Tuning time-domain pseudospectral computations of the self-force on a charged scalar particle

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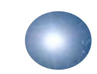
Outlook

- *A brief introduction to the Extreme-Mass Ratio Inspiral (EMRI) problem*
- *Description of the Particle-without-Particle (PWP) scheme.*
    - *Application to EMRIs with circular orbits*
    - *Application to EMRIs with Eccentric orbits.*
- *Summary.*

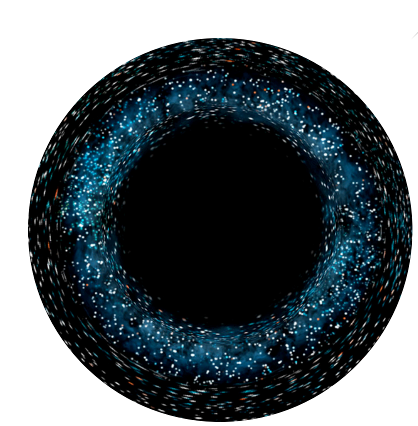
EMRIs

- Extreme-Mass-Ratio Inspirals (EMRIs) are formed when a massive black hole (MBH) captures a stellar-mass compact object (SCO).

$m \sim 1 - 50 \; M_\odot$



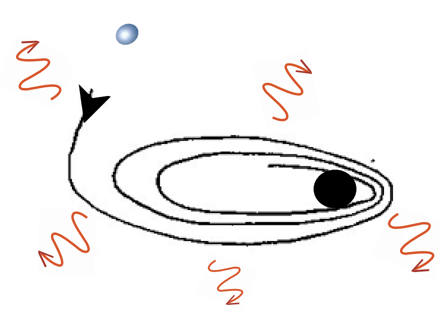
$M \sim 10^4 - 10^7 \; M_\odot$



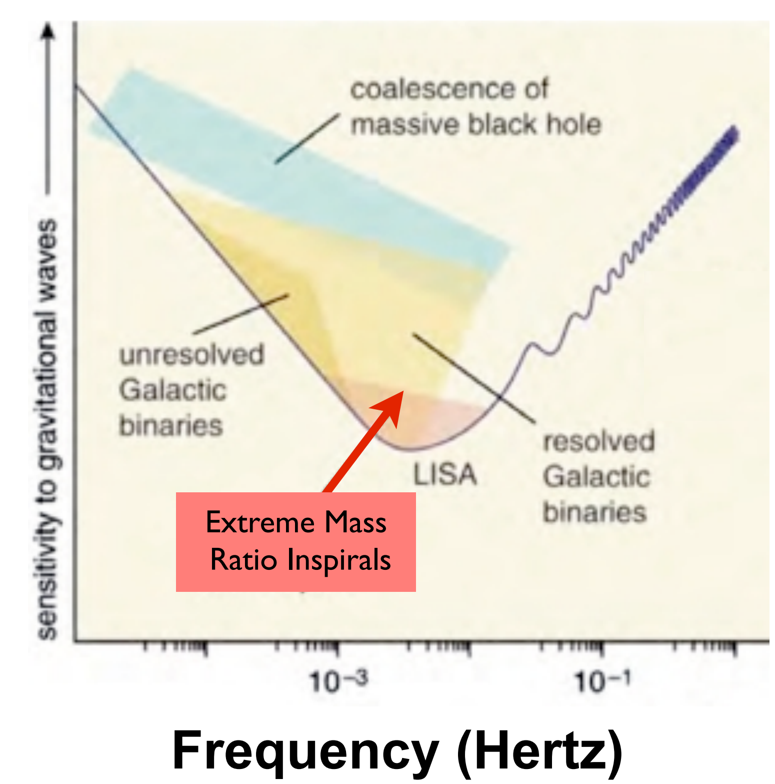
- These systems have extreme mass ratio:

$$\mu = m/M_\bullet \sim 10^{-7} - 10^{-3}$$

- They are one of the main sources of gravitational waves (GWs) for the future Laser Interferometer Space Antenna (LISA)



LISA sensitivity curve



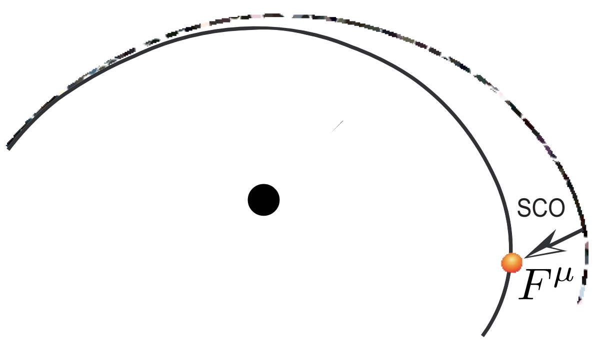
graphic (c) by LISA Science Case LISA-LIST-RP-436

Modelling EMRIs

- *Techniques to compute waveforms:*
  - *Frequency domain: They are efficient for e<<1, but they have convergence problems for high eccentric orbits.*
  - *Time domain: They handle in the same way circular and eccentric orbits.*

- *Modelling EMRIS implies deal with different spatial (SMBH and SCO) and temporal scales* ⟶ *Challenge for time-domain techniques.*

- *Perturbation theory: The SCO is pictured as point particle in a fixed SMBH background, which orbit is deviated by the action of a local self-force*



The Particle without Particle technique

- *The PwP technique avoids working with the singularity associated with the SCO.*
- *It employs time-domain techniques which allows us to deal easily with eccentric orbits.*
- *We employ the mode-sum regularisation scheme for regularise the field modes.*
- *Runge-Kutta algorithm for the time evolution.*
- *To test our technique we use a toy problem: A scalar charged particle falling in a geodesic of a Schwarzschild MBH spacetime.*

$$\Box\Phi^{ret} = -4\pi q \int \delta_4(x - z(\tau)) d\tau$$

$$F^{\mu} = q(g^{\mu\nu} + u^{\mu}u^{\nu})\nabla_{\nu}\Phi^{ret}$$

Modelling EMRIs

- *The retarded field can be decomposed into spherical harmonics:*

$$\Phi^{ret} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi^{lm}(t,r) Y^{lm}(\theta,\varphi)$$

