frac{1}{p!} V^{AB'}{}_{\alpha_1 \cdots \alpha_p}(x) \sigma^{\alpha_1}(x,x') \cdots \sigma^{\alpha_p}(x,x')$$



Radius of Convergence



Radius of convergence for time separated points (i.e. static particle) as a function of radial position

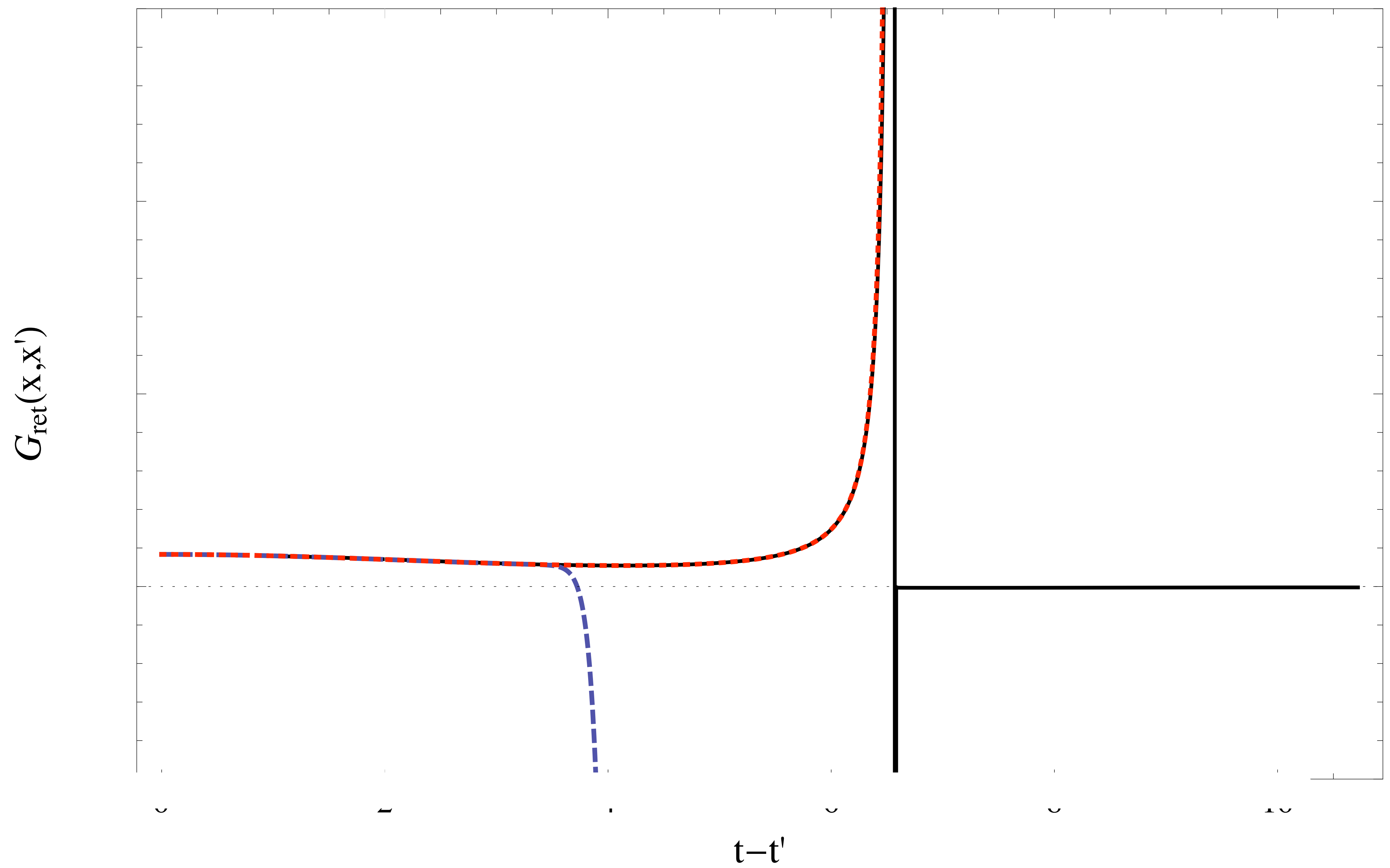
Padé Resummation

- Try to express V(x,x') as a rational function.

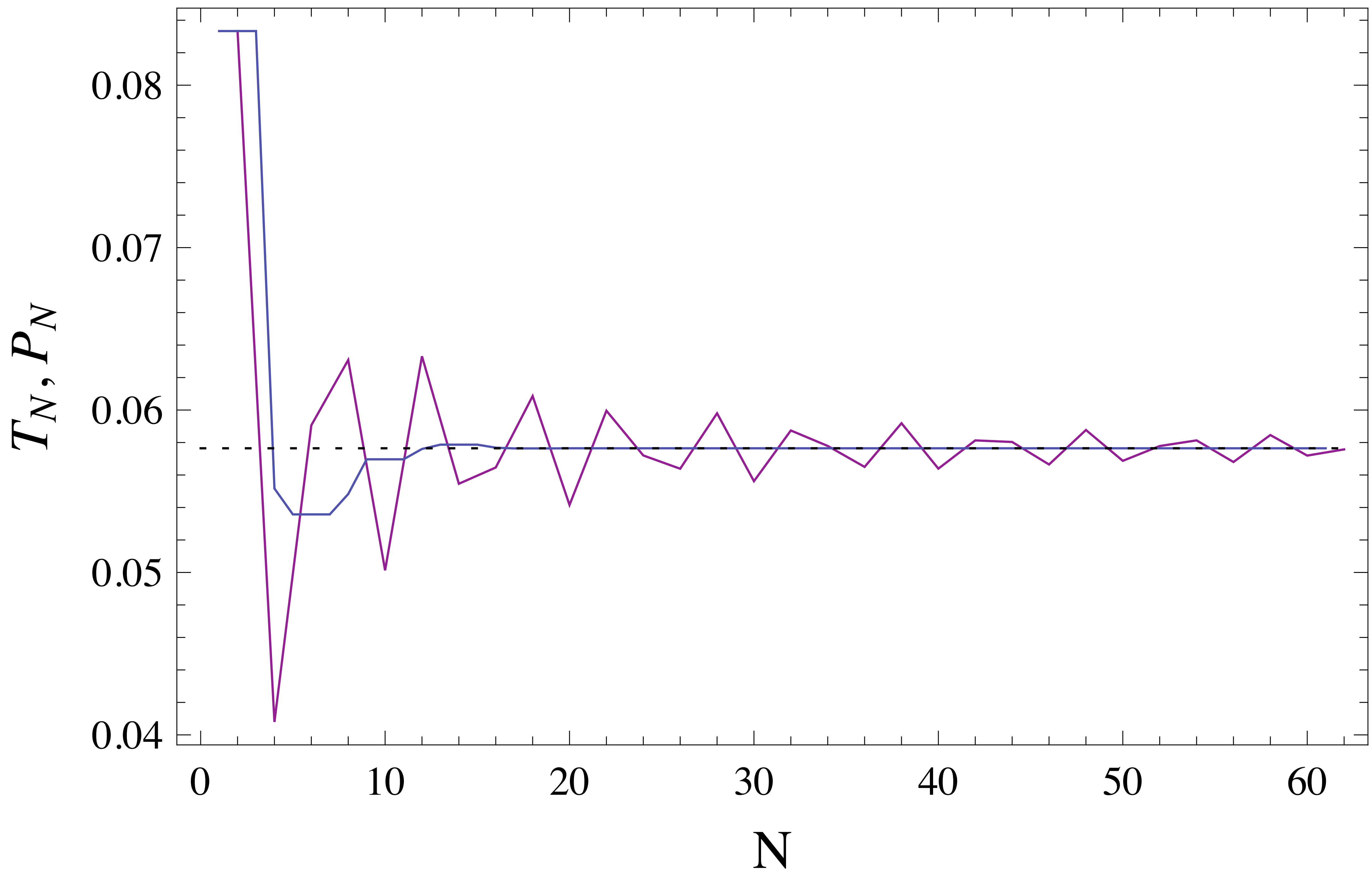
$$P_M^N(t-t') \equiv \frac{\sum_{n=0}^{N} A_n (t-t')^n}{\sum_{n=0}^{M} B_n (t-t')^n}$$

- Captures the behavior of the singularities in functional form.
- Accelerate convergence of the series.
- Extend domain of series.

Padé Resummation



Padé Resummation



Numerical Calculation

System of transport equations for $\Delta^{1/2}(x,x')$ and $V_r(x,x')$ along geodesics.

$$D' \equiv \sigma^{;\alpha'}\nabla_{\alpha'} = s'\left(\frac{d}{ds'} + \Gamma^{a'}_{b'\gamma'}u^{\gamma'} + \cdots\right)$$

$$D' \ln \Delta = (4 - \sigma^{\alpha'}{}_{\alpha'})$$

$$D'\sigma^{a'}{}_{b'} = -\sigma^{a'}{}_{\alpha'}\sigma^{\alpha'}{}_{b'} + \sigma^{a'}{}_{b'} - R^{a'}{}_{\alpha' b' \beta'}\sigma^{\alpha'}\sigma^{\beta'}$$

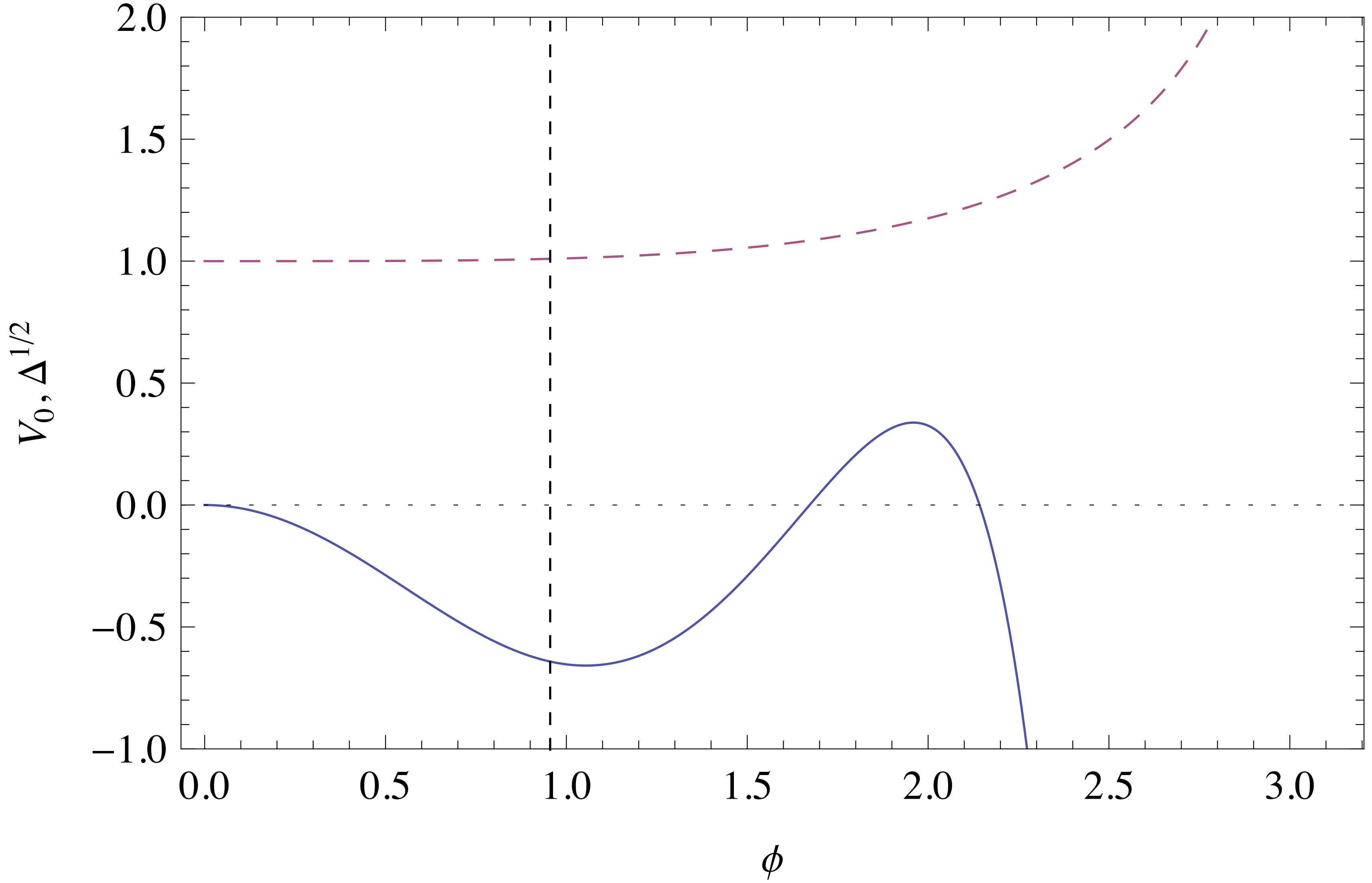
$$(D'+1)\,V_0 + \frac{1}{2}V_0\left(\sigma^{\mu'}{}_{\mu'} - 4\right) + \frac{1}{2}(\Box' - \xi R')\Delta^{1/2} = 0$$

