Progress on the Method of Mached Expansions in Schwarzschild

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Outline

- Method of Matched Expansions
- Past results in Nariai spacetime
- Branch Cut in Schwarzschild
- Conclusions

Scalar Self-force

- **Scalar field** $\phi(x)$ **with coupling const.** ξ **due to scalar charge q on worldline** γ **given by** $z(\tau)$ **:**

$$(\Box - \xi R)\phi(x) = -4\pi q \int_\gamma \delta_4(x, z(\tau))d\tau$$

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- **Self-force for scalar point charge q [after regularization]:**

$$f_\mu^{\rm self}(\tau) = \overbrace{-\frac{q^2}{12}(1-6\xi)Rv_\mu + q^2(g_{\mu\nu} + v_\mu v_\nu)\left(\frac{1}{3}\dot{a}^\nu + \frac{1}{6}R^\nu{}_\lambda v^\lambda\right)}^{\text{local}} +$$

$$q^2 \int_{-\infty}^{\tau^-} \nabla_\mu G_{ret}(z(\tau), z(\tau'))d\tau' \qquad \longleftarrow \quad \textbf{non-local}$$

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- It depends on the whole past of the worldline via the **'retarded' Green function**

$$(\Box - \xi R)G_{ret}(x, x') = -4\pi\delta_4(x, x')$$

with causality b.c.

Method of Matched Expansions

Non-local part of self-force: $$q^2 \int_{-\infty}^{\tau^-} \nabla_\mu G_{ret}(z(\tau), z(\tau'))d\tau'$$

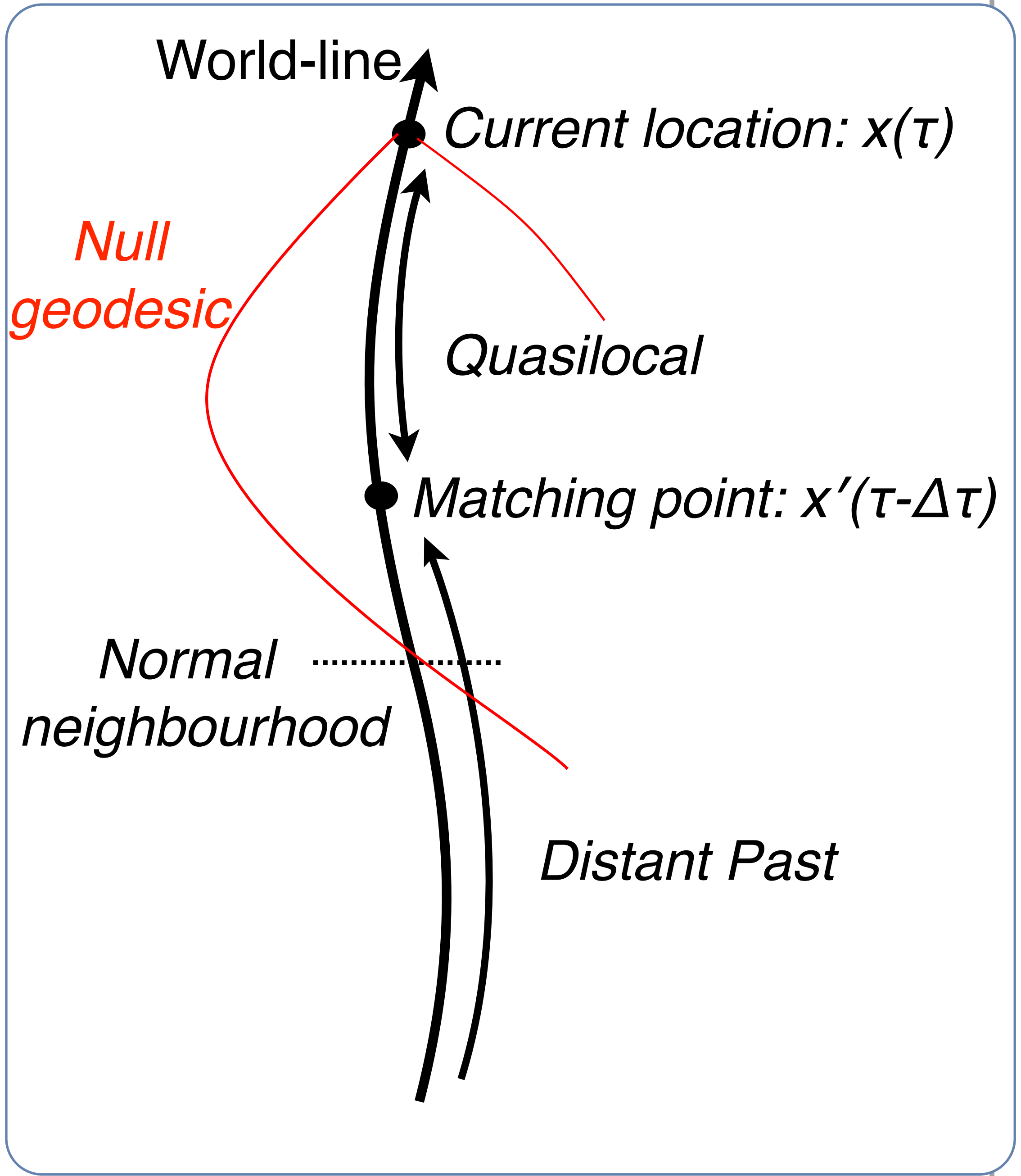
Choose point $\Delta\tau$ along worldline:

*** before that point ('Quasilocal' region)**

$$q^2 \int_{\tau-\Delta\tau}^{\tau^-} \nabla_\mu G_{ret} d\tau'$$

*** after that point ('Distant Past')**

$$q^2 \int_{-\infty}^{\tau-\Delta\tau} \nabla_\mu G_{ret} d\tau'$$



Quasilocal region - Hadamard form

$$G_{ret}\left(x,x'\right)=\theta\left(\Delta t\right)\left\{U\left(x,x'\right)\delta\left(\sigma\right)+V\left(x,x'\right)\theta\left(-\sigma\right)\right\}$$

- Using Hadamard form there are no calculational problems near x=x' since

$$q^2\int_{\tau-\Delta\tau}^{\tau^-}\nabla_\mu G_{ret}d\tau'=q^2\int_{\tau-\Delta\tau}^{\tau}\nabla_\mu V(x,x')d\tau'$$