- *The 1+1 equation for each harmonic coefficient takes the form:*

$$(-\partial_t^2 + \partial_{r^*}^2 - V_l)\psi^{lm} - S^{lm} = 0$$

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$$S^{lm} = A^{lm}\delta[r^* - r_p^*(t)]$$

$$A^{\ell m} = -4\pi q \frac{1 - 2M/r_p}{r_p E_p}$$

PwP technique

- *We perform a division of the spatial computational domain into two disjoints regions or subdomains, one at the left of the particle* $r^* > r^*_p$ *and other at the right of the particle* $r^* < r^*_p$

$$(-\partial_t^2 + \partial_{r^*}^2 - V_l)\psi^{lm} - S^{lm} = 0$$

$-\infty \leftarrow r^*$ $\qquad\qquad$ $r^* \rightarrow \infty$

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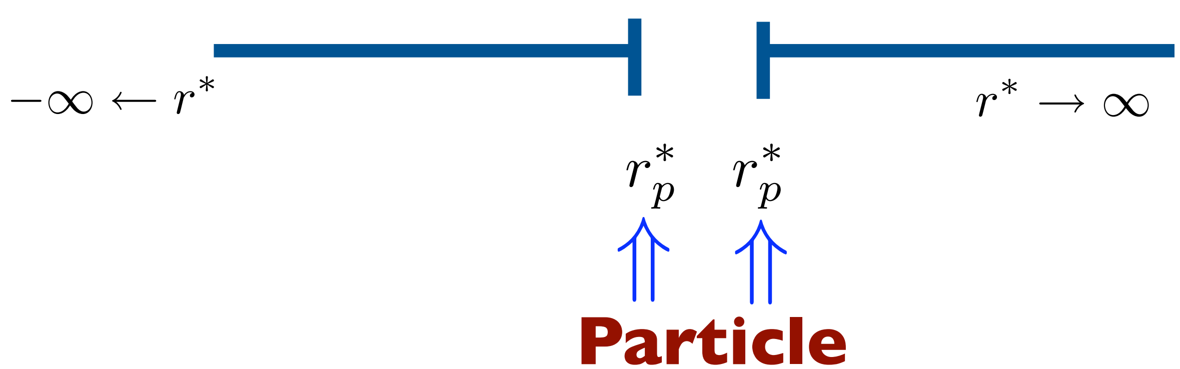

PwP technique

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$$(-\partial_t^2 + \partial_{r^*}^2 - V_l)\psi^{lm} - S^{lm} = 0$$

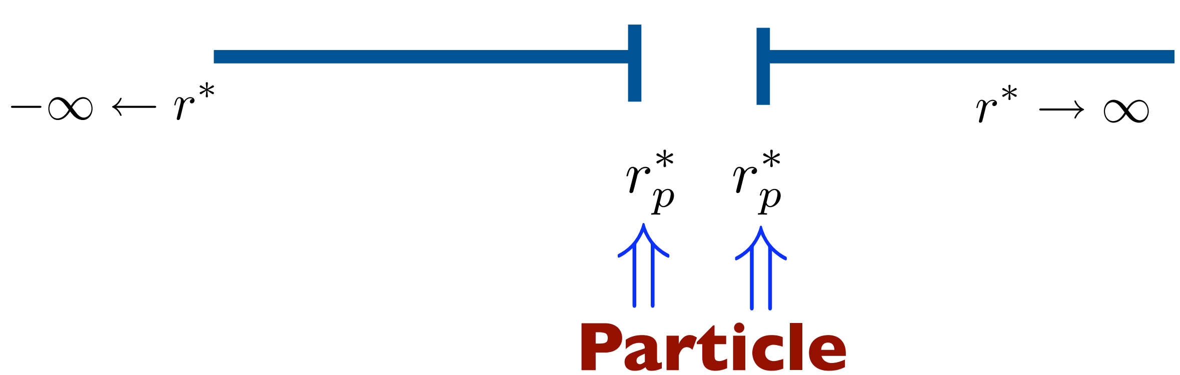
$$\downarrow$$

$$\partial_t \mathcal{U} = \mathbb{A} \cdot \partial_{r^*}\mathcal{U} + \mathbb{B} \cdot \mathcal{U} + \mathbf{S}$$

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbb{B} = \begin{pmatrix} 0 & 1 & 0 \\ -V_\ell & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{S} = (0, -A^{\ell m}\delta[r^* - r_p^*(t)], 0)$$

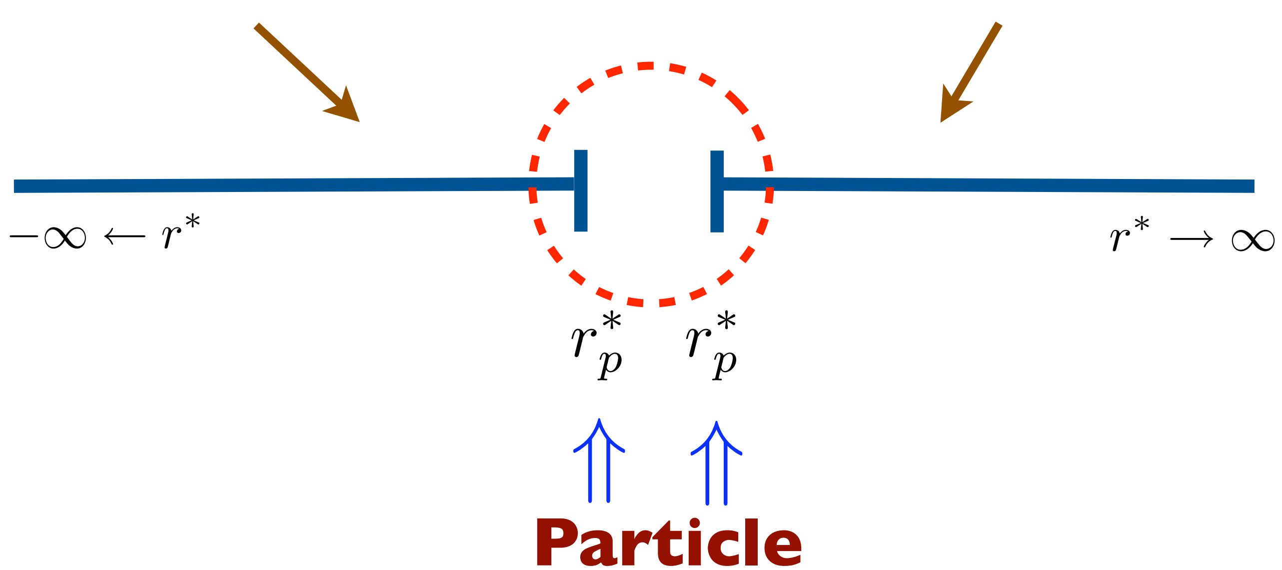
$$\psi^{\ell m} = r\,\Phi^{\ell m}$$
$$\phi^{\ell m} = \partial_t \psi^{\ell m}$$
$$\varphi^{\ell m} = \partial_{r^*}\psi^{\ell m}$$
$$\downarrow$$
$$\mathcal{U} = (\psi^{\ell m}, \phi^{\ell m}, \varphi^{\ell m})$$



PwP technique

- *We solve homogeneous wave equation inside each subdomain.*
- *During the evolution, the solutions have to be communicated through the boundaries*