$$(D'+r+1)\,V_r + \frac{1}{2}V_r\left(\sigma^{\mu'}{}_{\mu'} - 4\right) + \frac{1}{2r}(\Box' - \xi R')V_{r-1} = 0$$

$$\vdots$$

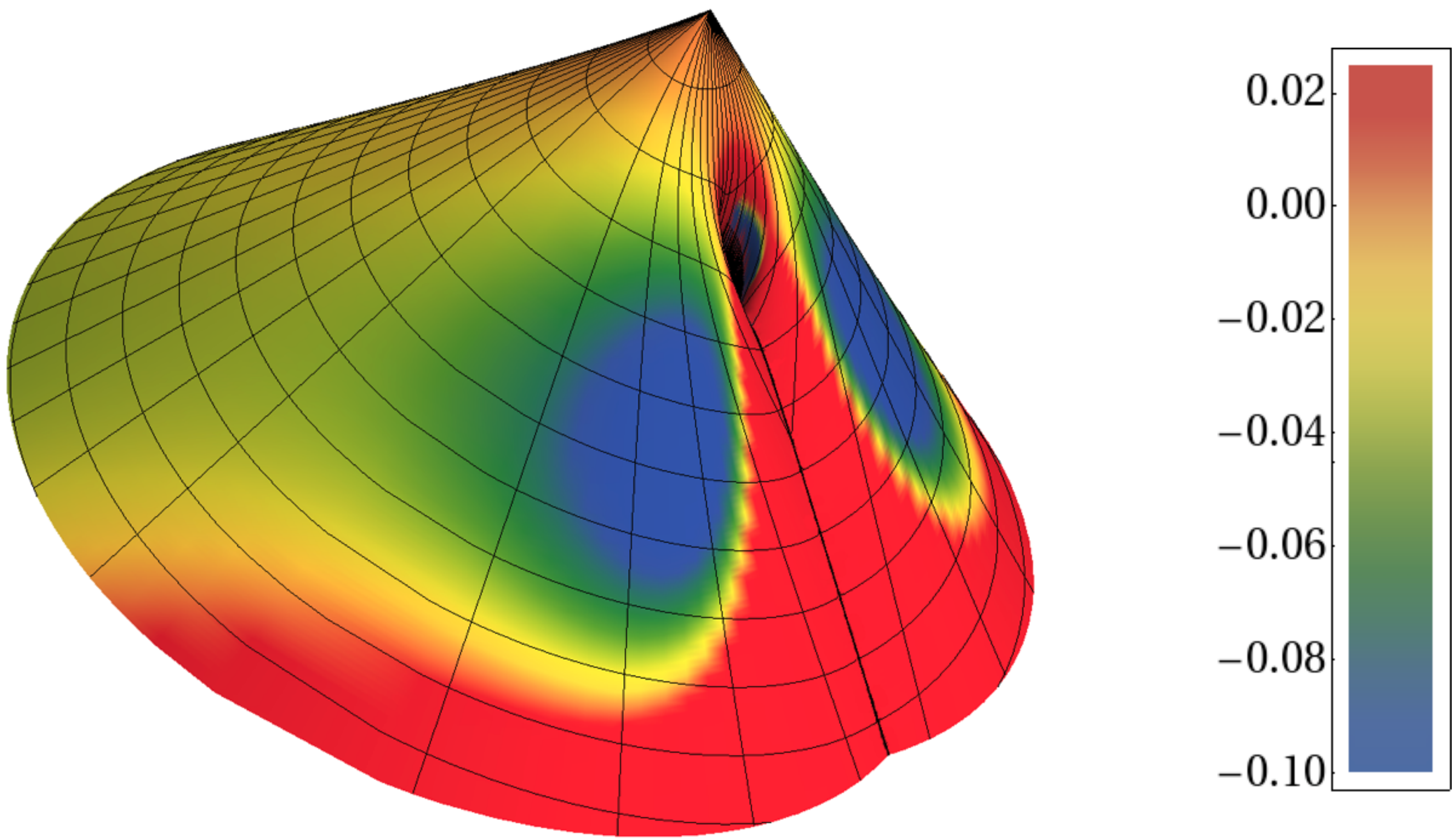
Numerical Calculation

Numerically integrate the transport equations (ODEs) for $V_r(x,x')$ along geodesics.



Numerical Calculation

Numerically integrate the transport equations (ODEs) for $V_r(x,x')$ along geodesics.



Is QL enough?

- No!
- Can calculate the Hadamard Green function everywhere in the normal neighborhood
- But, that's not enough for the self force - the DP Green function is crucial

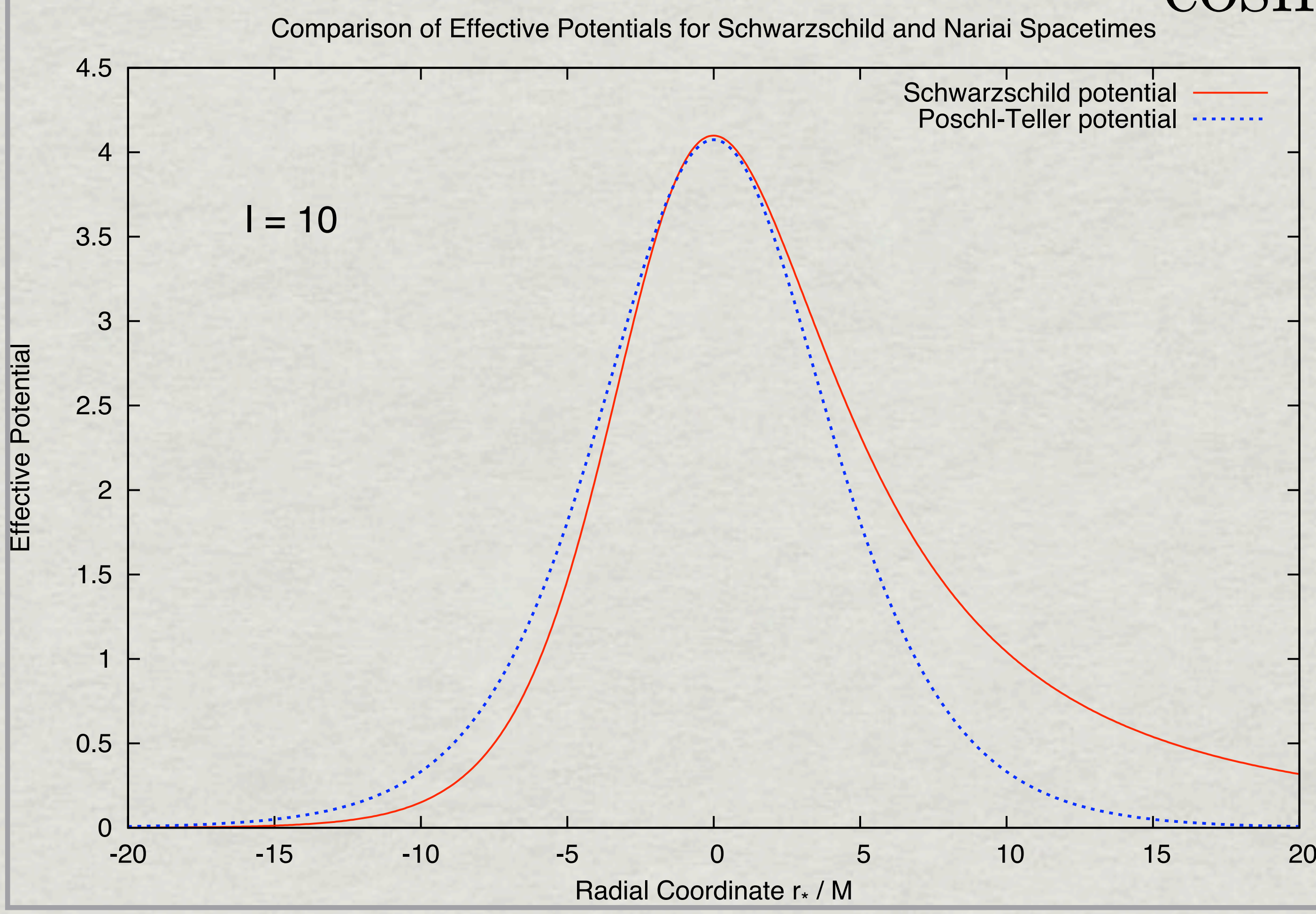
Distant past - Mode sum decomposition:

$$\Phi(x) = \int_{-\infty}^{\infty} d\omega \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} c_{lm\omega}\Phi_{lm\omega}(x) \qquad \Phi_{lm\omega}(x) = \frac{u_{l\omega}(r)}{r} Y_{lm}(\theta,\phi) e^{-i\omega t}$$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_l(r)\right] u_{l\omega}(r) = 0$$

Schwarzschild: $V_l^{(S)}(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) \longrightarrow \begin{matrix} O(r_*^{-2}) & r_* \to +\infty \\ O(e^{r_*/(4M)}) & r_* \to -\infty \end{matrix}$

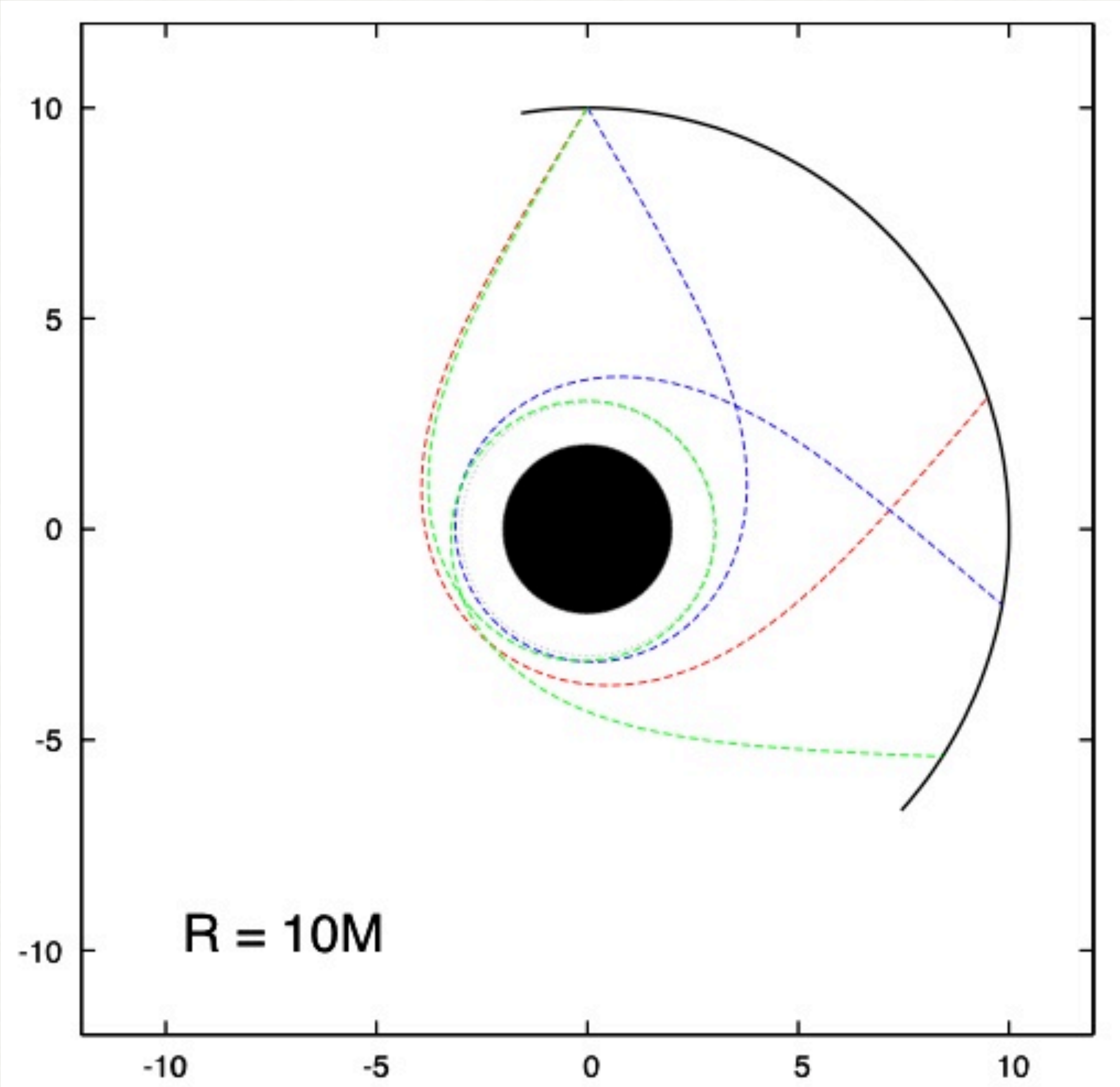
Poschl-Teller: $V_l^{(PT)}(r_*) = \frac{\alpha^2 \; l(l+1)}{\cosh^2(\alpha r_*)} \longrightarrow O(e^{\mp 2\alpha r_*}) \; r_* \to \pm\infty$