- Calculate V with, e.g., covariant Taylor series expansion

$$V(x,x')=\sum_{k=0}^{\infty}V_k(x,x')\sigma^k$$

Distant past - Mode sum decomposition

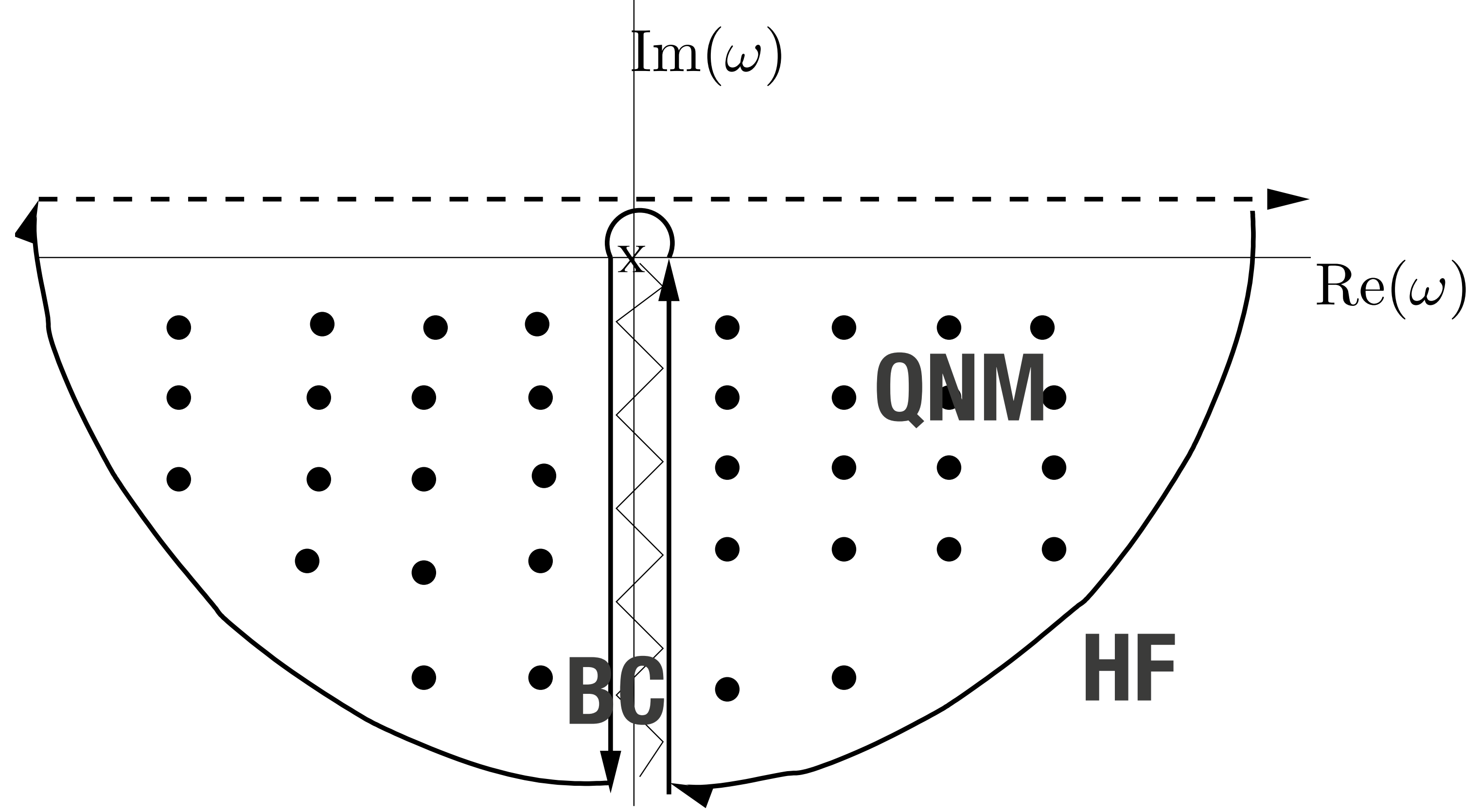
Green function in a spherically-symmetric spacetime:

$$G(r,r';t)=\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)G_\ell(r,r';t)$$

$$G_\ell(r,r';t)\equiv\int_{-\infty+ic}^{\infty+ic}d\omega\ G_\ell(r,r';\omega)e^{-i\omega t}$$

Deformation into the complex ω plane. Generally:

$$G_\ell(r,r',t)=G_\ell^{QNM}(r,r',t)+G_\ell^{HF}(r,r',t)+G_\ell^{BC}(r,r',t)$$



Distant past - Mode sum decomposition

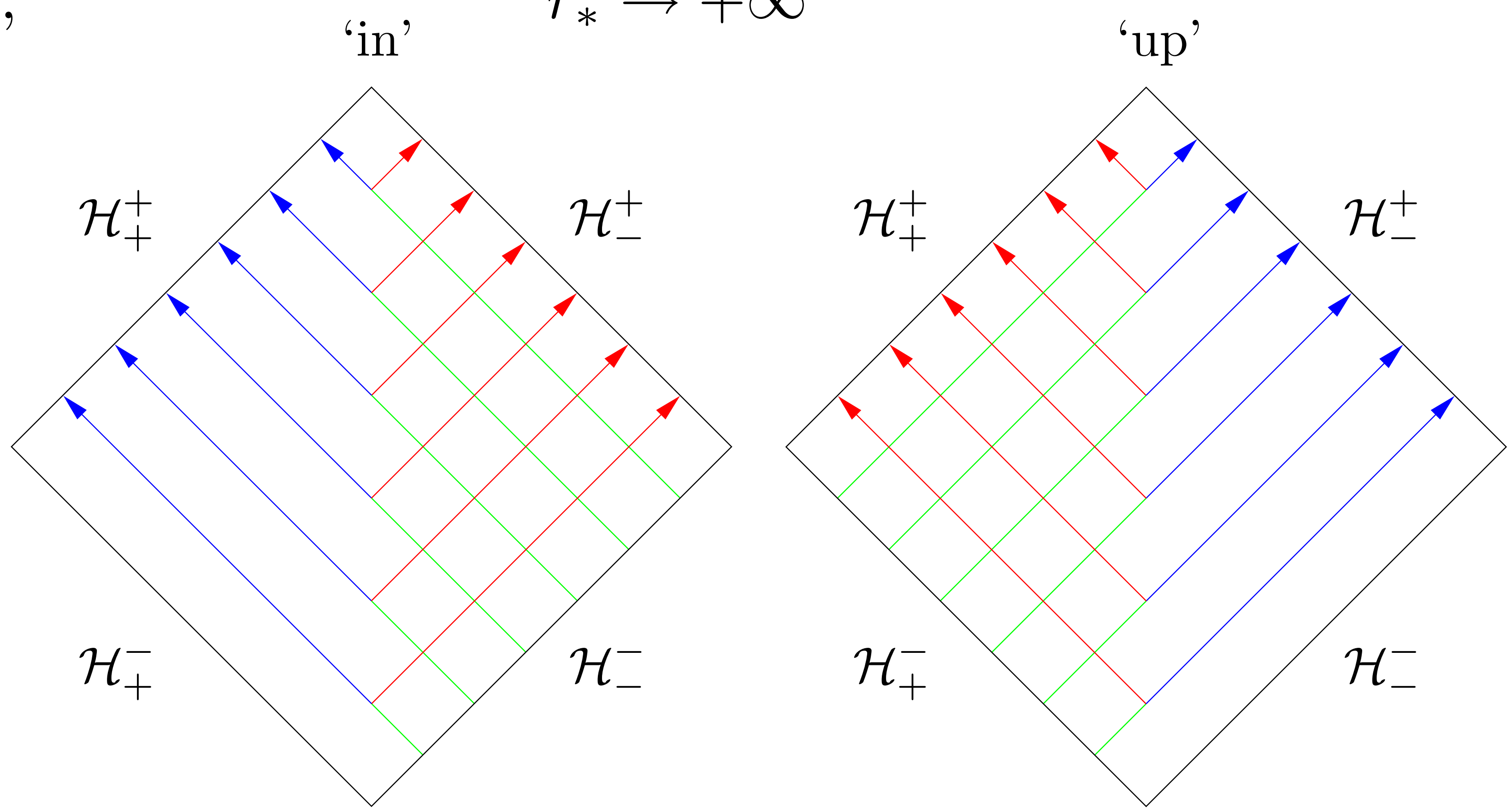
Green function modes: $$G_\ell(r,r';\omega) = \frac{R_\ell^{in}(r_<,\omega)R_\ell^{up}(r_>,\omega)}{W}$$

Radial ODE: $$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right] R_\ell(r,\omega) = 0$$

$$r_* = r_*(r) \in (-\infty, \infty)$$

$$R_\ell^{in}(r,\omega) \sim \begin{cases} e^{-i\omega r_*}, & r_* \to -\infty \\ A_\ell^{out} e^{+i\omega r_*} + A_\ell^{in} e^{-i\omega r_*}, & r_* \to +\infty \end{cases}$$

$$R_\ell^{up}(r,\omega) \sim \begin{cases} B_\ell^{in} e^{+i\omega r_*} + B_\ell^{out} e^{-i\omega r_*}, & r_* \to -\infty \\ e^{+i\omega r_*}, & r_* \to +\infty \end{cases}$$



Distant past - Mode sum decomposition

- If $V(r) \to 0$ exponentially as $r_* \to \infty$, then $R_\ell^{up}(r,\omega)$ has **poles** in ω-plane

- If $V(r) \to 0$ less fast than exponentially (except centrifugal barrier $V(r) = \ell(\ell+1)/r_*^2$) as $r_* \to \infty$, then $R_\ell^{up}(r,\omega)$ has **BC** in ω-plane