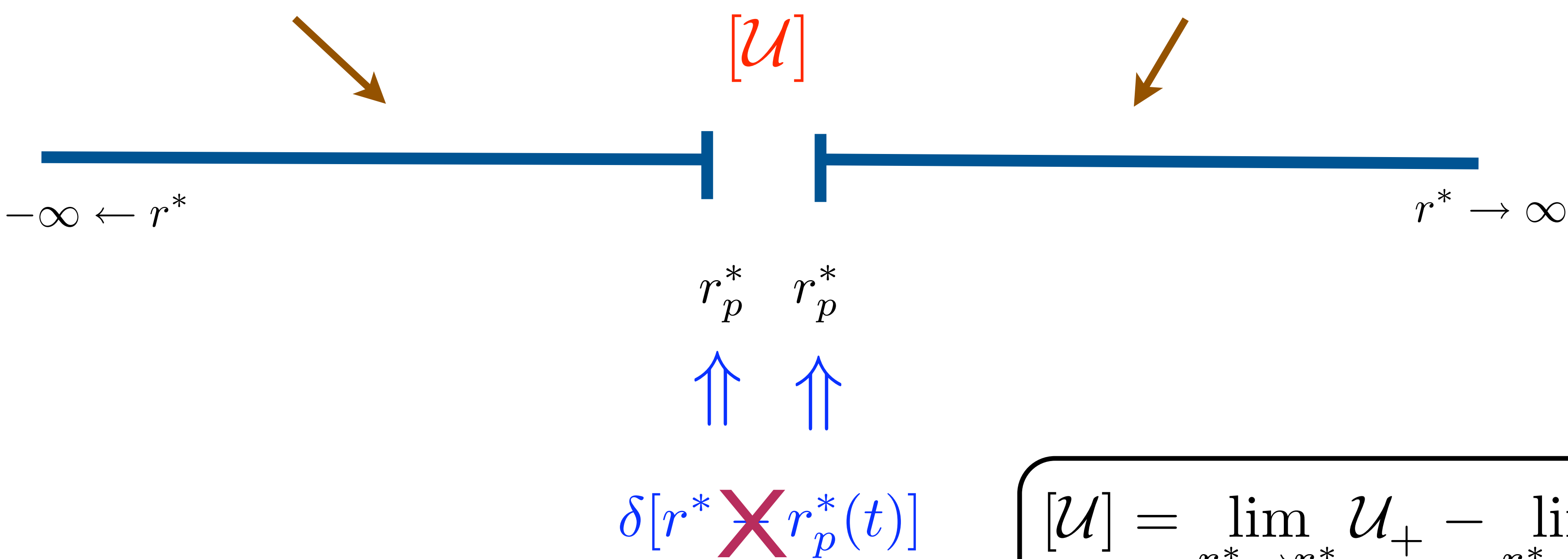
$$\partial_t \mathcal{U}_- = \mathbb{A} \cdot \partial_{r^*} \mathcal{U}_- + \mathbb{B} \cdot \mathcal{U}_- \qquad \partial_t \mathcal{U}_+ = \mathbb{A} \cdot \partial_{r^*} \mathcal{U}_+ + \mathbb{B} \cdot \mathcal{U}_+$$



PwP technique

- *The source term is replaced with the jumps of the field variables*

$$\partial_t \mathcal{U}_- = \mathbb{A} \cdot \partial_{r^*} \mathcal{U}_- + \mathbb{B} \cdot \mathcal{U}_- \qquad \partial_t \mathcal{U}_+ = \mathbb{A} \cdot \partial_{r^*} \mathcal{U}_+ + \mathbb{B} \cdot \mathcal{U}_+$$



$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbb{B} = \begin{pmatrix} 0 & 1 & 0 \\ -V_\ell & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\mathcal{U}] = \lim_{r^* \to r^*_p} \mathcal{U}_+ - \lim_{r^* \to r^*_p} \mathcal{U}_-$$

The PwP technique ensures smooth solutions

since we got rid of the singularity associated with the particle

PwP technique

- *The boundary conditions are imposed in our evolution equations*

1. **The penalty method**:

The system is dynamically driven to fulfil a set of additional conditions.

$$\partial_t \mathcal{U}_\pm = \mathbb{A} \cdot \partial_{r^*} \mathcal{U}_\pm + \mathbb{B} \cdot \mathcal{U}_\pm + \eta(\tau_{\mathcal{U}}[\mathcal{U}])$$

PwP technique

2. **The direct communication of the characteristic fields**:

We pass the value of the characteristic fields.