They share an unstable photon orbit (at $r_* = 0$)

$$\alpha = \frac{1}{\sqrt{27M}}$$

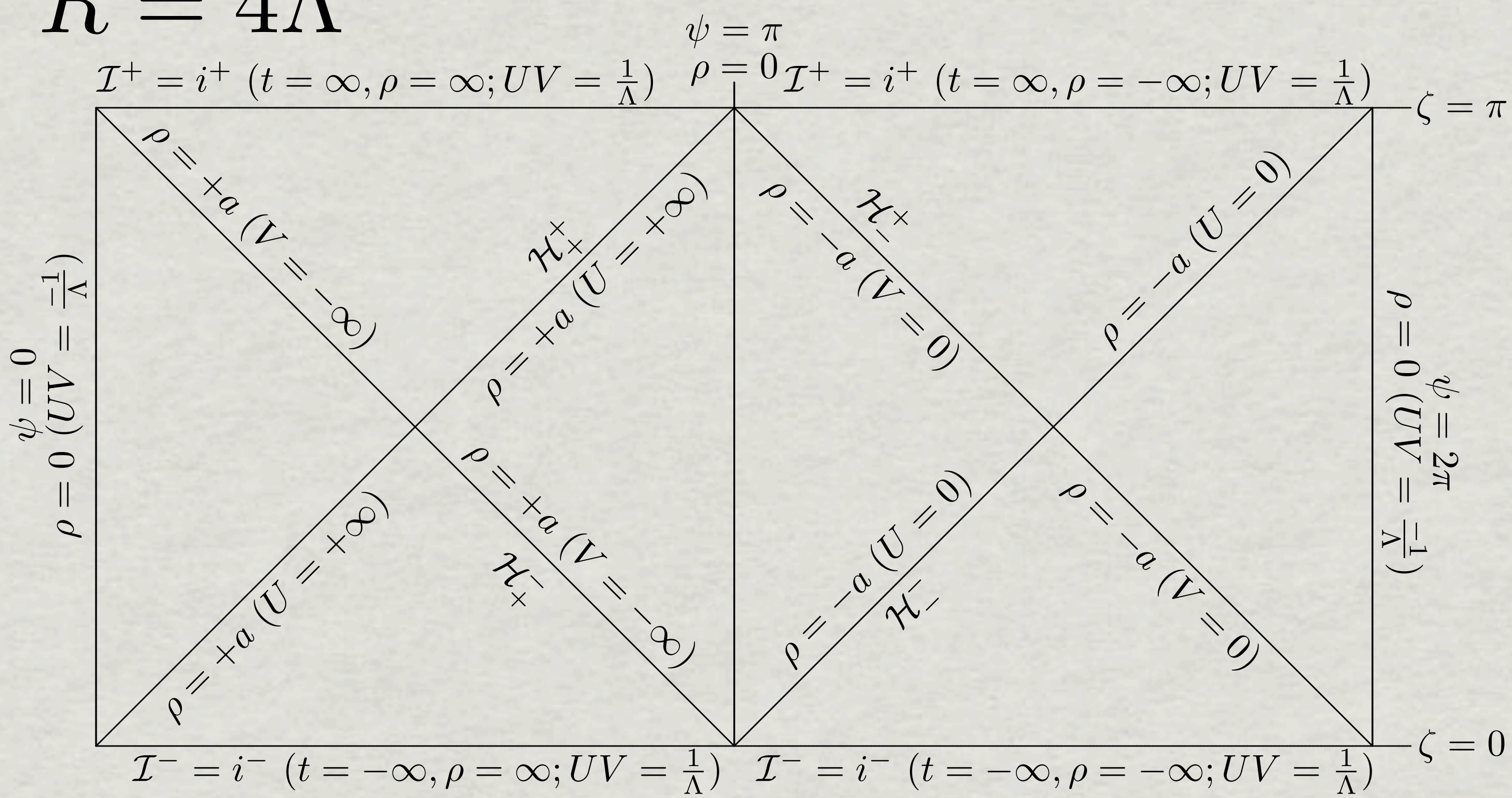
Unstable photon orbit -> null geodesics originating from a timelike worldline may reintersect it later



Is Poschl-Teller potential for radial component of wave eq. in any spacetime? yes!

Nariai (1950): $$ds^2 = -d\mathcal{T}^2 + \frac{1}{\Lambda}\cosh^2\left(\sqrt{\Lambda}T\right)d\psi^2 + \frac{1}{\Lambda}d\Omega_2^2$$

- homogeneous
- Topology: $dS_2 \times \mathbb{S}^2$
- Constant Ricci scalar: $R = 4\Lambda$



Static region: $$ds^2 = -(1-\rho^2)dt^2 + (1-\rho^2)^{-1}d\rho^2 + d\Omega_2^2$$

$$\rho \in (-1, +1), \;\; \alpha r_* \equiv \rho_* = \tanh^{-1}(\rho) \in (-\infty, +\infty) \;\; [\Lambda = 1]$$

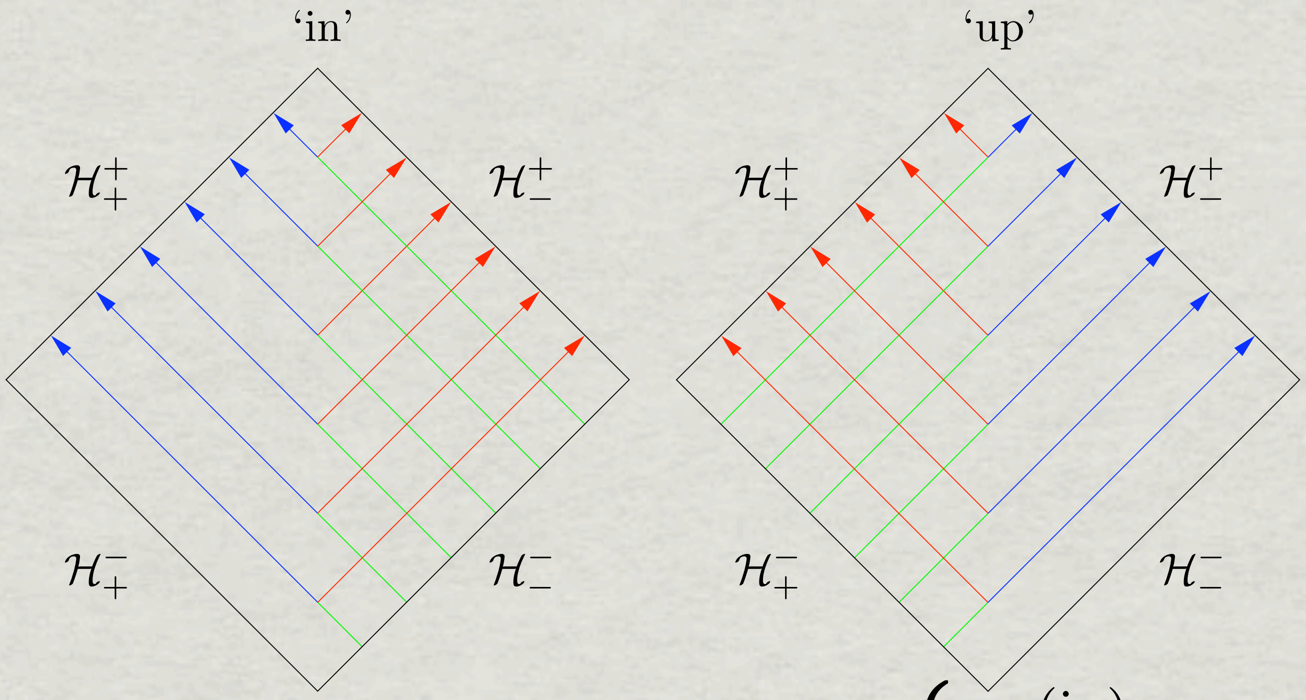
Retarded Green function

$$G_{ret}(t,\rho_*;t',\rho'_*;\gamma) = \frac{1}{2\pi}\int_{-\infty+ic}^{+\infty+ic} d\omega \sum_{l=0}^{+\infty} \tilde{g}_{l\omega}(\rho_*,\rho'_*)(2l+1)P_l(\cos\gamma)e^{-i\omega(t-t')}$$

$$\left[\frac{d^2}{d\rho_*^2} + \omega^2 - \frac{U_0}{\cosh^2\rho_*}\right]\tilde{g}_{l\omega}(\rho_*,\rho'_*) = -\delta(\rho_* - \rho'_*)$$

Can be solved analytically:

$$u^{(up)}_{l\omega}(-\rho) = u^{(in)}_{l\omega}(\rho) = \Gamma(1-i\omega)P^{i\omega}_{-1/2+i\lambda}(-\rho) \sim \begin{cases} e^{-i\omega\rho_*}, & \rho_* \to -\infty, \\ A^{(out)}_{l\omega}e^{i\omega\rho_*} + A^{(in)}_{l\omega}e^{-i\omega\rho_*}, & \rho_* \to +\infty, \end{cases}$$

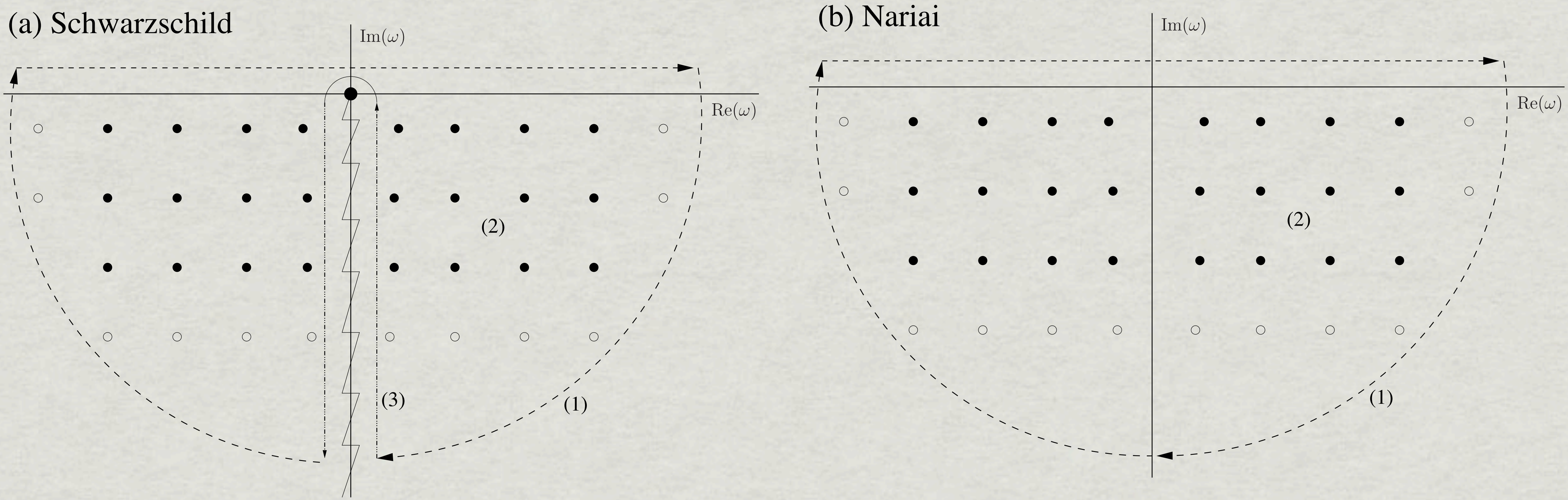


$$\lambda = \sqrt{(l+1/2)^2 + 4\xi - 1/2}$$

$$\tilde{g}_{l\omega}(\rho_*,\rho'_*) = -\frac{1}{W}\begin{cases} u^{(\mathrm{in})}_{l\omega}(\rho_*)u^{(\mathrm{up})}_{l\omega}(\rho'_*), & \rho_* < \rho'_*, \\ u^{(\mathrm{up})}_{l\omega}(\rho_*)u^{(\mathrm{in})}_{l\omega}(\rho'_*), & \rho_* > \rho'_*, \end{cases}$$

$$= \tfrac{1}{2}\Gamma(1/2+i\lambda-i\omega)\Gamma(1/2-i\lambda-i\omega)P^{i\omega}_{-1/2+i\lambda}(-\rho_<)P^{i\omega}_{-1/2+i\lambda}(\rho_>)$$

Leaver'86: deform contour in complex ω-plane



- Nariai potential decays exponentially at infinity->no branch cut (-> no power law tail)
- QNMs: poles of G_{ret}