Past results - Matched expansions in Nariai

with S.Dolan, A.Ottewill & B.Wardell (PRD 79,124043)

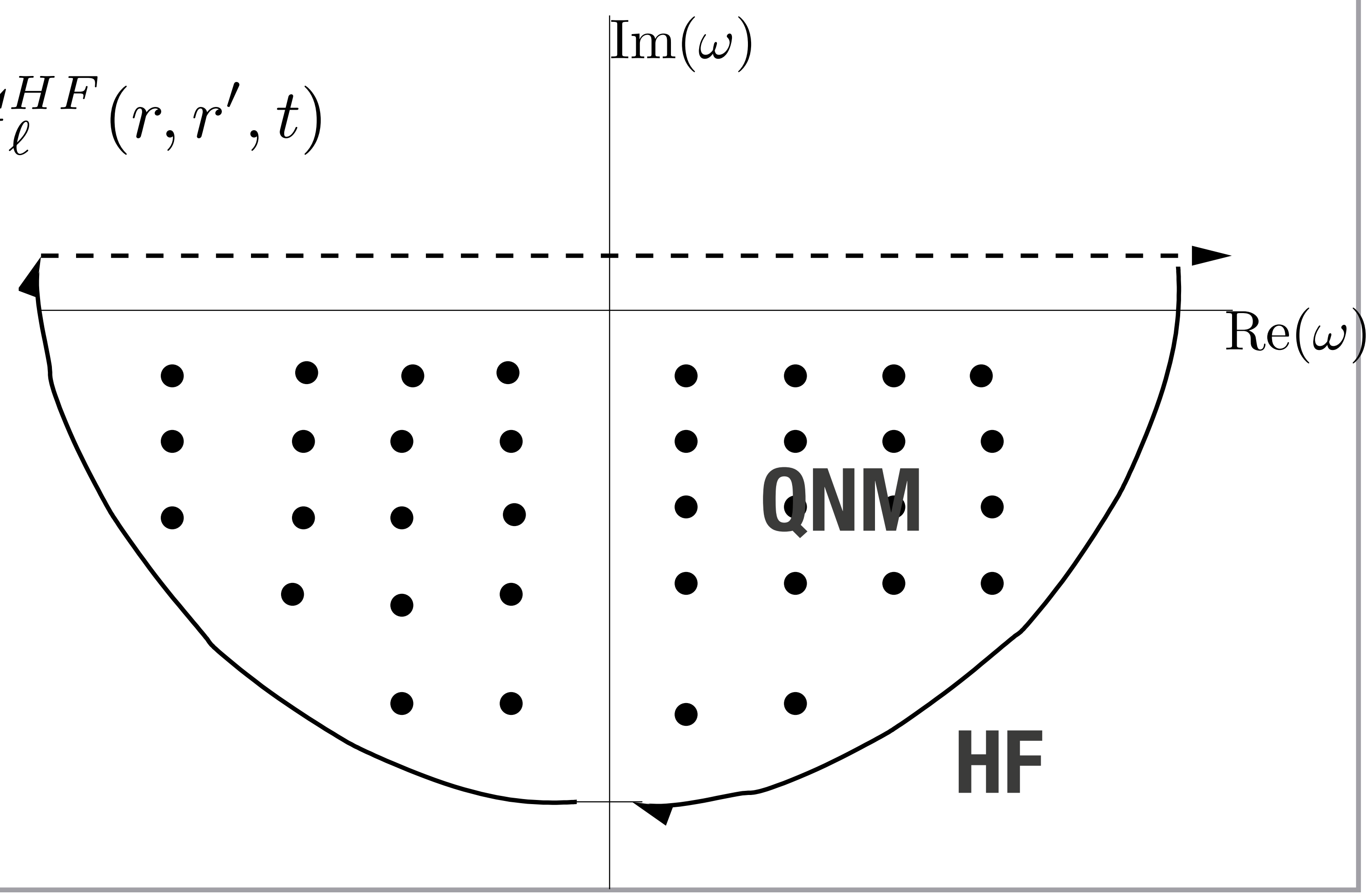
(static region of) Nariai spacetime ($dS_2 \times \mathbb{S}^2$):

$$ds^2 = -(1-\rho^2)dt^2 + (1-\rho^2)^{-1}d\rho^2 + d\Omega_2^2$$

It serves as a toy model for Schwarzschild

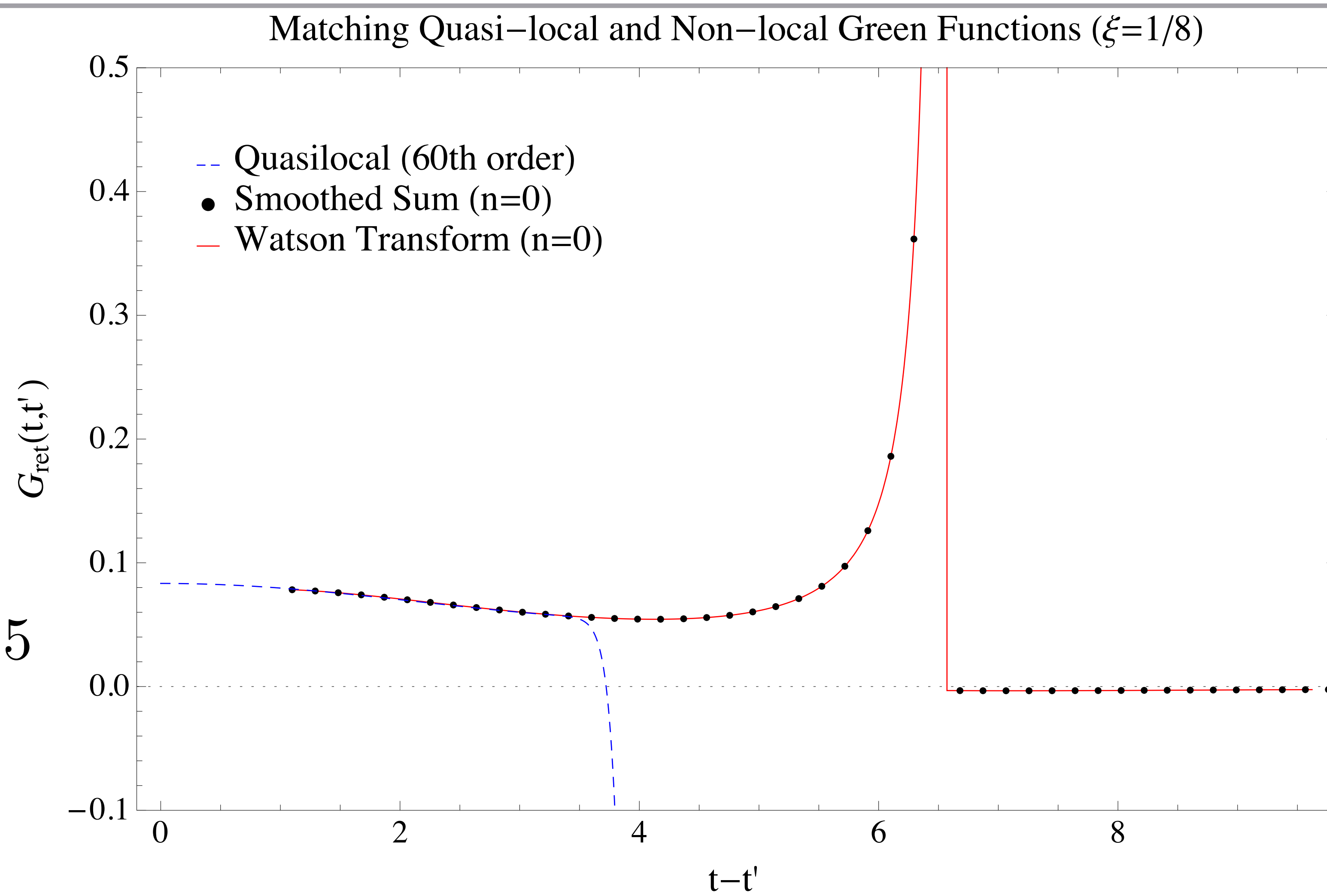
$$V(r) = \frac{\ell(\ell+1)}{\cosh^2(r_*)} = O\left(e^{\mp 2r_*}\right), \quad r_* \to \pm\infty \implies R_\ell^{in/up}$$ **have no BC**