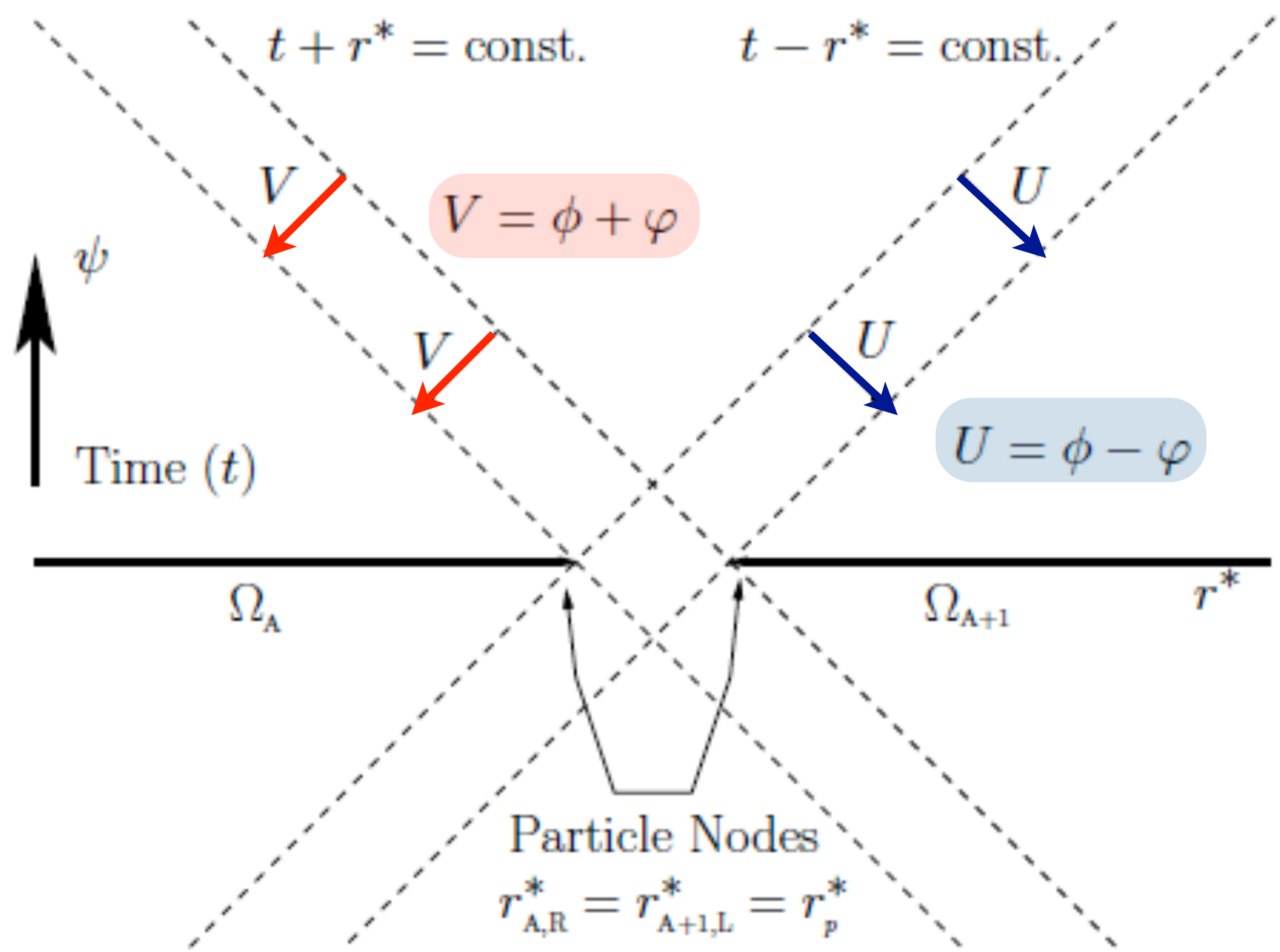
$$\psi^{\ell m} = r\,\Phi^{\ell m}$$

$$U^{\ell m} = \phi^{\ell m} - \varphi^{\ell m}$$

$$V^{\ell m} = \phi^{\ell m} - \varphi^{\ell m}$$

$$\downarrow$$

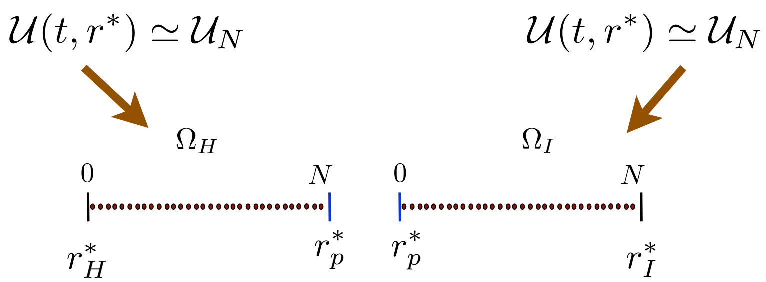
$$\mathcal{U} = (\psi^{\ell m}, U^{\ell m}, V^{\ell m})$$



Numerical implementation

- *To implement the PwP scheme numerically we use the pseudospectral collocation (PSC)*

$$\mathcal{U}_N(X_i) = \mathcal{U}(X_i)$$



Discretisation points
$X_i \in [-1,1], i = 0, ..., N$

Spectral coefficients $a_n(t)$

Chebyshev polynomials $\{T_n(X)\}$

Cardinal function $C_i(X_j) = \delta_{ij}$

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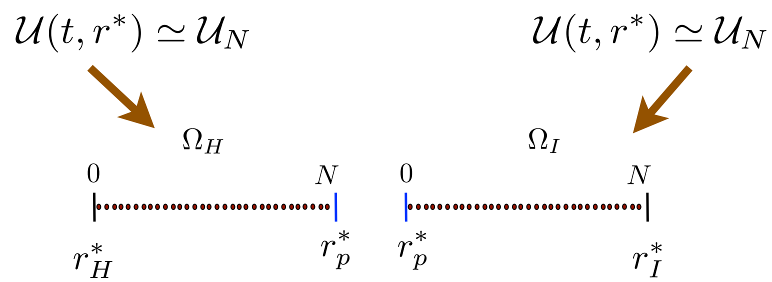

Spectral representation

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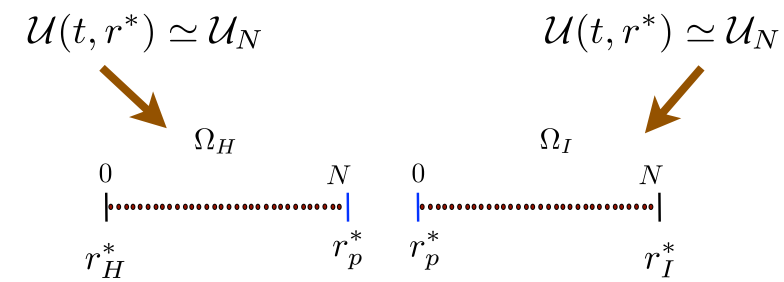

$$\mathcal{U}_N(X_i) = \mathcal{U}(X_i)$$

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Spectral representation

$$\mathcal{U}_N = \sum_{n=0}^{N} \mathcal{U}_i(X) C_i(X)$$

Physical representation

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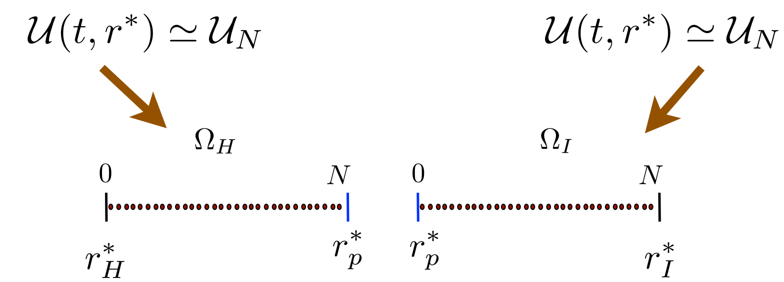

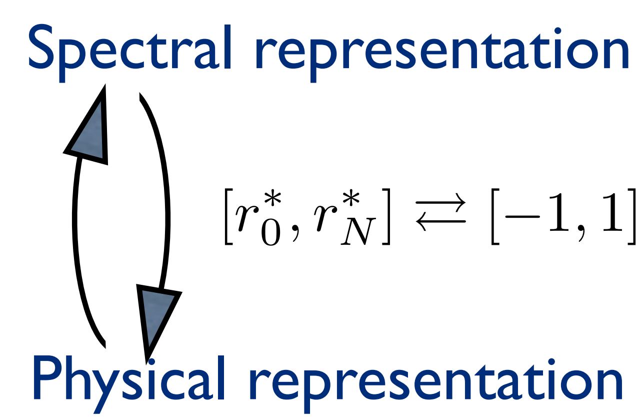

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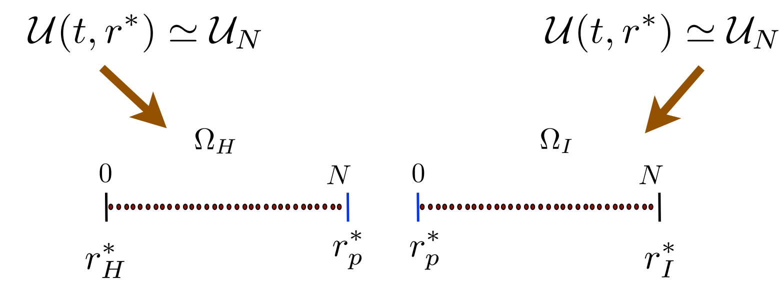


Spectral representation

$[r^*_0, r^*_N] \rightleftarrows [-1,1]$

Physical representation

The PSC method converges exponentially with N for smooth functions

Numerical implementation

Employing a Chebyshev basis there are some paybacks:

FFT

- *Physical representation* ⟷ *Spectral representation*

$\{\mathcal{U}_i\}$ $\{a_i\}$

- *Differentiation is cheaper in the spectral domain:*

$$\mathcal{U}'_N = \sum_{j=0}^{N} D_{ij}\,\mathcal{U}_j(X) \qquad \mathcal{U}'_N = \sum_{j=0}^{N} b_j\,T_j(X)$$

N*²** *operations* ~ ***NLn(N) *operations*

$$\partial_{r*} : \{\boldsymbol{U}_i\} \xrightarrow{FFT} \{\boldsymbol{a}_n\} \xrightarrow{\partial_{r*}} \{\boldsymbol{b}_n\} \xrightarrow{FFT} \{(\partial_{r*}\boldsymbol{U})_i\}$$

Number Computations

$$\Phi^{ret} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi^{lm}(t,r) Y^{lm}(\theta,\varphi)$$