Schwarzschild: $M\omega_{ln}^{(S)} \approx \frac{1}{\sqrt{27}}\left[\pm(l+1/2) - i(n+1/2)\right]$ as $l \gg n$

Nariai: $\omega_{ln} = -\lambda - i(n+1/2)$ $\qquad \lambda = \sqrt{(l+1/2)^2 + 4\xi - 1/2}$

QNM sum

$$G^{QNM}_{ret}(t,\rho;t',\rho';\gamma) = 2\,\mathrm{Re}\sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}(2l+1)P_l(\cos\gamma)\mathcal{B}_{ln}\tilde{u}_{ln}(\rho)\tilde{u}_{ln}(\rho')e^{-i\omega_{ln}T},\ \ \tilde{u}_{ln}(\rho) = \frac{u^{\mathrm{in}}_{l\omega_{ln}}(\rho)}{A^{(out)}_{l\omega_{ln}}e^{i\omega_{ln}\rho_*}}$$

Excitation factors (~residues at QNMs): $\mathcal{B}_{ln} \equiv \dfrac{A^{(out)}_{l\omega_{ln}}}{2\omega_{ln}\left.\dfrac{dA^{(in)}_{l\omega}}{d\omega}\right|_{\omega_{ln}}}$

- QNM n-sum divergent[convergent] for $T \equiv t - t' - \rho_* - \rho'_* < [>] 0$

(T=0 ~ time for light ray to go from ρ' to 0 and out to ρ)

- Beyer'99: QNMs form a complete basis if $T > 0$

- Modes in QNM l-sum (fixed n) ~ $O((l+1/2)^n)$

=> l-sum is divergent

- QNM n-sum can be done analytically at $\rho,\rho' \to 1$

$$G^{QNM}_{ret}(T,\gamma) \sim \frac{e^{-T/2}}{\sqrt{\pi}}\,\mathrm{Re}\sum_{l=0}^{+\infty}\frac{(l+1/2)\Gamma(i\lambda)}{\Gamma(1/2+i\lambda)}P_l(\cos\gamma)e^{i\lambda(T+2\ln 2)}{}_2F_1(1/2+i\lambda,1/2+i\lambda;1+2i\lambda;-e^{-T}),$$

Singularity structure of the Green function outside the normal neighbourhood

Kay, Radzikowski,Wald (1997) [based on “propagation of singularities” theorems by Duistermaat and Hormander]:

“if such a distributional bisolution [Hadamard elementary function] is singular for sufficiently nearby pairs of points on a given null geodesic, then it will necessarily remain singular for all points on that null geodesic.”

Use Poisson sum formula: $\sum_{l=0}^{+\infty} g(l+1/2) = \sum_{s=-\infty}^{+\infty} (-1)^s \int_0^{+\infty} d\nu g(\nu) e^{2\pi i s \nu}$

and large-ν asymptotics (not valid for $\gamma = 0, \pi$) :

$$G_{ret}^{QNM}(T,\gamma) \sim \sum_{N=0}^{+\infty} G_{ret,N}^{\rho\sim 1}, \quad \rho, \rho' \to 1$$

$$G_{ret,N}^{\rho\sim 1} \sim \left(\frac{e^{-T}}{2\sin\gamma\sqrt{1+e^{-T}}}\right)^{1/2} \begin{cases} (-1)^{N/2}\,\delta\left(t-t'-t_N^{(\rho\sim 1)}\right), & N \text{ even,} \\ \dfrac{(-1)^{(N+1)/2}}{\pi\left(t-t'-t_N^{(\rho\sim 1)}\right)}, & N \text{ odd.} \end{cases}$$

where

$$t_N^{(\rho\sim 1)} \sim \rho_* + \rho'_* + \begin{cases} \ln\left(\sinh^2\left([N\pi+\gamma]/2\right)\right), & N \text{ even,} \\ \ln\left(\sinh^2\left([(N+1)\pi-\gamma]/2\right)\right), & N \text{ odd,} \end{cases}$$

is the time it takes for a null geodesic to go from ρ' to ρ after orbiting around the unstable photon orbit N/2 times -> N is the number of caustics ($\gamma = 0, \pi$) that the null geodesic has crossed.

- **Four-fold singularity structure:** $\delta(\sigma),\ 1/\pi\sigma,\ -\delta(\sigma),\ -1/\pi\sigma,\ \delta(\sigma), \ldots$
- **Known in other fields of Physics; first noted in GR by A.Ori**
- **Characteristic of $\mathbb{S}^2$ topology (-> also in Schwarzschild)**

Alternative (non-rigorous) derivation

Hadamard form of Feynman propagator:

$$G_F(x,x') = \frac{i}{2\pi}\left[\frac{U(x,x')}{\sigma + i\epsilon} + V(x,x')\ln(\sigma + i\epsilon) + W(x,x')\right]$$

$$G_{\text{ret}}(x,x') = 2\theta_-(x,x')\ \text{Re}\left(G_F(x,x')\right)$$

Tentatively, let the 'direct' part remain in Hadamard form outside normal neighbourhood:

$$G^{\text{dir.}}_{ret}(x,x') = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\text{Re}\left[i\frac{U(x,x')}{\sigma + i\epsilon}\right] = \text{Re}\left[U(x,x')\left(\delta(\sigma) + \frac{i}{\pi\sigma}\right)\right]$$

$$U(x,x') = \Delta^{1/2}(x,x')$$

Choose motion on ϕ - plane & factorize Van Vleck determinant:

$$\Delta = \Delta_{\phi}\Delta_{t\rho}$$

$$\theta\frac{d\ln\Delta_\phi}{d\theta} = 1 - \sigma^{\phi}{}_{\phi},$$

$$\theta\frac{d\ln\Delta_{t\rho}}{d\theta} = 2 - \sigma^{t}{}_{t} - \sigma^{\rho}{}_{\rho}$$

Transport eq. -> $\ln \Delta_\phi = \ln\left(\frac{\theta}{\theta_0}\right) - \int_{\theta_0}^{\theta} d\theta' \cot\theta'$

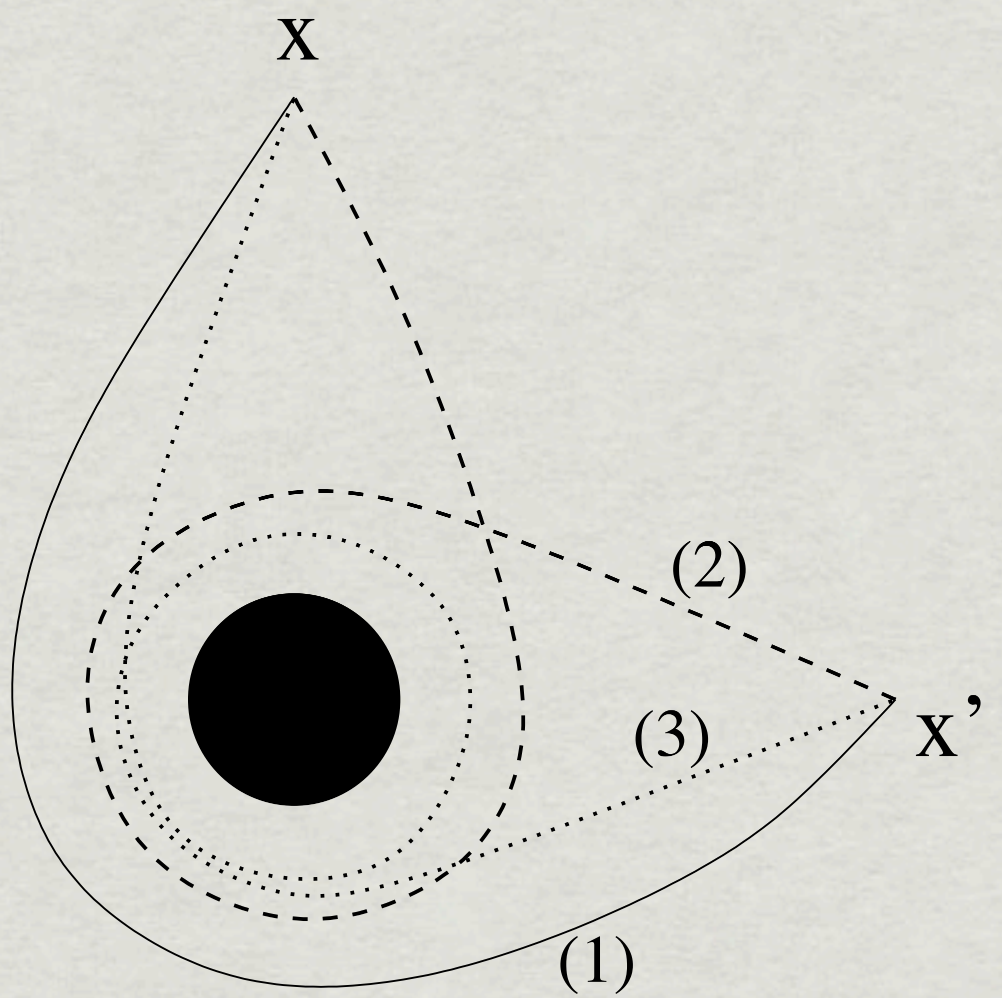
Landau contour -> $\Delta_\phi = \left|\frac{\theta}{\sin\theta}\right| e^{-iN\pi}$

A phase is picked up at each caustic $\theta' = N\pi,\ N \in \mathbb{N}$

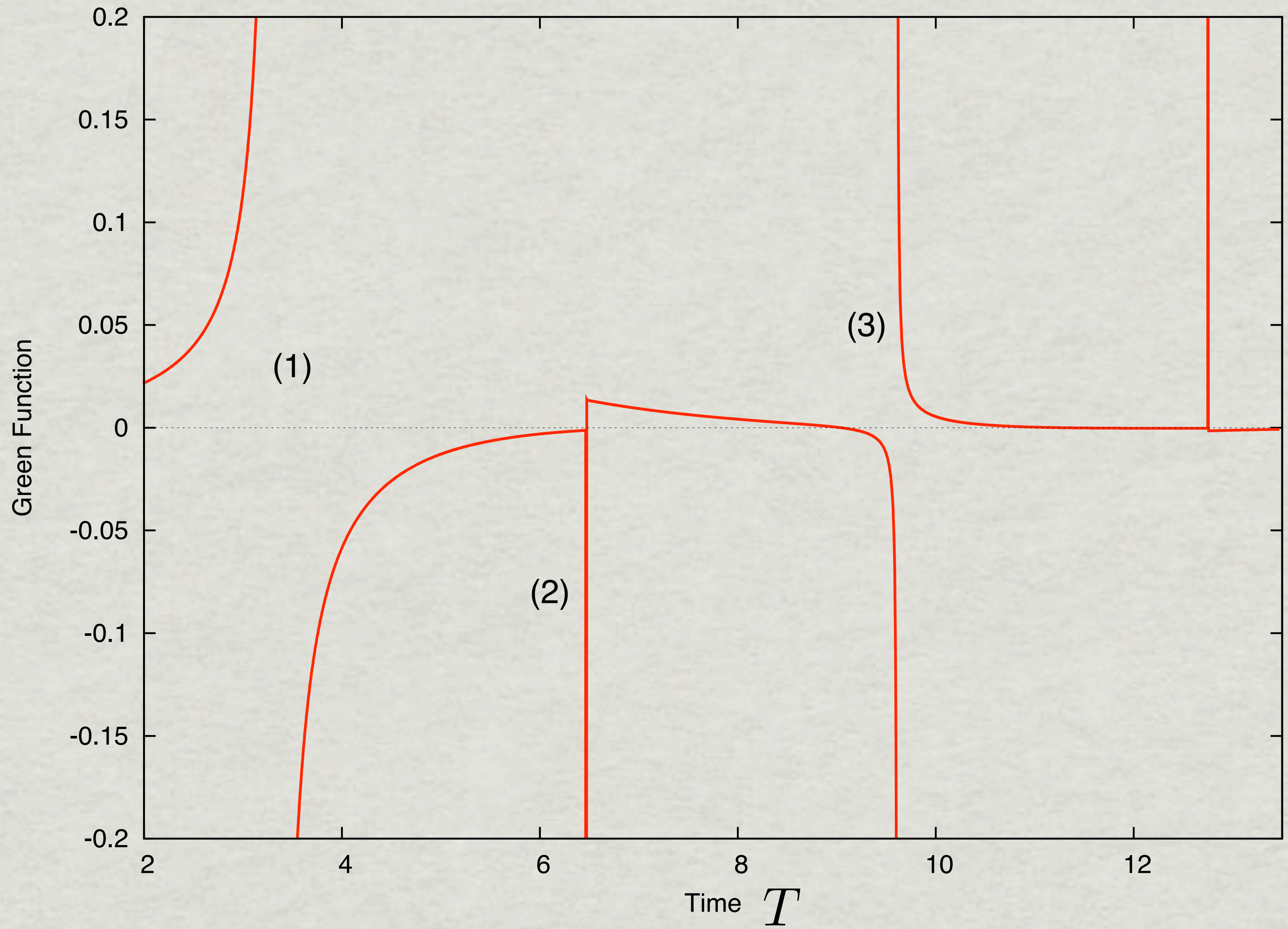
$$G_N^{dir} \sim \left(\frac{\eta}{\sinh\eta}\right)^{1/2}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\begin{cases}(-1)^{N/2}\delta(\sigma), & N \text{ even,}\\ \frac{(-1)^{(N-1)/2}}{\pi\sigma}, & N \text{ odd.}\end{cases}$$