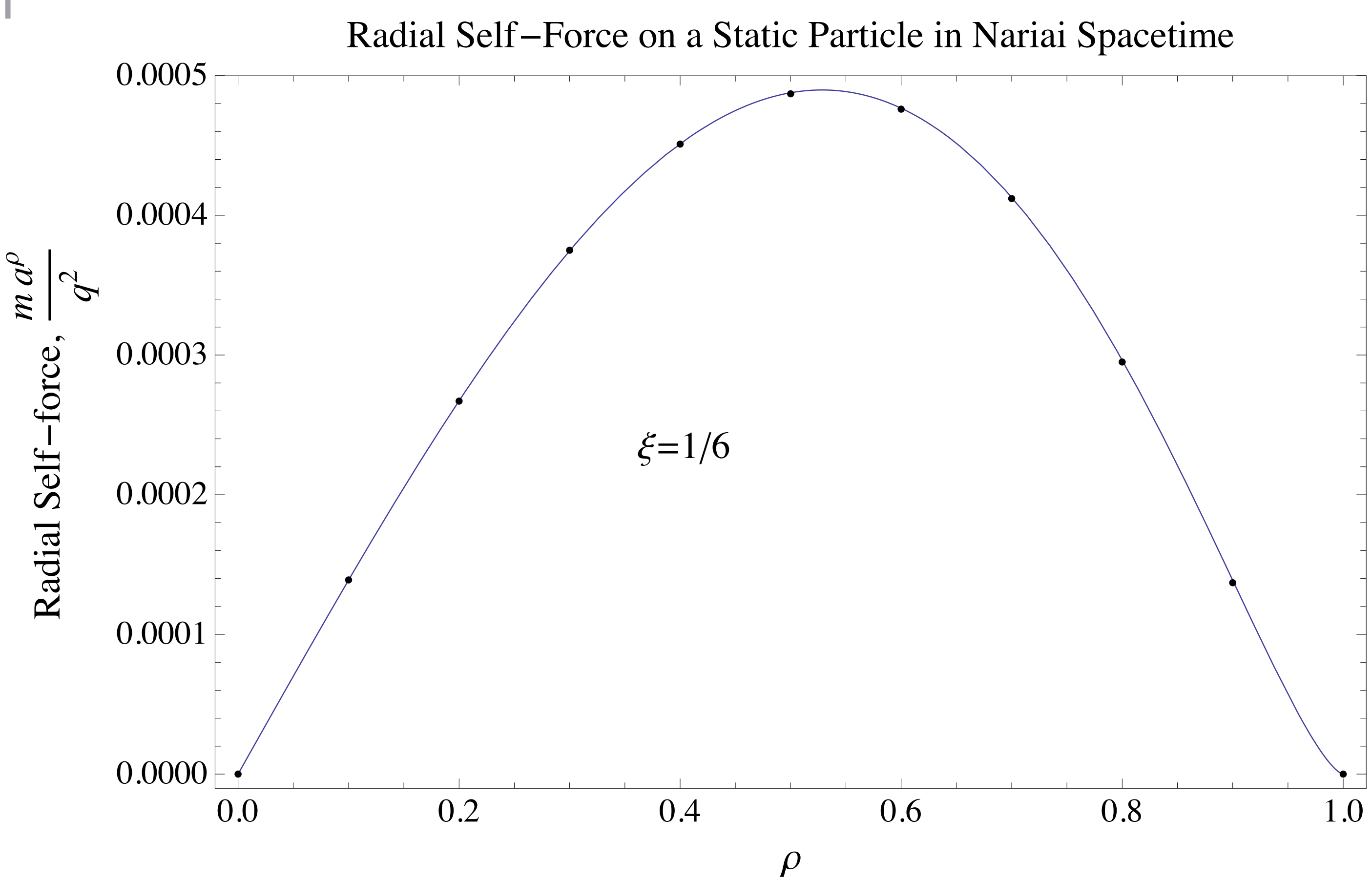
$$G_\ell(r,r',t) = G_\ell^{QNM}(r,r',t) + G_\ell^{HF}(r,r',t)$$



Matching QL and QNM:

$$\gamma = 0$$
$$\rho = \rho' = 0.5$$





Self-force for static scalar charge

Branch Cut in Schwarzschild

Radial ODE ('R-W eq.'): $s = 0, 1, 2$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right] R_\ell(r,\omega) = 0 \qquad V(r) = \left(1 - \frac{r_h}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{r_h(1-s^2)}{r^3}\right]$$

$$V(r) \to \frac{\left[\ell(\ell+1)+1-s^2\right]}{r_h^2} e^{r_*-r_h}, \quad r_* \to -\infty \implies R_\ell^{in}$$ **has poles in** ω

$$V(r) \to \frac{\ell(\ell+1)}{r_*^2} + \frac{2\ell(\ell+1) r_h \ln(r_*/r_h)}{r_*^3}, \quad r_* \to \infty \implies R_\ell^{up}$$ **has BC on NIA**

$$R_\ell^{up}(+i\nu) \in \mathbb{R}, \quad R_\ell^{up}(-i\nu) \in \mathbb{C} \qquad \omega \equiv -i\nu \qquad \nu \geq 0$$

Symmetry: $R_\ell^{in/up}(r,\omega) = R_\ell^{in/up^*}(r,-\omega^*)$

BC 'strength' ↓

$$\Delta R_\ell^{up}(r,-i\nu) \equiv \lim_{\epsilon\to 0}\left(R_\ell^{up}(r,\epsilon - i\nu) - R_\ell^{up}(r,-\epsilon-i\nu)\right) = iq(\nu) R_\ell^{up}(r,+i\nu)$$

Wronskian has BC along NIA but |W| has no BC

BC Green function:

$$G^{BC}(r,r';t) = \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\gamma)\Delta G_\ell(r,r';t)$$

$$\Delta G_\ell(r,r';t) \equiv \frac{1}{2\pi}\int_0^\infty d\nu\ \Delta G_\ell(r,r';-i\nu)e^{-\nu t}$$

BC Green function modes:

$$\Delta G_\ell(r,r',-i\nu) = -2i\nu\frac{q(\nu)R_\ell^{in}(r,-i\nu)R_\ell^{in}(r',-i\nu)}{|W|^2}$$

$$q(\nu) \equiv -i\ \frac{\Delta R_\ell^{up}(r,-i\nu)}{R_\ell^{up}(r,+i\nu)}$$

Calculation of ΔG_ℓ in 3 regimes:

- **Small**-ν: Sasaki&Tagoshi Liv.Rev. asymptotics
  (reformulation of Leaver'86)

- **Mid**-ν: a new series expression for the BC on the NIA

- **Large**-ν: Continuation into the complex-r plane
  (Maassen van den Brink'2004 for spin=2)

$\bar{\omega} \equiv r_h \omega$

Mid-ν expression

Jaffé series for Rin (Leaver'86):

$$R_\ell^{in}(r,\omega) = (\bar{r}-1)^{-i\bar{\omega}}\,\bar{r}^{2i\bar{\omega}} e^{i\omega r} e^{-2i\bar{\omega}} \sum_{n=0}^{\infty} a_n \left(1-\frac{1}{\bar{r}}\right)^n$$

Asymptotics: $$a_n = O\left(n^{-\bar{\nu}-3/4} e^{\pm 2\sqrt{-2\bar{\nu} n}}\right), \quad n \to \infty$$

Jaffé series is absolutely convergent $\forall \omega \in \mathbb{C}$ **and** $\forall r \in [r_h, \infty)$