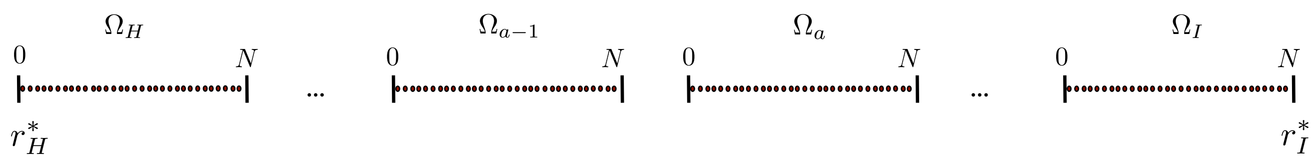
For a given ℓ_{max} the number of modes of evolutions that we need to perform goes as:

$$N_{\text{evolutions}} = \frac{1}{2}\left(\ell_{\text{max}} + 1\right)\left(\ell_{\text{max}} + 2\right)$$

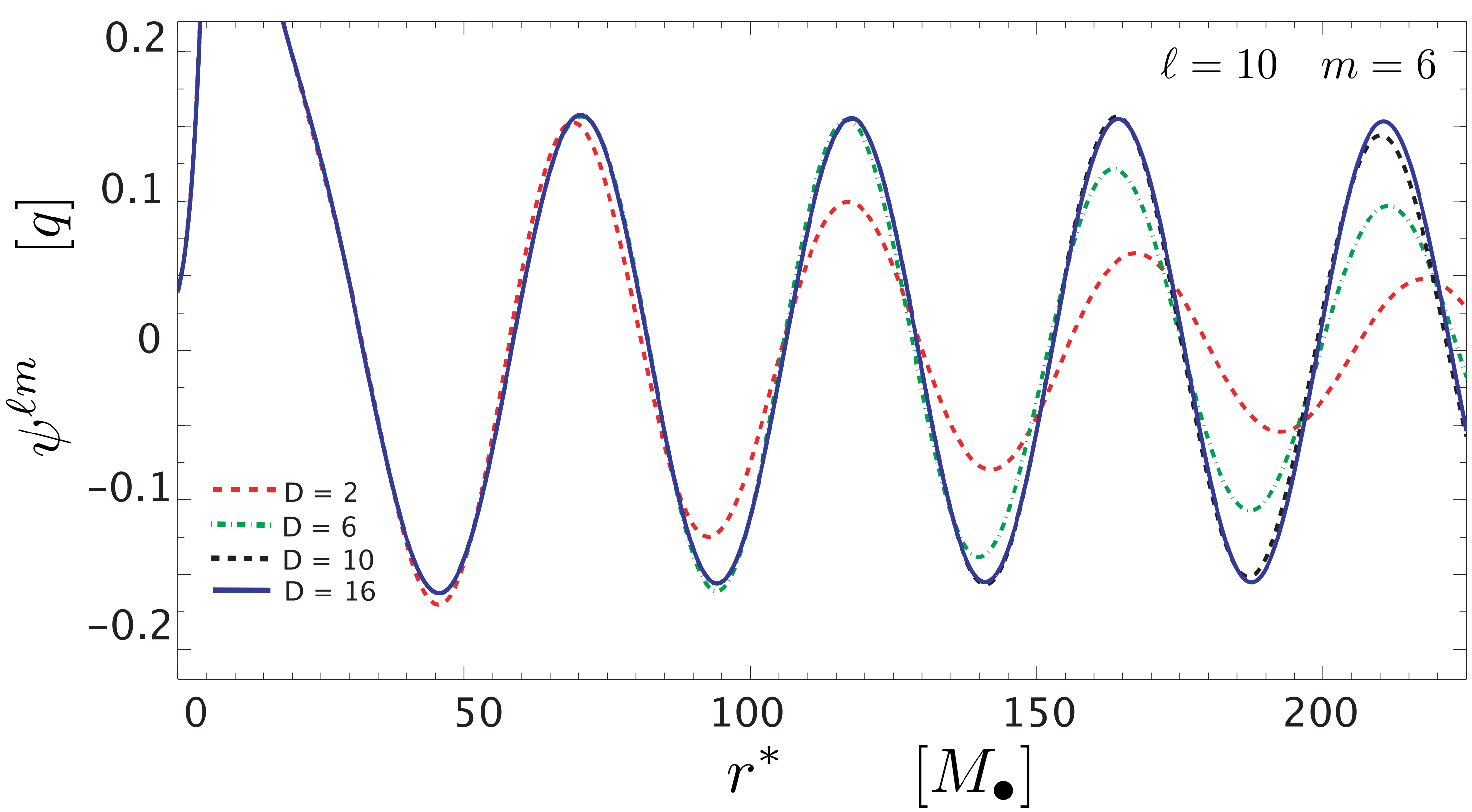
For instance for ℓ_{max} = 20, we need 231 evolutions of the wave-type equations instead of $N^2_{evolutions}$ = 400.

Code Validation

- *Covering the spatial domain with a given number of subdomains (D) we improve the field resolution with a relatively small N.*