η **: geodesic distance in** dS_2

Null geodesics joining x & x’



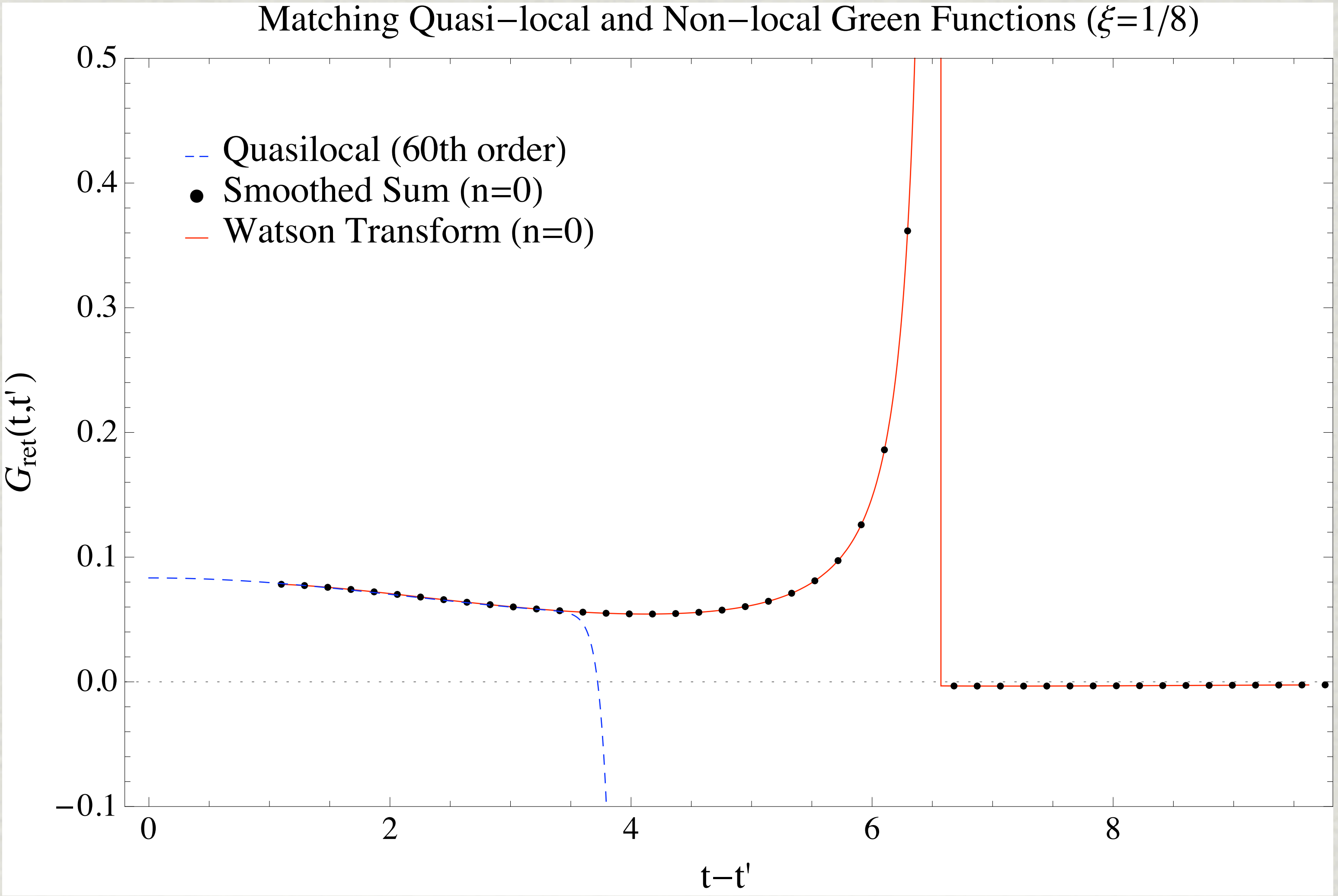
QNM Green function in Nariai

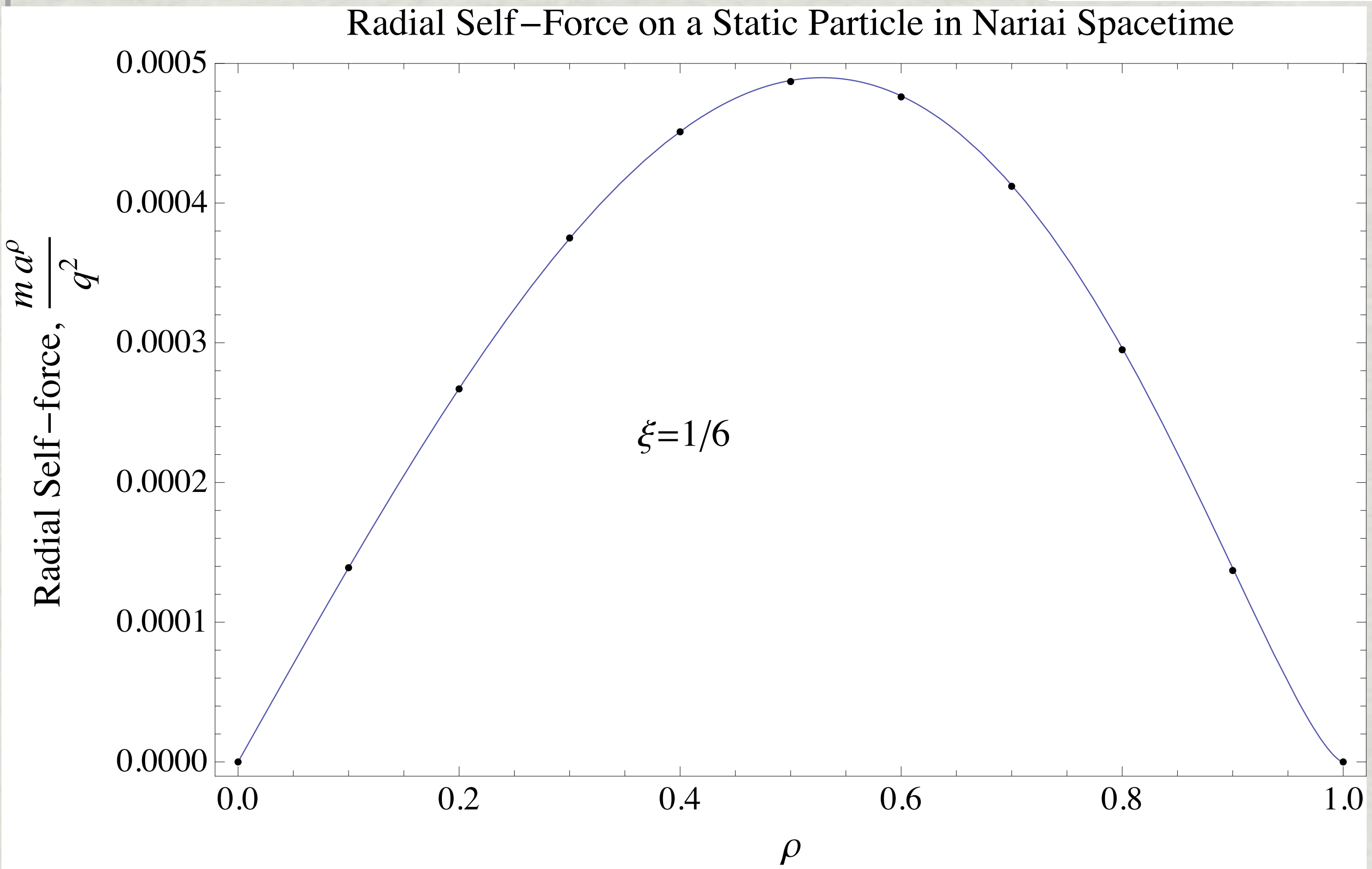


$\gamma = \pi/2$ $\xi = 1/6$

Matching QL and QNM:

$$\gamma = 0$$





Self-force for static scalar particle

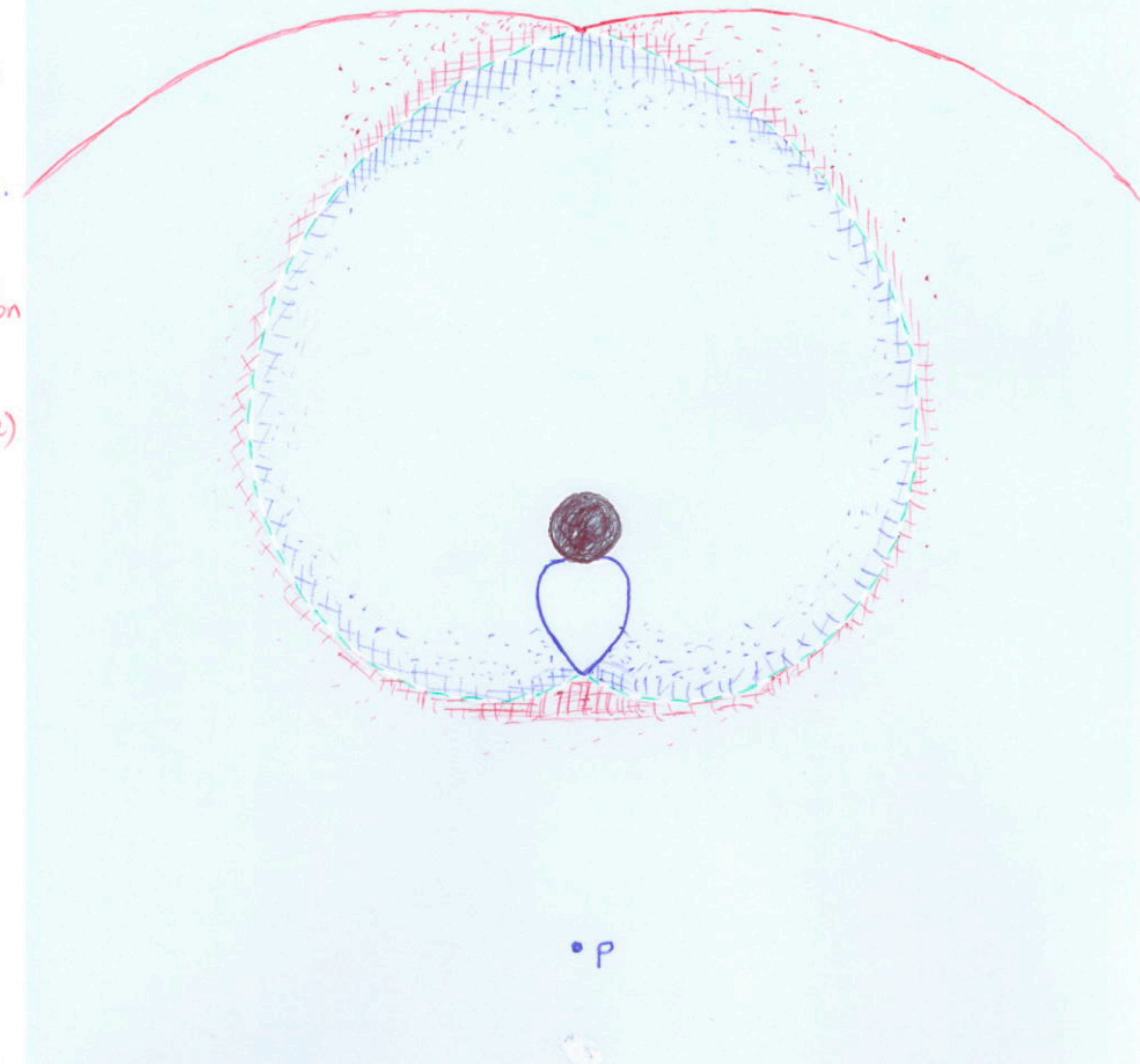
Snapshot of scalar Green Function on Schwarzschild spacetime

Sketch of the scalar Green function and cross-section of light cone of an event near a Schwarzschild black hole, at $\Delta t = \text{const}$, where Δt is large enough that two caustics have formed. In $\theta = \pi/2$ plane.

Key:
- — red line = +ve delta function
- — blue line = −ve δ-fn
- XXX red hatching = finite $1/\sigma$ contribution (+ve)
- XX blue hatching $1/\sigma$ contribution (−ve)
- - - - dotted green line singular part of $1/\sigma$

Wavefront emitted from point p below black hole

The wavefronts were computed using Kirill Ignatiev's code.
Insight into the GF comes from (i) asymptotics of QNM part of GF, (ii) Hadamard's form + Maslov index.