Mid-ν expression

a_n satisfy a 3-term **recurrence rln.**:

$$(n+1)(n-2\bar{\nu}+1)a_{n+1} - [2n^2 + (2-8\bar{\nu})n + 8\bar{\nu}^2 - 4\bar{\nu} + l(l+1) + 1 - s^2]a_n + \left[n^2 - 4\bar{\nu}n + 4\bar{\nu}^2 - s^2\right] a_{n-1} = 0$$

No dominant/subdominant slns. to recurrence rln. on the NIA

a_n have simple **poles** $\forall n \geq k$ when $\bar{\nu} = k/2$ for some $k \in \mathbb{N}$

Exception is the **algebraically-special frequency** for $s=2$:

$$\bar{\nu} = \bar{\nu}_{AS} \equiv \frac{\ell(\ell-1)(\ell+1)(\ell+2)}{6}$$

So better calculate the radial function: $\hat{R}_\ell^{in} \equiv \sin(2\pi\bar{\nu}) R_\ell^{in}$

Wronskian in the Green function cancels out this factor

Mid-ν expression

In Leaver'86, as a series of confluent hypergeometric functions:

$$R_\ell^{up}(r,\omega) = (-2i\bar\omega)^{s+1-2i\bar\omega}\bar r^{1+s}(\bar r - 1)^{-i\bar\omega} e^{+i\omega r}\sum_{n=0}^{\infty} a_n T_n^+$$

$$T_n^+ \equiv (-2i\bar\omega+1)_n\, U(s+1-2i\bar\omega+n, 2s+1, -2i\omega r)$$

U has branch cut along $\omega r : 0 \to -\infty i$

$$\left(ze^{-2\pi i}\right)^a U(a,b,ze^{-2\pi i}) - z^a U(a,b,z) = \frac{2\pi i e^z(-z)^a}{\Gamma(a)\Gamma(1+a-b)} U(b-a,b,-ze^{-2\pi i})$$

New series:

$$\Delta R_\ell^{up}(r,\omega) = 2\pi i \bar r^{1+s}(\bar r - 1)^{-\bar\nu} e^{+\nu r}\frac{e^{-2\nu r}e^{\pi i(s+1-2\bar\nu)}}{\Gamma(1-2\bar\nu)}\sum_{n=0}^{\infty} a_n T_n^-$$

$$T_n^- \equiv \frac{(-1)^n\Gamma(1+n-2\bar\nu)U(s-n+2\bar\nu, 2s+1, 2\nu r)}{\Gamma(1+s+n-2\bar\nu)\Gamma(1-s+n-2\bar\nu)}$$

this can be evaluated *on* the NIA (using principal branch for U) !

(only evaluation of BC in the literature uses extrapolation to the NIA)

Mid-ν expression

Both T_n^- and T_n^+ satisfy the same recurrence rlns.:

(no surprise, since R_ℓ^{up} and ΔR_ℓ^{up} satisfy same ODE)

$$(n+1-2\bar{\nu})(n-2\bar{\nu})T_{n-1} - (n+1-2\bar{\nu})(2n+1-4\bar{\nu}-2\nu r)T_n + (n+1-2\bar{\nu}+s)(n+1-2\bar{\nu}-s)T_{n+1} = 0$$

Asymptotics: $T_n^{\pm} \to O\left(n^{-1/4}e^{-2i\sqrt{2\nu r n}}\right), \quad n \to \infty$

so no dominant/subdominant slns. to recurrence rlns. on the NIA so no need for Miller's algorithm of backward recursion (as would be required off the NIA)

From the 'integral test', $\sum_n a_n T_n^{\pm}$ is absolutely convergent $\forall \nu > 0$

However, on PIA R_ℓ^{up} corresponds to the subdominant sln. and so the dominant sln. would 'creep in'. Better use an integral representation for U

Large-ν asymptotics

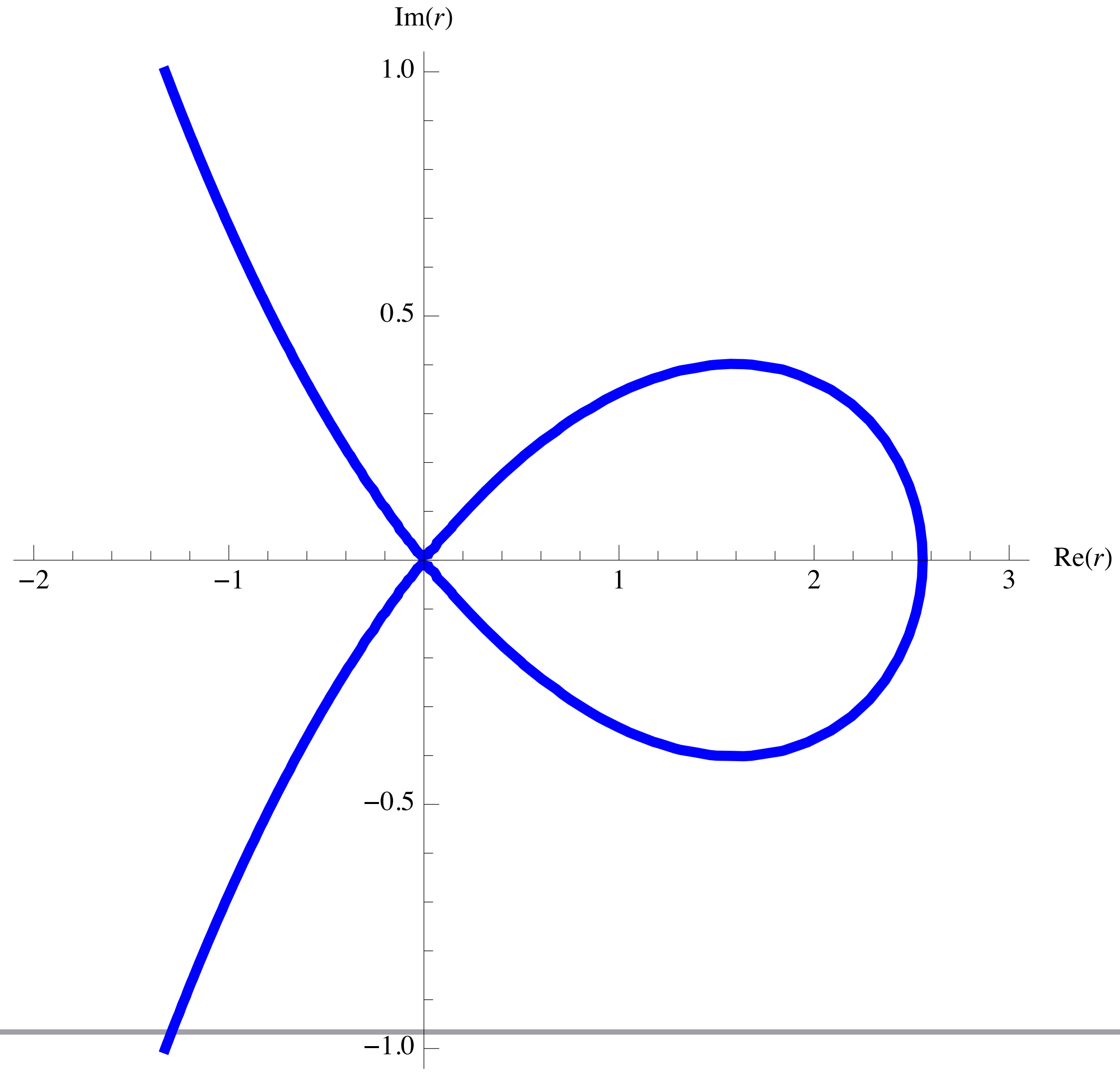
- Problem: if one integrates ODE with b.c. that have the dominant sln. then initial error is magnified exponentially

- Sln.: continuation into the **complex-r plane** . Follow curves where neither sln. is dominant.

- Done by Maassen van den Brink'2004 for spin=2. We extended it to spin=0,1.

Anti-Stokes lines: $\mathrm{Re}(r_*) = 0$

(no dominant sln. $e^{\pm i\omega r_*}$ on NIA)



Large-ν asymptotics

- Find 2 l.ind. WKB expansions $g_a(r,\omega)$ for $|r\sqrt{\nu}| \gg 1$

  Valid away from r=0,rh and anti-Sokes.