$$\Delta r^*_a = r^*_N - r^*_0$$

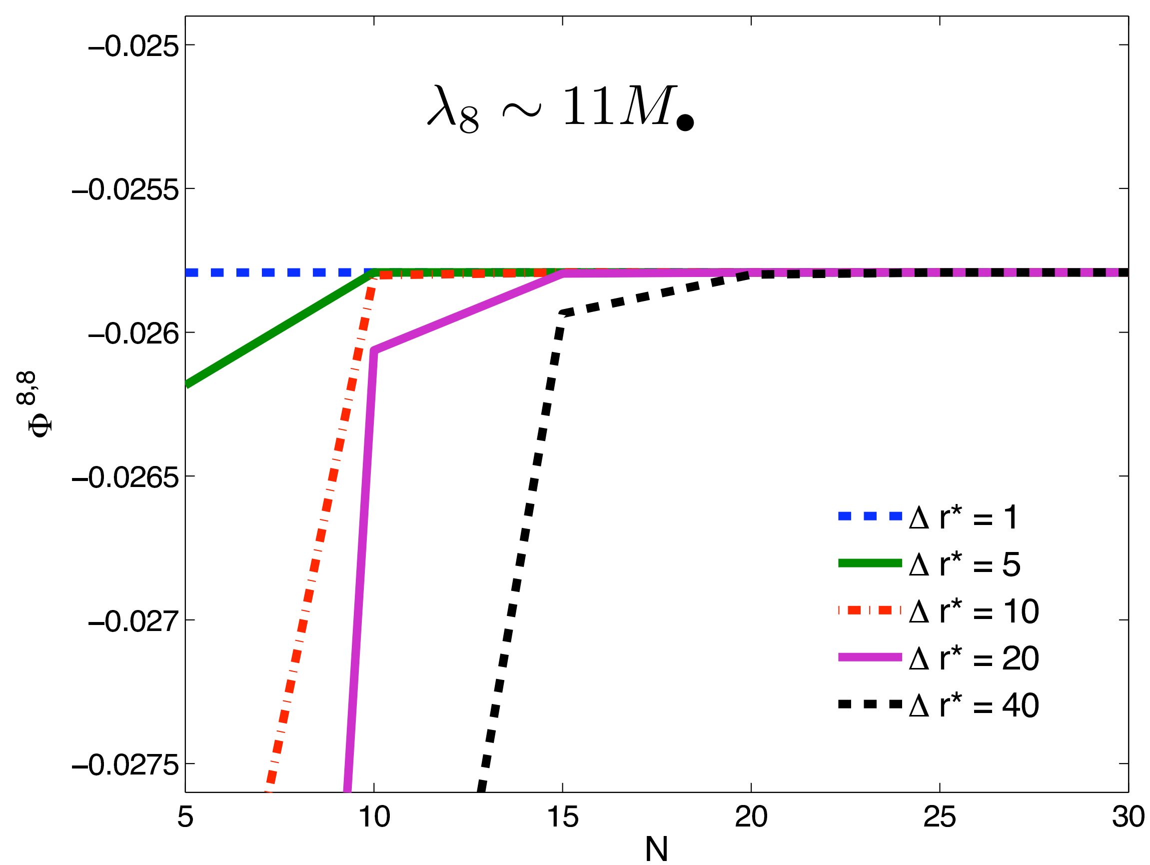


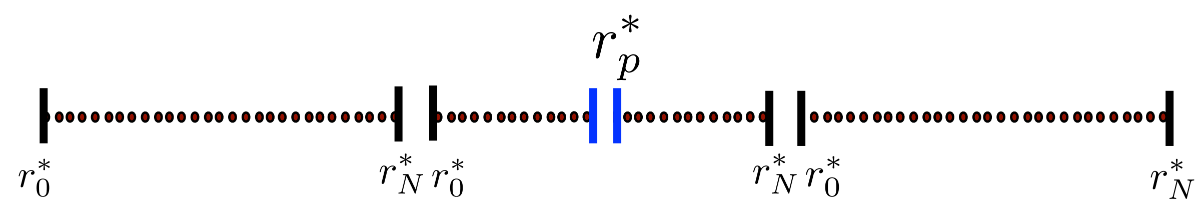
Resolution = $\begin{cases} D \uparrow, & N \downarrow \\ D \downarrow, & N \uparrow \end{cases}$

$$\Delta r^* = 50 M_\bullet \qquad N = 50$$

Improving the Mode Resolution

- *Different harmonic modes need different resolution*
- *We adjust the size of the subdomain around the particle location to the smaller mode wavelength*





$$\Delta r^* = r_N^* - r_0^*$$

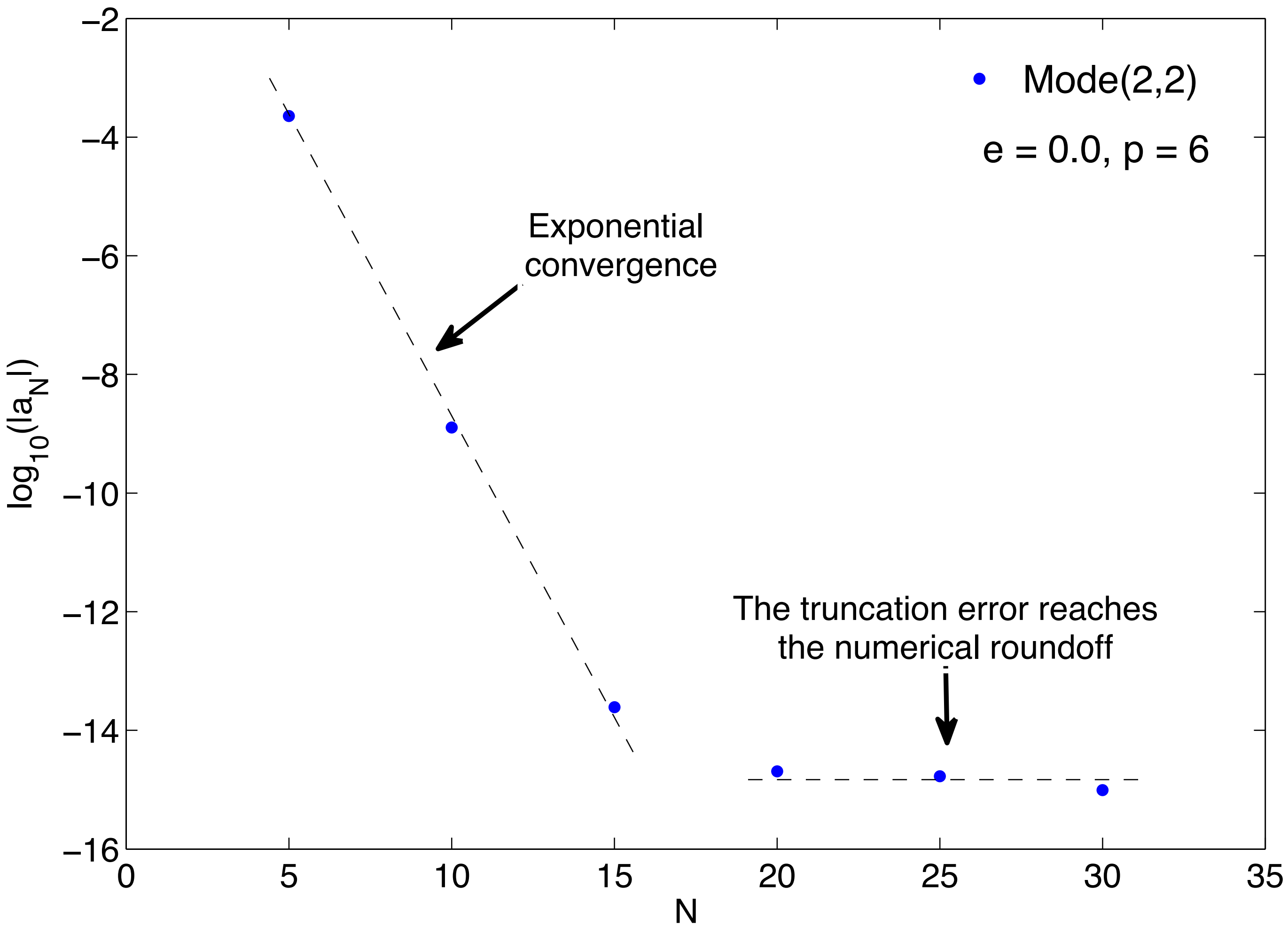
$$\boxed{\Delta r^* \sim \lambda_m}$$

$$\rho \sim \frac{\Delta r^*}{N}$$

[P. Canizares & Carlos F. Sopuerta (2011)].