Singular Structure of Retarded GF: Expectations

- Retarded GF is singular everywhere on lightcone, and nowhere else.
- Singular behaviour encapsulated by large-l asymptotics of QNM sum
- QNM sum on Schwarzschild predicts four-fold singularity pattern upon passing through caustics:

$$\delta(\sigma) \quad \rightarrow \quad -\frac{1}{\pi\sigma} \quad \rightarrow \quad -\delta(\sigma) \quad \rightarrow \quad +\frac{1}{\pi\sigma} \quad \rightarrow \quad \delta(\sigma)$$

Schwarzschild QNM sum

$$G_{\text{ret}}^{\text{QNM}} = \frac{4}{rr'}\text{Re}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(l+\tfrac{1}{2})P_l(\cos\gamma)\mathcal{B}_{ln}\tilde{u}_{ln}(r)\tilde{u}_{ln}(r')e^{-i\omega_{ln}T}$$

LEAVER (1986)

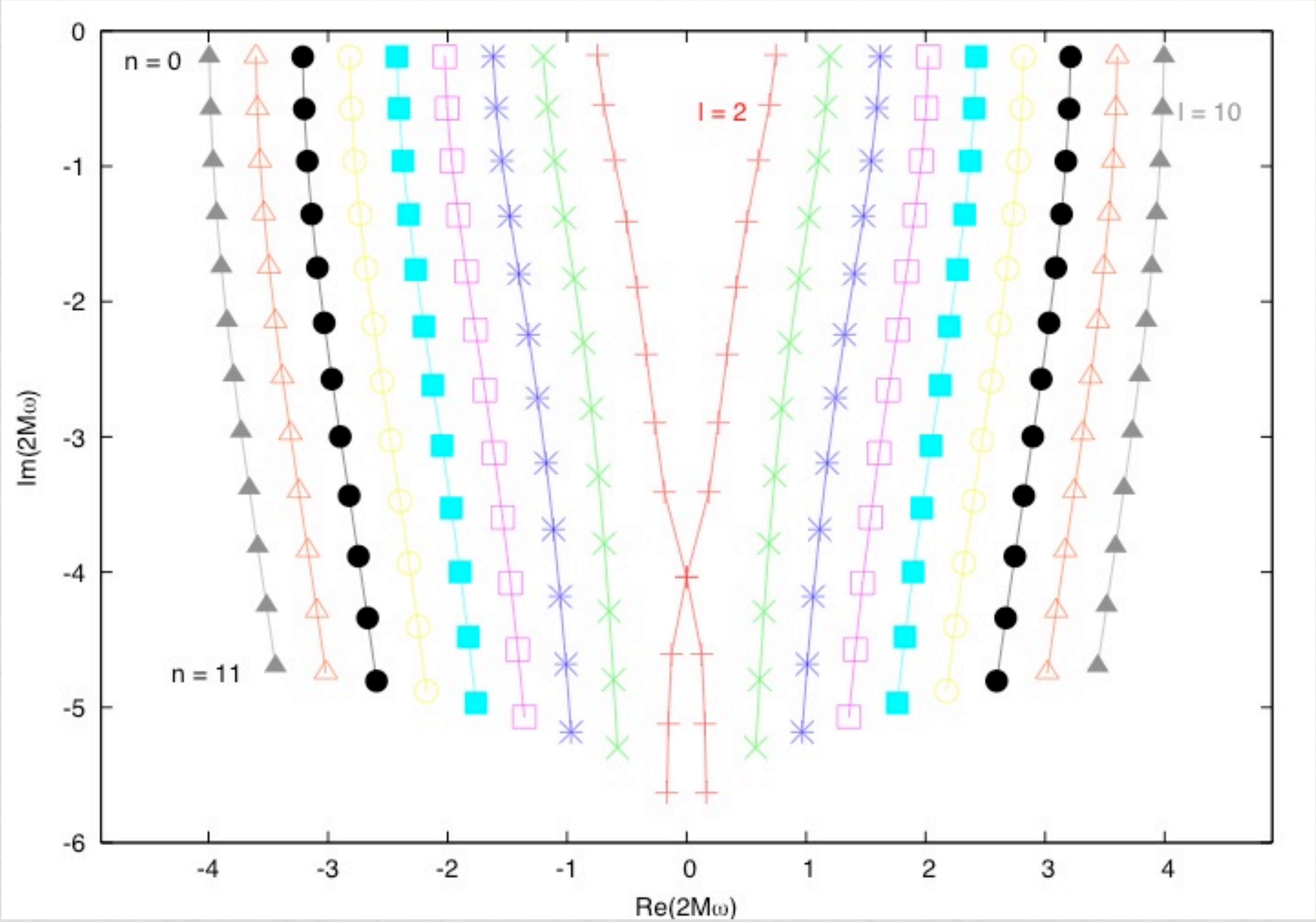
Ingredients:

- QNM frequencies ω_{ln} labelled by integers *l* (multipole) and *n* (overtone number).
- $T = \Delta t - r_* - r'_*$ convergent in n for T > 0
- Normalised mode functions, $\tilde{u}_{ln}(r) = u_{ln}(r)/[A_{\text{out}}e^{i\omega_{ln}r_*}]$

$$u_{ln}(r) \sim \begin{cases} e^{-i\omega_{ln}r_*}, & r_* \to -\infty \\ A_{\text{out}}e^{i\omega_{ln}r_*} + A_{\text{in}}e^{-i\omega_{ln}r_*}, & r_* \to +\infty \end{cases}$$

- QNM Excitation Factors: $\mathcal{B}_{ln} = \dfrac{A_{\text{out}}}{2\omega_{ln}\frac{dA_{\text{in}}}{d\omega}}$

Schw. QNM Frequency Spectrum



$$\omega_{ln} \approx \frac{1}{\sqrt{27}\,M}\left[\pm(l+1/2) - i(n+1/2)\right]$$

Finding Schw. QNMs: a new method (I)

- QNMs <==> unstable null orbit at r=3M
- Consider 'critical' null geodesic, incident from infinity, which asymptotes to null orbit.
- Expand around this geodesic in powers of 1/L, where L = *l*+1/2
- Obtain high-order asymptotic expansion, e.g.,

$$\begin{aligned}\sqrt{27}M\omega_{l,n=0} \quad = \quad & L - 0.5i - 1.30092593L^{-1} + 0.20460391iL^{-2} \\ & -0.56376775L^{-3} + 0.25454392iL^{-4} - 0.19978348L^{-5} \\ & -0.05688153iL^{-6} + 0.14611217L^{-7} - 1.16950214iL^{-8} \\ & +0.04477903L^{-9} - 3.49565902iL^{-10} - 1.50293894L^{-11} \\ & -6.56185182iL^{-12} - 4.40093979L^{-13} + \mathcal{O}(L^{-14})\end{aligned}$$

Finding QNMs: a new method (II)

RADIAL EQ:

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - f\left(\frac{(L^2 - 1/4)}{r^2} + \frac{2\beta}{r^3}\right)\right] u_{l\omega}(r) = 0$$

ANSATZ:

$$u_{l\omega}(r) = \exp\left(i\omega \int^{r_*} \left(1 + \frac{6}{r'}\right)^{1/2} \left(1 - \frac{3}{r'}\right) dr'_*\right) v_{l\omega}(r)$$

NEW EQ:

$$(fv')' + \left[2i\omega \left(1 + \frac{6}{r}\right)^{1/2} \left(1 - \frac{3}{r}\right)\right] v' +$$

$$\left[\frac{27\omega^2 - L^2}{r^2} + \frac{27i\omega}{r^3}\left(1 + \frac{6}{r}\right)^{-1/2} + \frac{1}{4r^2} - \frac{2\beta}{r^3}\right] v \quad = \quad 0$$

which is amenable to WKB-like expansion :

$$\omega = \omega_0 L + \omega_1 + \omega_2 L^{-1} + \ldots$$

$$v(r) = \exp\left(S_0(r) + S_1(r) L^{-1} + S_2(r) L^{-2} + \ldots\right)$$

Finding QNMs: a new method (III)

$$L^2 : \quad 27\omega_0^2 - 1 = 0 \qquad \Rightarrow \omega_0 = \pm 1/\sqrt{27}$$

$$L^1 : \quad 2i\omega_0 \left(1 + \frac{6}{r}\right)^{1/2} \left(1 - \frac{3}{r}\right) S_0' + \frac{54\omega_0\omega_1}{r^2} + \frac{27i\omega_0}{r^3}\left(1 + \frac{6}{r}\right)^{-1/2} = 0$$

$$L^0 : \quad 2i \left(1 + \frac{6}{r}\right)^{1/2} \left(1 - \frac{3}{r}\right) (\omega_0 S_1' + \omega_1 S_0') + f(S_0'' + (S_0')^2) + f' S_0' + \frac{27(2\omega_0\omega_2 + \omega_1^2)}{r^2} + \frac{27i\omega_1}{r^3}\left(1 + \frac{6}{r}\right)^{-1/2} + \frac{1}{4r^2} - \frac{2\beta}{r^3} = 0$$

- group terms by powers of L
- impose a **continuity condition** at null circular orbit, r=3M
- e.g. at order L^1, fix w_1 by insisting S_0 is continuous & differentiable at r = 3M. Then sub in w_1 to obtain first-order ODE for S_0. Solve.

- Obtain asymptotic expansion of wavefunction in inverse powers of L
- Also get outgoing coefficient A_{out}
- Straightforward to extend to higher overtone modes

Large-L Asymptotics of QNM Wavefunctions

- To leading order in L, the normalised wavefunctions are

$$\tilde{u}_{ln}(r) \approx U(r)\rho^n(r)\exp\left(iL\mathcal{R}(r)/\sqrt{27}\right)$$

- where

$$\begin{aligned}
U(r) &= x^{-1/2}e^{-r(x-1)^2/(4\sqrt{27})}\left(\frac{1+x}{4-2x}\right)^{2/\sqrt{27}}\left(\frac{1+\sqrt{3}}{x+\sqrt{3}}\right) \\
\mathcal{R}(r) &= 4\ln\left(\frac{1+x}{4-2x}\right) - \tfrac{1}{2}r(x-1)^2 \\
\rho(r) &= \left(1-\frac{3}{r}\right)\left(\frac{1+x}{4-2x}\right)^{4/\sqrt{27}}\left(\frac{1+\sqrt{3}}{x+\sqrt{3}}\right)^2 e^{-\frac{1}{2}r(x-1)^2/\sqrt{27}}
\end{aligned}$$

- and $x = \left(1+\dfrac{6}{r}\right)^{1/2}$

Large-L Asymptotics of Excitation Factors

$$\mathcal{B}_{ln} = \frac{A_{\rm out}}{2\omega_{ln}\frac{dA_{\rm in}}{d\omega}}$$: i.e. need to take omega derivative

- Combine new "asymptotic expansion method" with standard WKB methods to obtain large-L asymptotics:

$$\mathcal{B}_{ln} \approx i^{1/2} L^{-1/2} B e^{2iyL/\sqrt{27}} \frac{(-i\kappa L)^n}{n!}$$

- with the following 'geometric constants':

$$\begin{aligned} y &= 3 - \sqrt{27} + 4\ln 2 - 6\ln(2+\sqrt{3}) = -7.325311084 \\ B &= \frac{\sqrt{27}\, e^{y/\sqrt{27}}}{(2+\sqrt{3})\sqrt{\pi}} = 0.19182703317 \\ \kappa &= \frac{216\, e^{2y/\sqrt{27}}}{(2+\sqrt{3})^2} = 0.92482482643 \end{aligned}$$

Scalar QNM Green Function

- Mix all the ingredients together:

$$G^{\mathrm{QNM}}_{\mathrm{ret}} \sim \mathcal{G}(r,r';T)\,\mathrm{Re}\sum_{l=0}^{\infty}(iL)^{1/2}P_l(\cos\gamma)e^{-iL\Phi_{(0)}}\sum_{n=0}^{\infty}\frac{\left[-i\kappa\rho(r)\rho(r')e^{-T/\sqrt{27}}L\right]^n}{n!}$$

$$\sim \mathcal{G}(r,r';T)\,\mathrm{Re}\sum_{l=0}^{\infty}(iL)^{1/2}P_l(\cos\gamma)e^{-iL\Phi_{(1)}}$$