- Find 2 l.ind. slns. for fixed $|r\sqrt{\nu}|$ and up to next-to-leading order for large-ν

  They're expressed in terms of Bessel functions. Therefore, one knows how to analytically-continue them.

Large-ν asymptotics

- Exact **monodromy** around the horizon:

$$R_\ell^{in}(r,-i\nu) \sim e^{-i\omega r_*}, \quad r_* \to -\infty \implies R_\ell^{in}\left((r-r_h)e^{2\pi i},\omega\right) = e^{2\pi\omega} R_\ell^{in}(r-r_h,\omega)$$

Ansatz: $R_\ell^{in}(r,-i\nu) \sim g_a(r,i\nu) + c(\nu) g_a(r,-i\nu)$

$c(\nu)$:unknown

Match to the 2 l.ind. slns. (with Bessel functions) for fixed $|r\sqrt{\nu}|$

These can be analyticallly-continued around r=rh along anti-Stokes lines

$$R_\ell^{in}\left((r-r_h)e^{2\pi i},-i\nu\right) \sim e^{-2\pi i\nu} g_a(r-r_h,i\nu) + \left[(-1)^{s/2+1} 2i\left(1-\frac{\alpha}{\sqrt{\nu}}\right) + c(\nu)e^{2\pi i\nu}\right] g_a(r-r_h,-i\nu)$$

Comparing with exact monodromy: $c(\nu) \sim (-1)^{s/2} \dfrac{\left(1-\alpha/\sqrt{\bar{\nu}}\right)}{\sin\left(2\pi\bar{\nu}\right)}$

Large-ν asymptotics

Similar idea for R_ℓ^{up} :

$$s=0,2: \quad R_\ell^{up}(r,-i\nu) \sim g_a(r,-i\nu) - (-1)^{s/2} 2e^{2\pi i\nu}\left[i + \frac{\alpha}{\sqrt{\nu}}\right] g_a(r,+i\nu)$$

$$s=1: \quad R_\ell^{up}(r,-i\nu) \sim \left[1 + O\left(\frac{1}{\nu}\right)\right] g_a(r,-i\nu) + O\left(\frac{1}{\nu}\right) 2e^{2\pi i\nu} g_a(r,+i\nu)$$

$$s=0: \quad \alpha = -\frac{\Gamma\left(\frac{1}{4}\right)^4}{48\pi^{3/2}}\left[1+3\ell(\ell+1)\right]$$

$$s=2: \quad \alpha = \frac{\Gamma\left(\frac{1}{4}\right)^4}{48\pi^{3/2}}\left[1-\ell(\ell+1)\right]$$

$$s=1: \quad \alpha = -\frac{\ell(\ell+1)\sqrt{\pi}}{2}$$

$$g_a(r,\omega) \equiv e^{i\omega r_*}\left\{1 + \frac{i\left[2\ell(\ell+1)\bar{r} + 1 - s^2\right]}{4\bar{r}^2\bar{\omega}} + \frac{8(\bar{r}-1)\left[\ell(\ell+1)\bar{r}+1-s^2\right] - \left[2\ell(\ell+1)\bar{r}+1-s^2\right]^2}{32\bar{r}^4\bar{\omega}^2} + \ldots\right\}$$

Large- ν asymptotics

BC 'strength' $$q(\nu) \equiv -i \, \frac{\Delta R_\ell^{up}(r,-i\nu)}{R_\ell^{up}(r,+i\nu)} \qquad s=1: \quad q_s(\nu) \sim O\left(\frac{1}{\sqrt{\bar{\nu}}}\right)$$

$$s=0,2: \quad q_s(\nu) \sim (-1)^{s/2+1} 4 \left[\cos(2\pi\bar{\nu}) + \frac{\alpha}{\sqrt{\bar{\nu}}}\sin(2\pi\bar{\nu})\right]$$

Wronskian:

$$W[R_\ell^{up}, R_\ell^{in}; -i\nu] \sim \frac{W[g_a(-i\nu), g_a(i\nu)]}{\sin(2\pi\bar{\nu})} \left[\sin(2\pi\bar{\nu}) + 2ie^{2\pi i\bar{\nu}}\left(1 - \frac{(1+i)\alpha}{\sqrt{\bar{\nu}}}\right)\right], \quad \bar{\nu} \gg 1$$

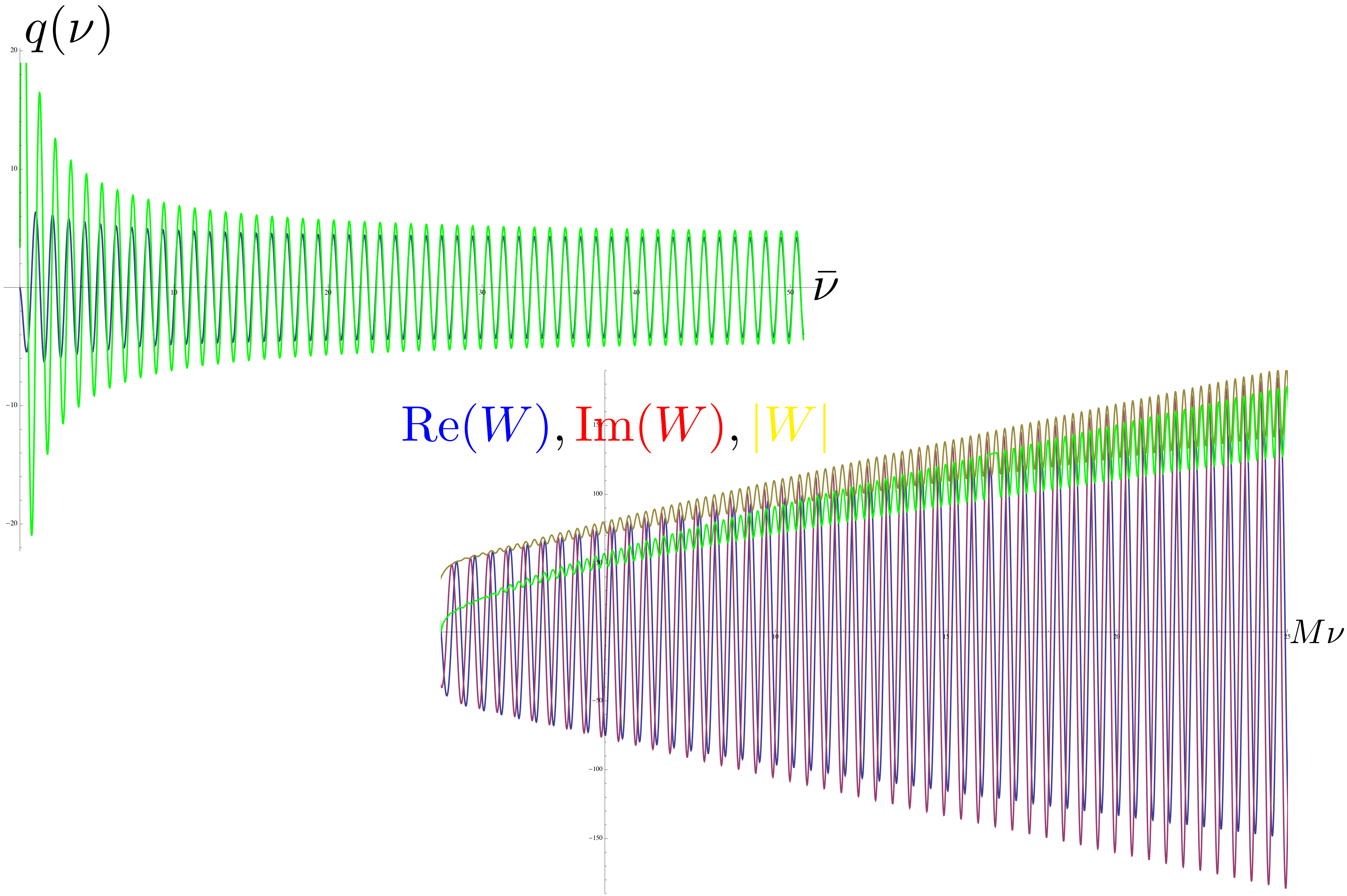
$s=0,2$

QNMs approach NIA asymptotically for s=1 -> unknown connection to BC?