Code Validation

The dependence of the truncation error (~ $|a_N|$) with respect increasing numbers of collocation points, N, give us an estimation of the exponential convergence of the code: e^{-N}

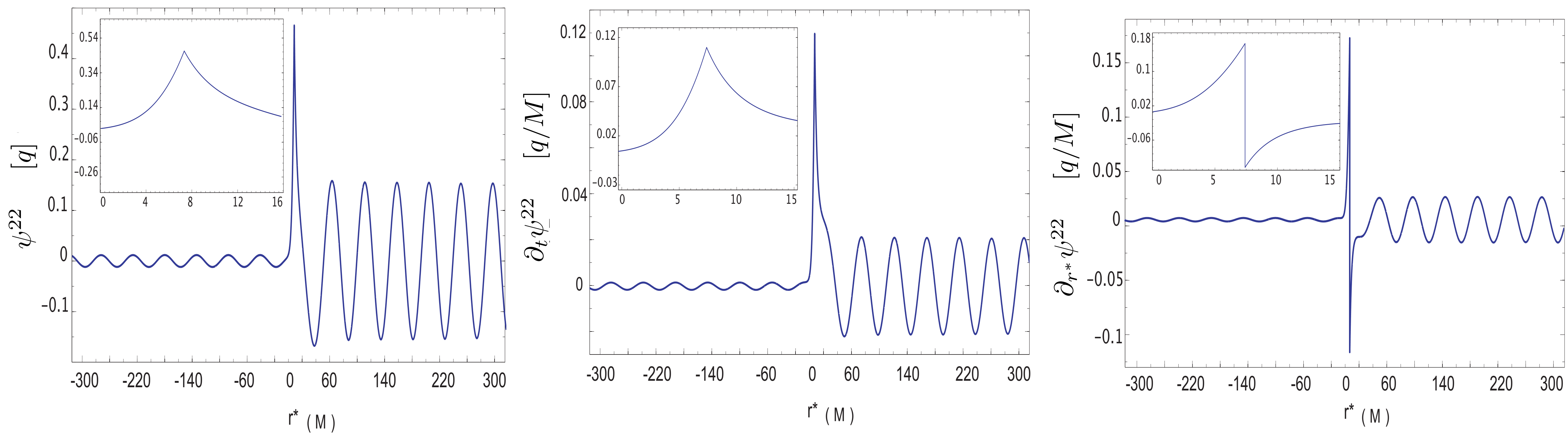


[P. Canizares & Carlos F. Sopuerta (2011)].

Some numerical Results

circular case

Snapshots from the Circular case (D=12, N=50)



$$[\psi^{\ell m}]_p = 0 \,, \qquad \left[\partial_t \psi^{\ell m}\right]_p = 0 \qquad \left[\partial_{r^*} \psi^{\ell m}\right]_p = A^{\ell m}$$

[P. Canizares & Carlos F. Sopuerta (2009)].

Some numerical Results:
Circular case

- *Results for the self-force components*

$\ell_{max} = 40$	PwP scheme	Frequency domain (a,b)
$\mathcal{F}^t$	3.609002 E-4	3.609072 E-4
$\mathcal{F}^r$	1.677282 E-4	1.67728 E-4
$\mathcal{F}^\phi$	-5.304234 E-3	-5.304231 E-3

$\Delta r^*_{min} \approx 2.5 M_\bullet$ $\quad D = 43$ $\quad N = 50$

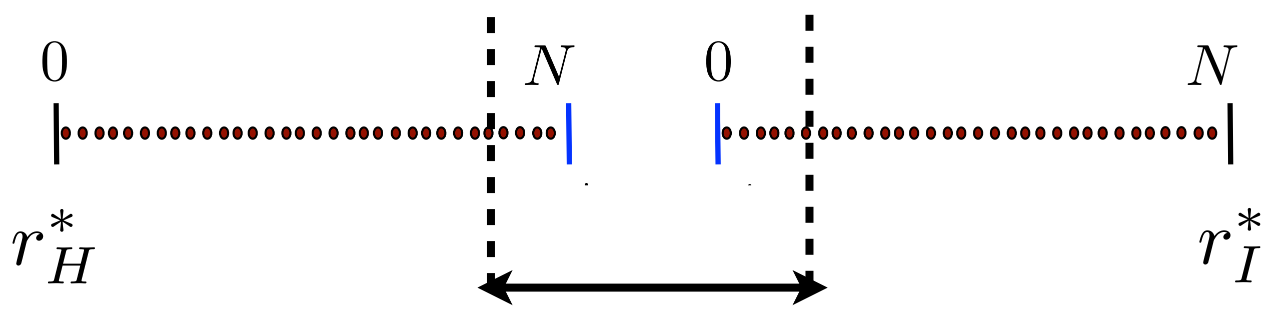
Our time domain results agree within 10^{-4}% with the frequency domain employing a small amount of computational resources.

(a) [Diaz-Rivera et al. PRD 70, 124018 (2004)] , (b) [Haas, Poisson. PRD 74, 044009 (2006)]

[P. Canizares & Carlos F. Sopuerta (2009)].

PwP with eccentric generic orbits

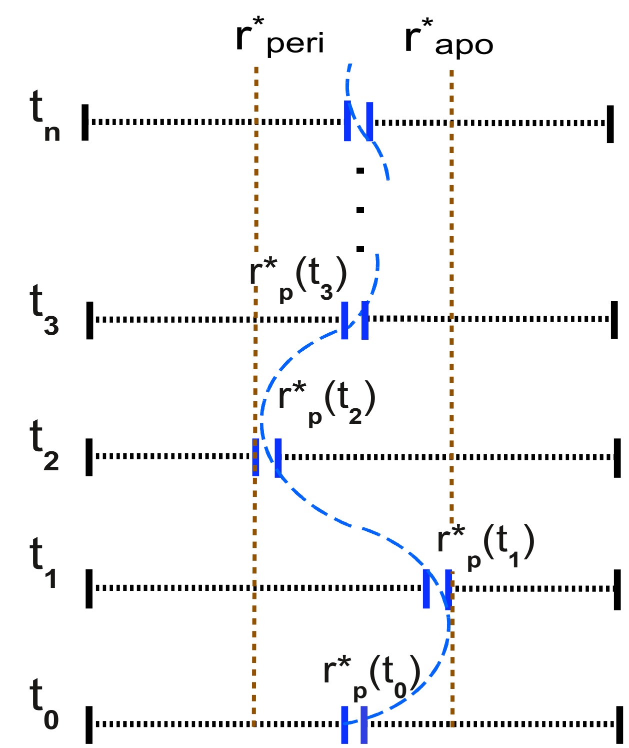
- *The key point of the PwP method is to keep the particle at the interface between subomains:*



$$r^*_p = r^*_p(t)$$

For eccentric orbits we use a time dependent linear mapping between the physical and spectral domains.

$$\mathcal{T} \times [r^*_0, r^*_N] \leftrightarrow \mathcal{T} \times [-1, 1]$$



[P. Canizares, Carlos F. Sopuerta & José L. Jaramillo (2010)].