- where

$$\Phi_{(0)} = \left[T-\mathcal{R}(r)-\mathcal{R}(r')-2y\right]/\sqrt{27}$$

$$\Phi_{(1)} = \Phi_{(0)}+\kappa\rho(r)\rho(r')e^{-T/\sqrt{27}}$$

- and

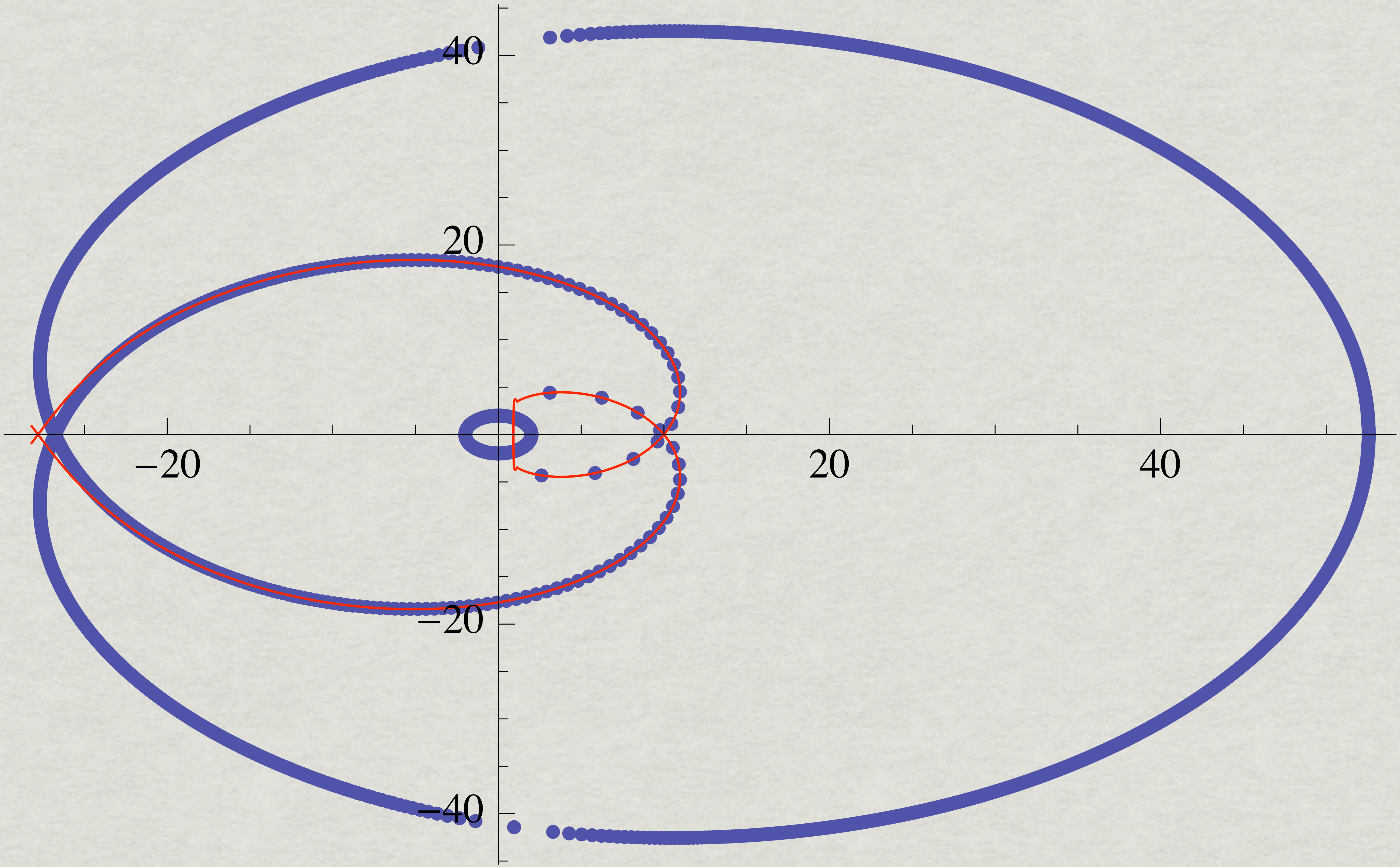
$$\mathcal{G}(r,r';T) = \frac{4BU(r)U(r')e^{-T/(2\sqrt{27})}}{rr'}$$

When is GF singular?

$$G^{\text{QNM}}_{\text{ret}} \sim \mathcal{G}(r, r'; T)\,\mathrm{Re}\sum_{l=0}^{\infty}(iL)^{1/2}P_l(\cos\gamma)e^{-iL\Phi_{(1)}}$$

- Series is oscillatory and *divergent*, but still summable ... provided we avoid a *coherent phase condition:*
$$\Phi_{(1)} \pm \gamma = 2\pi N, \qquad N \in \mathbb{Z}$$
- ... which (approx.) corresponds to the null cone.
- e.g. time to go once around the black hole
$$T = 2\pi\sqrt{27} + 2y - 2\mathcal{R}(r) - \sqrt{27}\kappa\rho^2(r)e^{-T/\sqrt{27}}$$

	T_{exact}	T_1	$T_1 - T_{\text{exact}}$
$r = 4$	25.6449315	25.6449191	−0.00001246
$r = 10$	20.7019188	20.7017749	0.00014394
$r = 100$	18.1280305	18.1285177	0.00048718


40
20
−20
−40
−20
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40

Asymptotic Analysis

Poisson Sum Formula

$$\mathrm{Re} \sum_{l=0}^{\infty} (iL)^{1/2} P_l(\cos\gamma) e^{-iL\Phi_{(1)}} = \sum_{N=-\infty}^{\infty} \mathcal{I}_N$$

$$\mathcal{I}_N = \mathrm{Re} \int_0^{\infty} dL (iL)^{1/2} e^{-iL\Phi} R_N(L,\gamma)$$

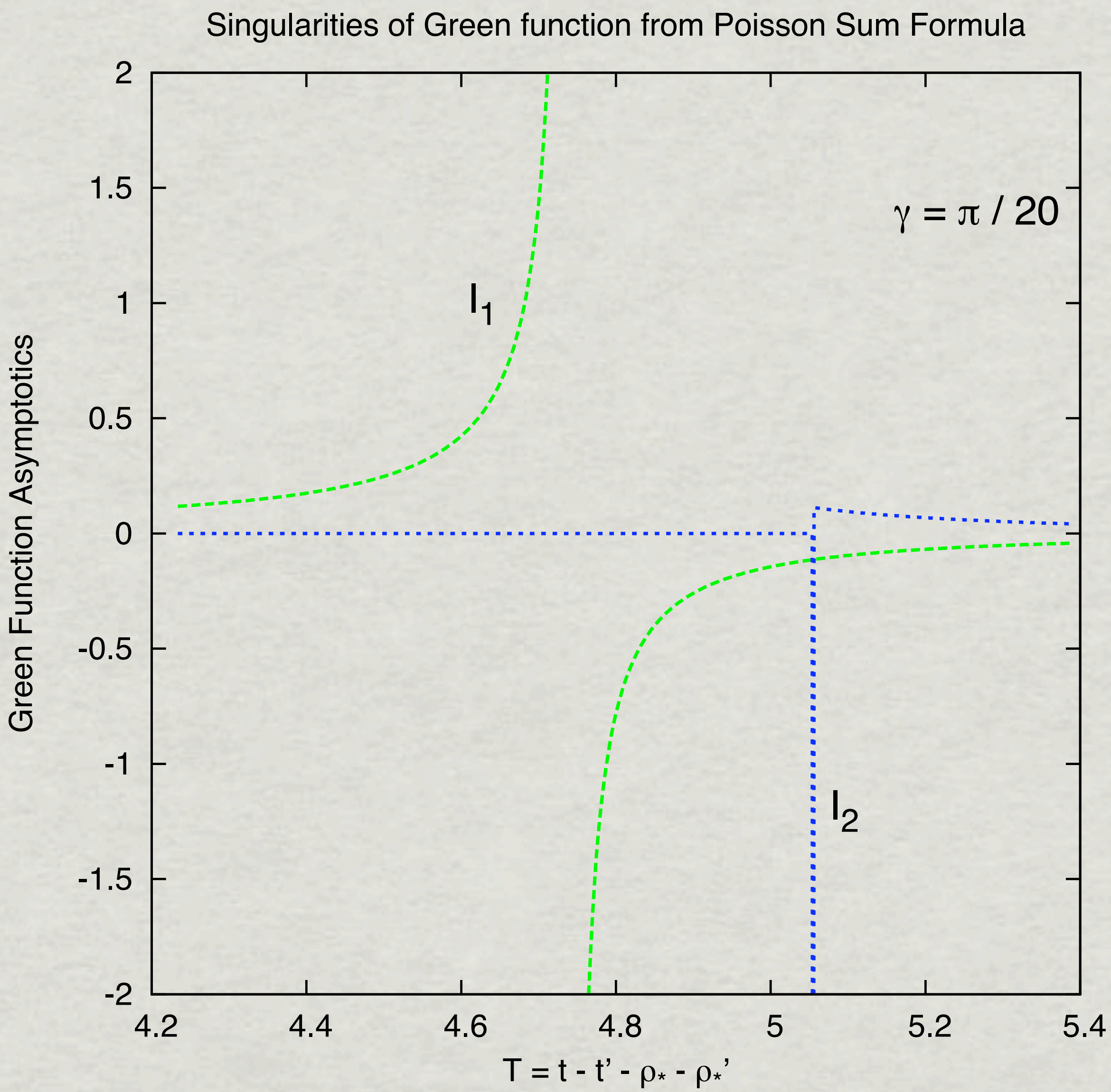
$$R_N = \begin{cases} (-1)^{N/2} \left[\mathcal{Q}^{-}_{L-1/2}(\cos\gamma) e^{iN\pi L} + \mathcal{Q}_{L-1/2}(\cos\gamma) e^{-iN\pi L} \right] \\ (-1)^{(N+1)/2} \left[\mathcal{Q}^{+}_{L-1/2}(\cos\gamma) e^{i(N+1)\pi L} + \mathcal{Q}^{-}_{L-1/2}(\cos\gamma) e^{-i(N+1)\pi L} \right] \end{cases}$$

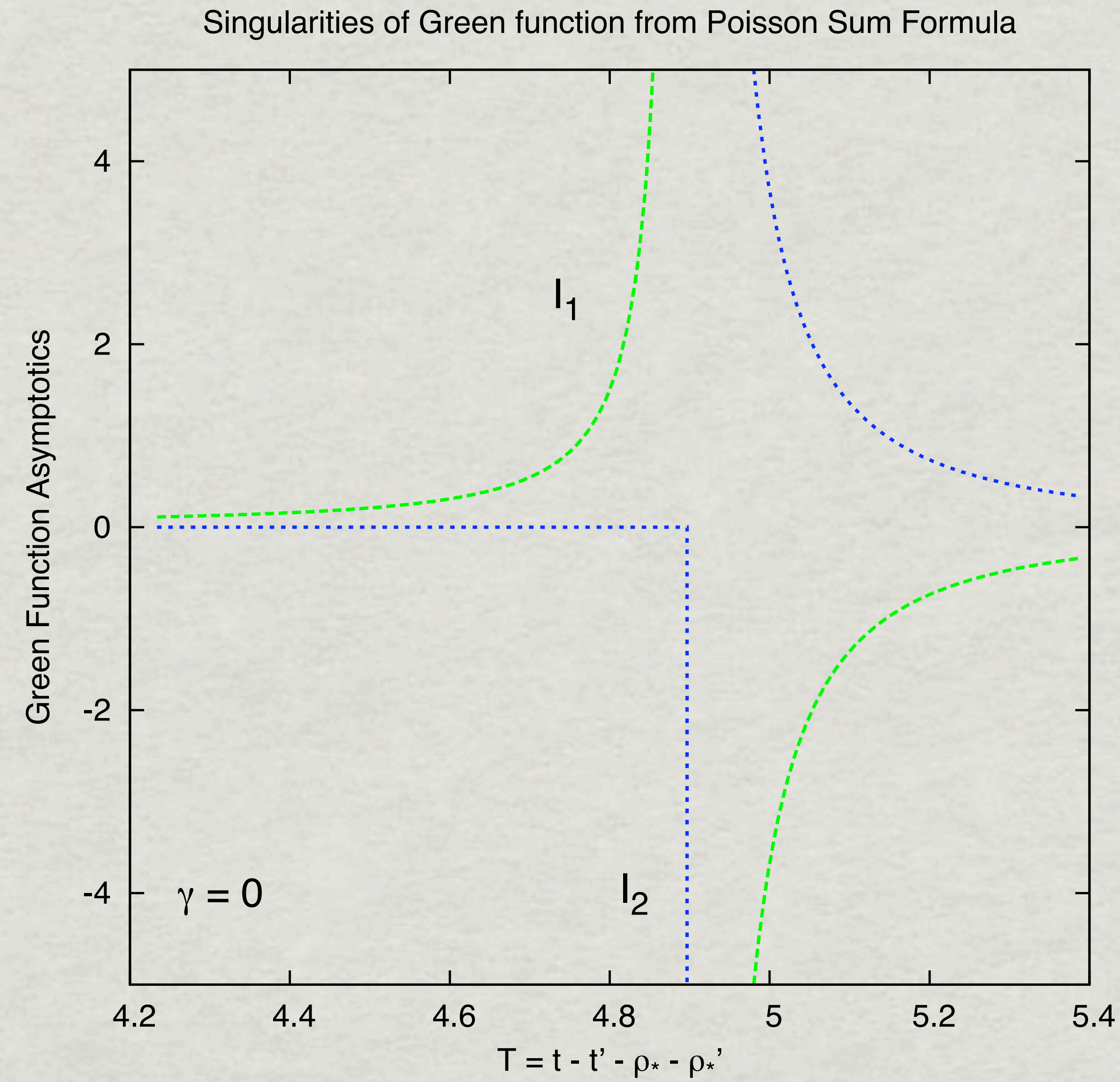
$$\begin{aligned} \mathcal{Q}^{\pm}_{L-1/2}(\cos\gamma) &= \frac{1}{2}\left[P_{L-1/2}(\cos\gamma) \pm \frac{2i}{\pi} Q_{L-1/2}(\cos\gamma) \right] \\ &\approx \left(\frac{\gamma}{\sin\gamma} \right)^{1/2} H_0^{(\mp)}(\gamma L) \approx \left(\frac{1}{2\pi L \sin\gamma} \right)^{1/2} e^{\pm i\pi/4} e^{\mp iL\gamma} \end{aligned}$$

and $\displaystyle \lim_{\epsilon \to 0} \int_{L=0}^{\infty} e^{iL(\sigma + i\epsilon)} dL = i/\sigma + \pi\delta(\sigma)$ **=> four fold behaviour**