BC Green function to leading order:

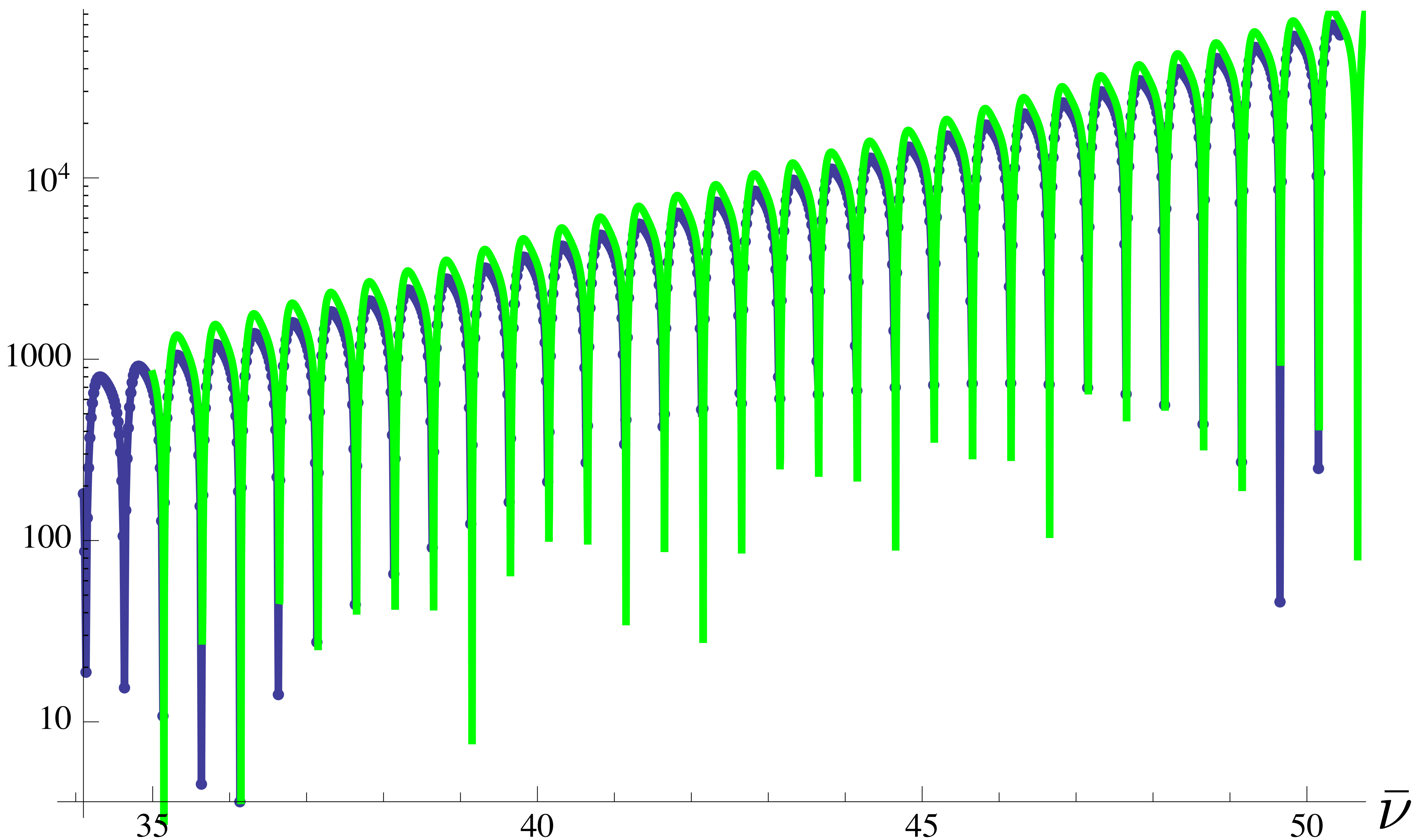
$$\Delta G_\ell(r,r';t) = O\left(\frac{e^{-\nu(t-|r_*|-|r'_*|)}}{\bar{\nu}}\right), \quad \bar{\nu} \to \infty$$

Matching of the mid- and large- regimes in ν $(s=0, \ell=1)$



Matching of the mid- and large- regimes in ν $(s = 0, \ell = 1)$

$$\Delta G_\ell(r_* = 0.4M, r_* = 0.2; -i\nu)$$



Small-ν series

Sasaki&Tagoshi , Liv.Rev., series of hypergeometric functions:

$$R_\ell^{in}(r,\omega) = R_0^{\mu} + R_0^{-\mu-1}$$

$$R_0^{\mu} = e^{i\bar{\omega}(1-\bar{r})}(\bar{r}-1)^{-s-i\bar{\omega}}\bar{r}^{i\bar{\omega}+\mu}.$$

$$\sum_{n=-\infty}^{\infty} a_n^{\mu} \frac{\Gamma(1-s-2i\bar{\omega})\Gamma(2n+2\mu+1)}{\Gamma(n+\mu+1-i\bar{\omega})\Gamma(n+\mu+1-s-i\bar{\omega})} \bar{r}^n {}_2F_1\left(-n-\mu-i\bar{\omega}, -n-\mu-s-i\bar{\omega}, -2n-2\mu; \frac{1}{\bar{r}}\right)$$

this series converges $\forall r \neq \infty$

a_n^{μ} **satisfies a 3-term recurrence rln.**

Parameter μ **chosen so that** a_n^{μ} **is minimal sln.**

As a series of confluent hypergeometric functions:

$$R_\ell^{up}(r,\omega) = 2^{\mu} e^{-\pi\bar{\omega}} e^{-i\pi(\mu+1+s)} e^{i\omega r} (\omega r)^{\mu+i\bar{\omega}} (\omega r - \bar{\omega})^{-s-i\bar{\omega}}$$

$$\sum_{n=-\infty}^{\infty} i^n \frac{(\mu+1+s-i\bar{\omega})_n}{(\mu+1-s+i\bar{\omega})_n} a_n^{\mu} (2\omega r)^n U(n+\mu+1+s-i\bar{\omega}, 2n+2\mu+2, -2i\omega r)$$

this series converges $\forall r > r_h$

Small-ν series

To leading order as $\bar{\nu} \to 0$:

$$\mu \sim \ell$$

$$q(\nu) \sim (-1)^{\ell} 2\pi\nu$$

$$W \sim \frac{(-1)^{\ell} 2^{-\ell} \Gamma(2\ell+1)\Gamma(2\ell+2)}{r_h \Gamma^3(\ell+1)} \bar{\nu}^{-\ell}$$