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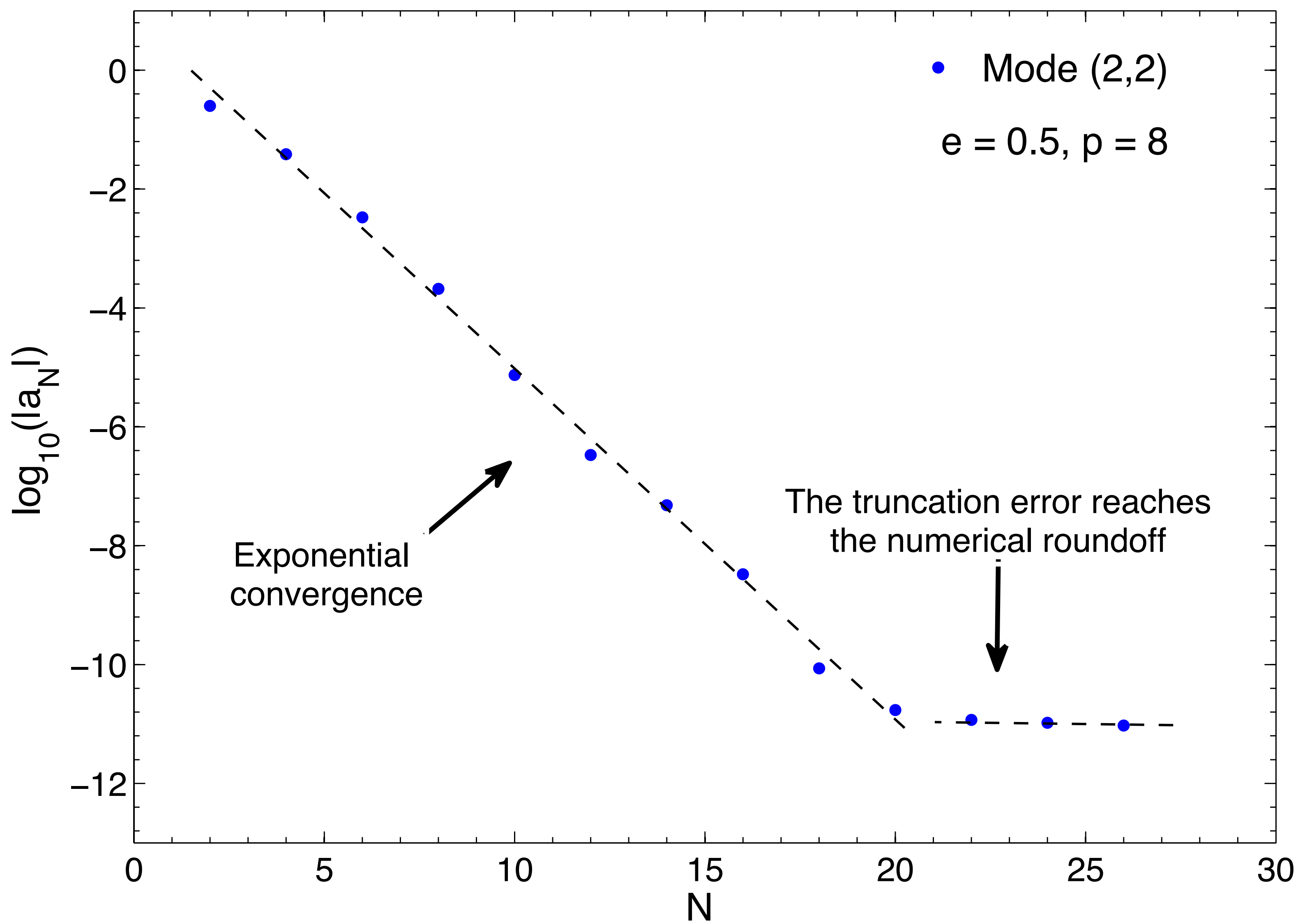
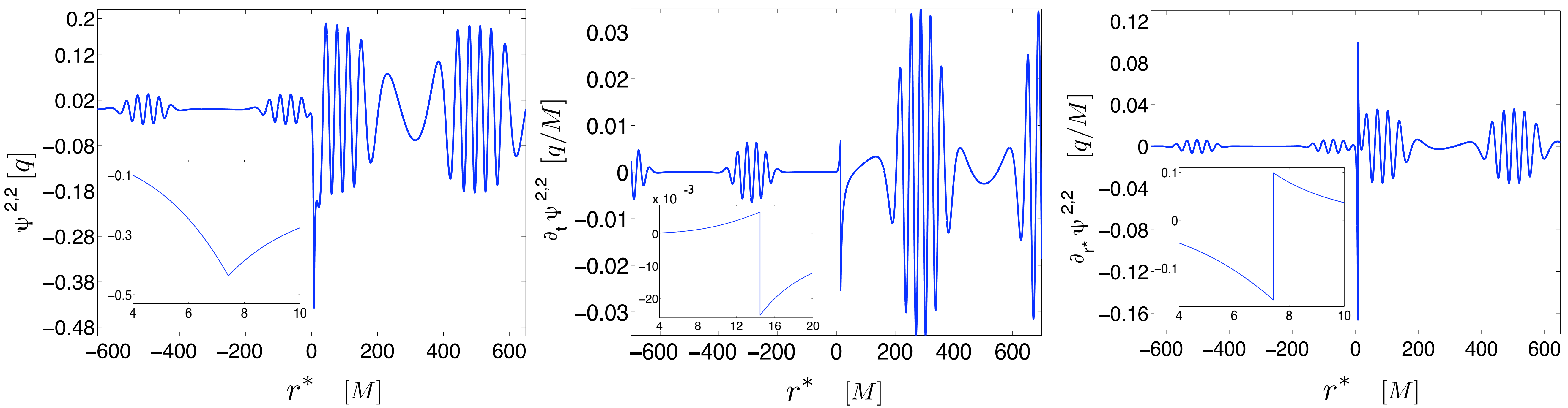


[P. Canizares & Carlos F. Sopuerta (2011)].

Code Validation

Snapshots from the Eccentric (e=0.5, p= 7.1) case (D=10, N= 100)



$$[\psi^{\ell m}]_p = 0\ , \qquad \left[\partial_t \psi^{\ell m}\right]_p = -\frac{\dot{r}_p^*\, A^{\ell m}}{(1-\dot{r}_p^{*2})}\ , \qquad \left[\partial_{r^*} \psi^{\ell m}\right]_p = \frac{A^{\ell m}}{(1-\dot{r}_p^{*2})}$$

[P. Canizares, Carlos F. Sopuerta & José L. Jaramillo (2010)].

Some numerical Results:

Eccentric case

(e,p) = (0.1, 63) $\ell_{max} = 17$

After tune modes

D = 80, N = 50

$$\partial_t \Phi = 4.5171 \cdot 10^{-4} q/M_\bullet^2$$

$$\partial_r \Phi = 2.1250 \cdot 10^{-4} q/M_\bullet^2$$

$$\partial_\varphi \Phi = -6.2040 \cdot 10^{-3} q/M_\bullet$$

Before tune modes

D = 41, N = 50

$$\partial_t \Phi = 4.5284 \cdot 10^{-4} q/M_\bullet^2$$

$$\partial_r \Phi = 2.1227 \cdot 10^{-4} q/M_\bullet^2$$

$$\partial_\varphi \Phi = -6.2086 \cdot 10^{-3} q/M_\bullet$$

The results differ with a fractional error of 0. 2%, 0.1% and 0.07%

[P. Canizares & Carlos F. Sopuerta (2011)].

Summary

We have developed a robust time-domain technique for modelling EMRIs:

- *Avoids the resolution of the small scale associated with the SCO,*
- *Provides precise determination of the retarded field and its derivatives near an on the SCO.*
- *It is suitable to deal with moderate to high eccentric EMRI orbits.*
- *It is an efficient method to make time-domain computations of the self-force.*