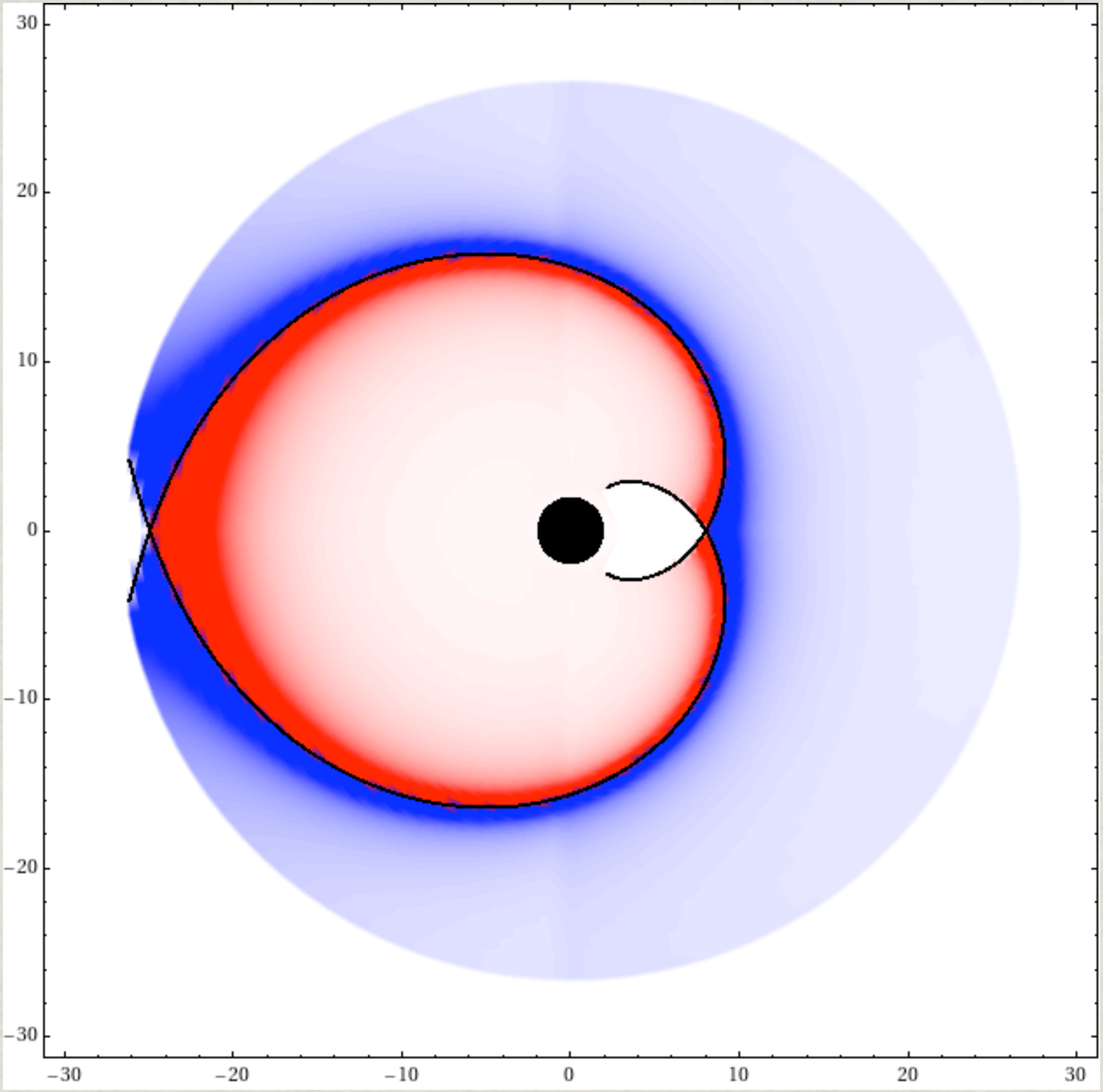
Green function near caustic

more accurate asymptotics near caustics found using Hankel function approximation (uniform convergence)



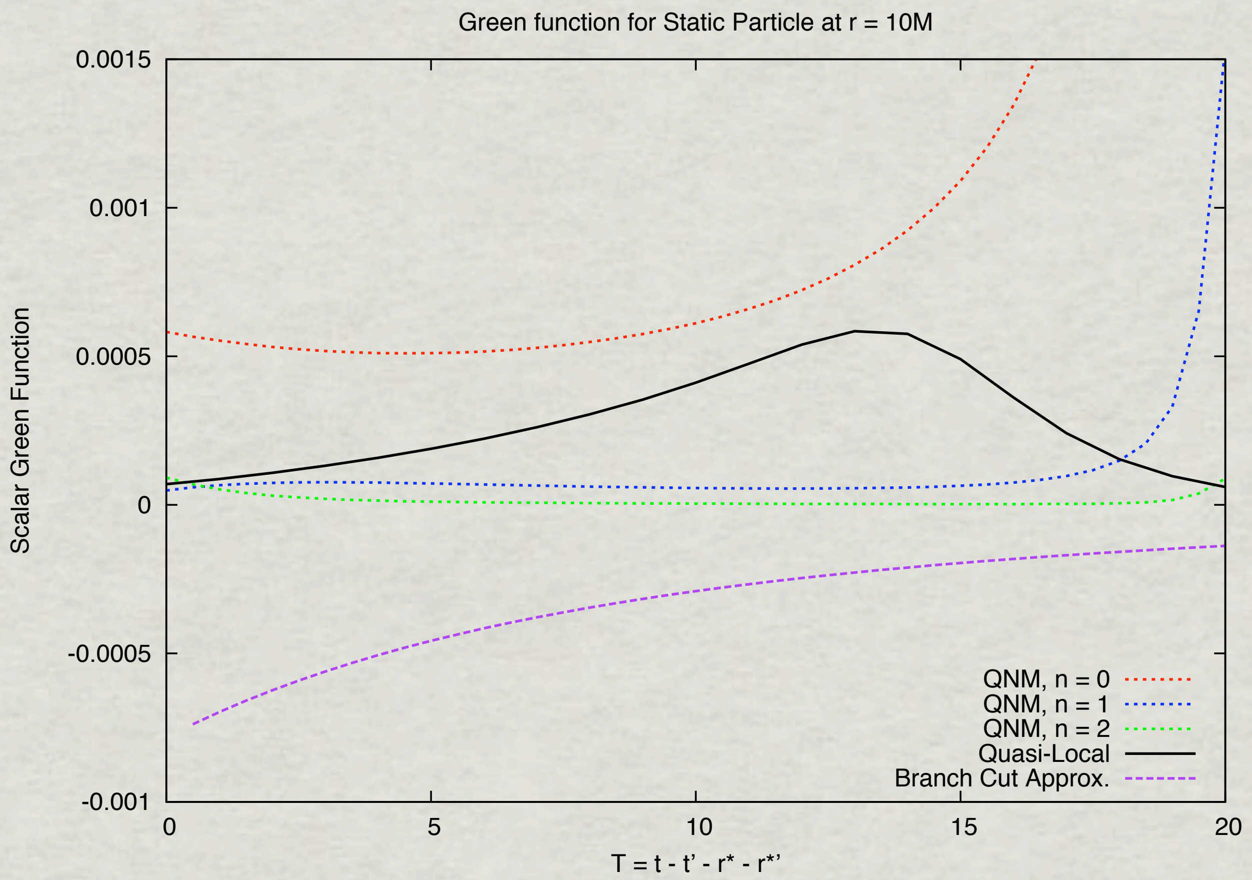


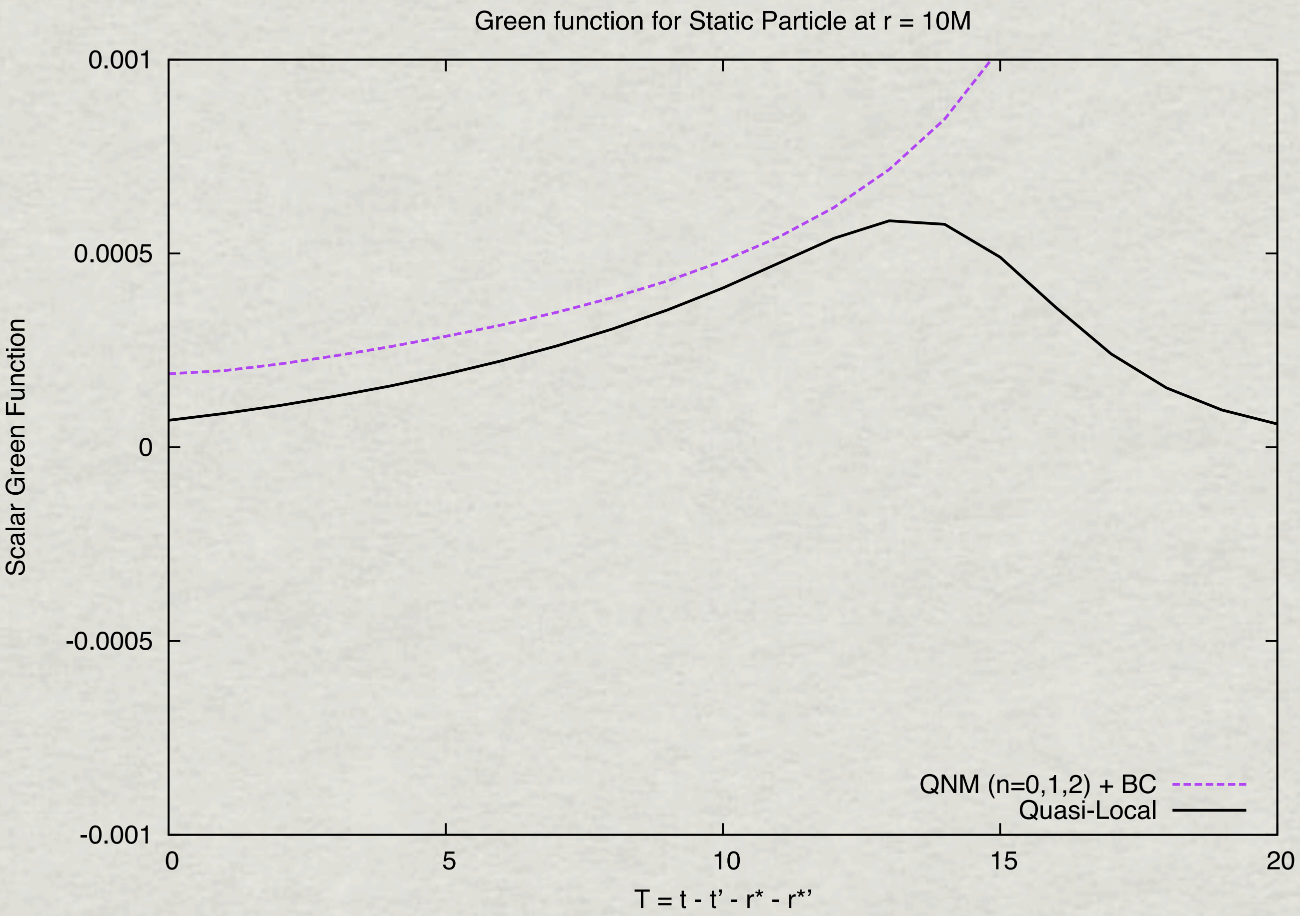
Light cone + Scalar Green function from QNM asymptotics



Schw. Green Function Matching

- Simple case: static particle at r = 10M
- Compare Quasi-Local against QNM + branch cut





Conclusion: QNM + QL are in good shape, but Branch Cut needs more work