(W has a BC but only to higher order)

$$R_{\ell}^{in} \sim \bar{r}\,{}_2F_1(-\ell, \ell+1, 1, 1-\bar{r})$$

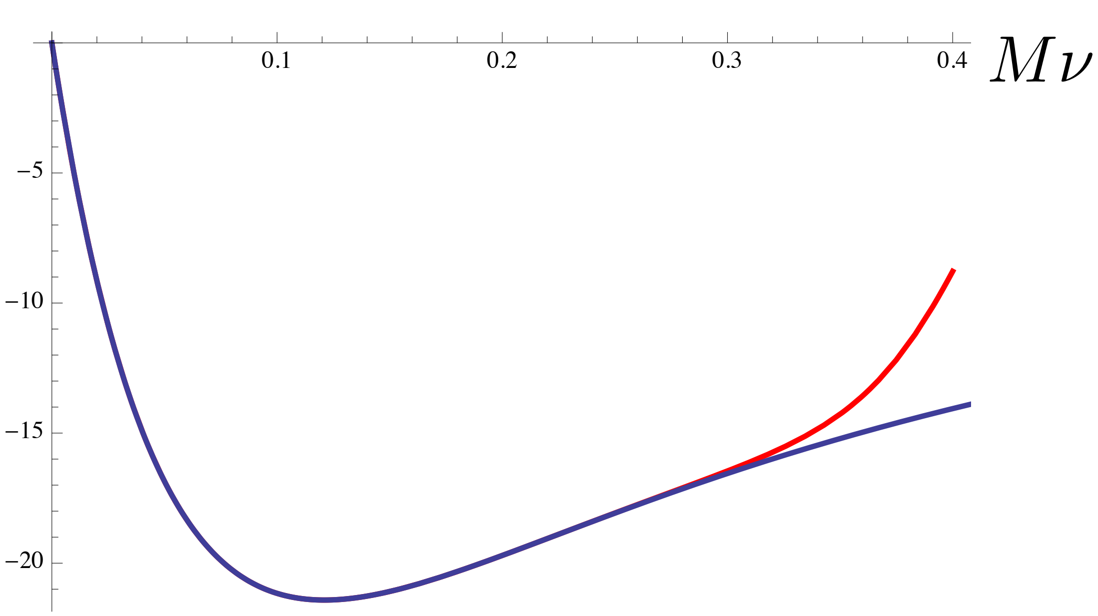
$$\Delta G_{\ell}(r, r'; -i\nu) \sim \frac{(-1)^{\ell+1} 2^{2(\ell+1)} \pi \Gamma^6(\ell+1) \bar{r}\bar{r}'\,{}_2F_1(-\ell, \ell+1, 1, 1-\bar{r})\,{}_2F_1(-\ell, \ell+1, 1, 1-\bar{r}')}{\Gamma^2(2\ell+1)\Gamma^2(2\ell+1)} \bar{\nu}^{2\ell+2}$$

which yields the power-law tail:

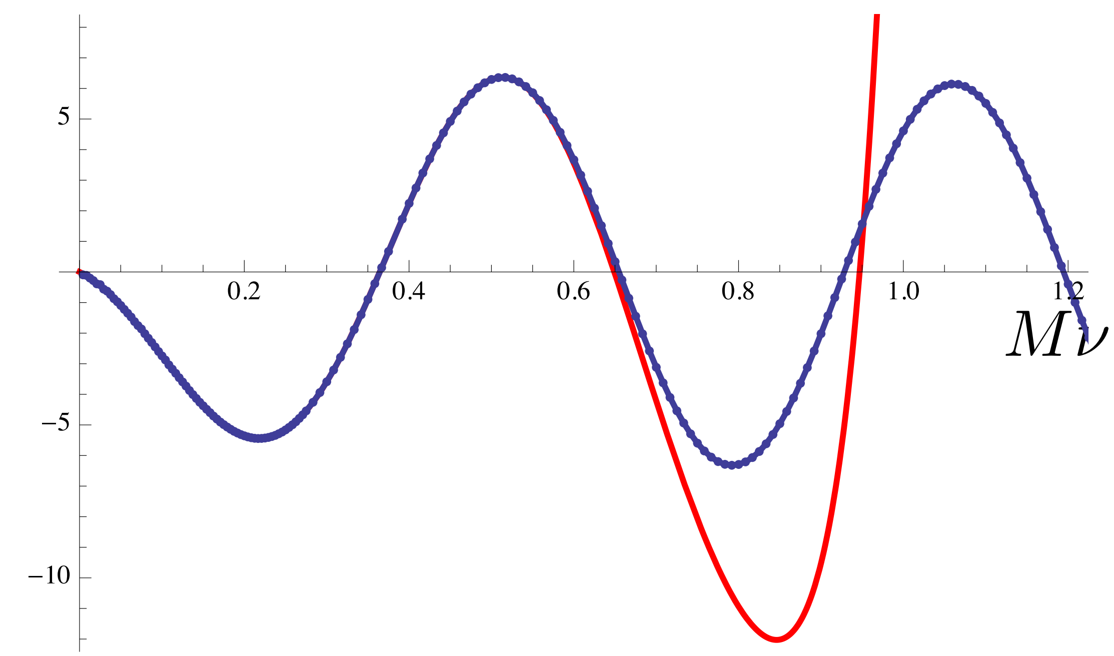
$$\Delta G_{\ell}(r, r'; t) = O\left(\bar{t}^{-2\ell-3}\right), \quad \bar{t} \to \infty$$

Matching of the mid-ν and small-ν expressions $(s = 0, \ell = 1)$

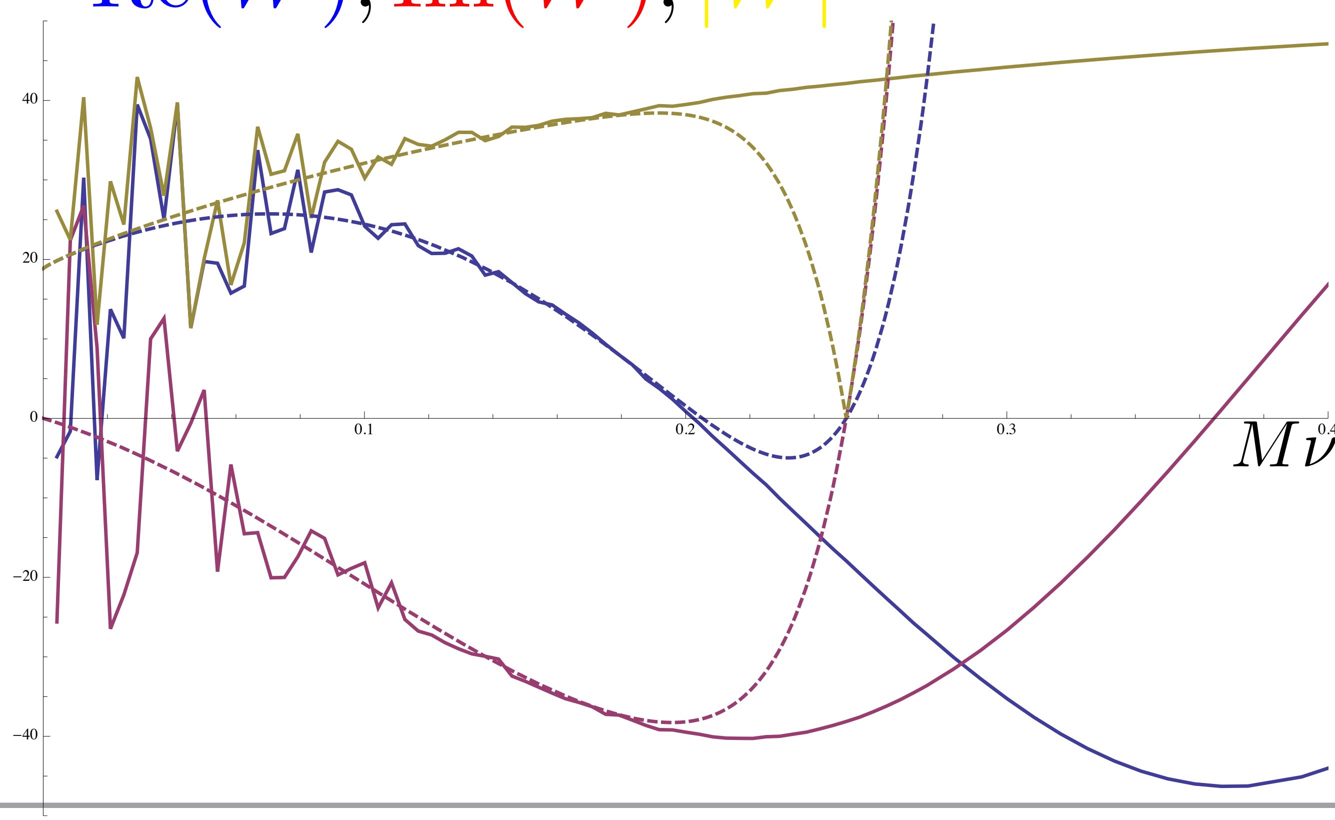
$e^{-i\omega r_*}\hat{R}_\ell^{in}(r = 10M)$



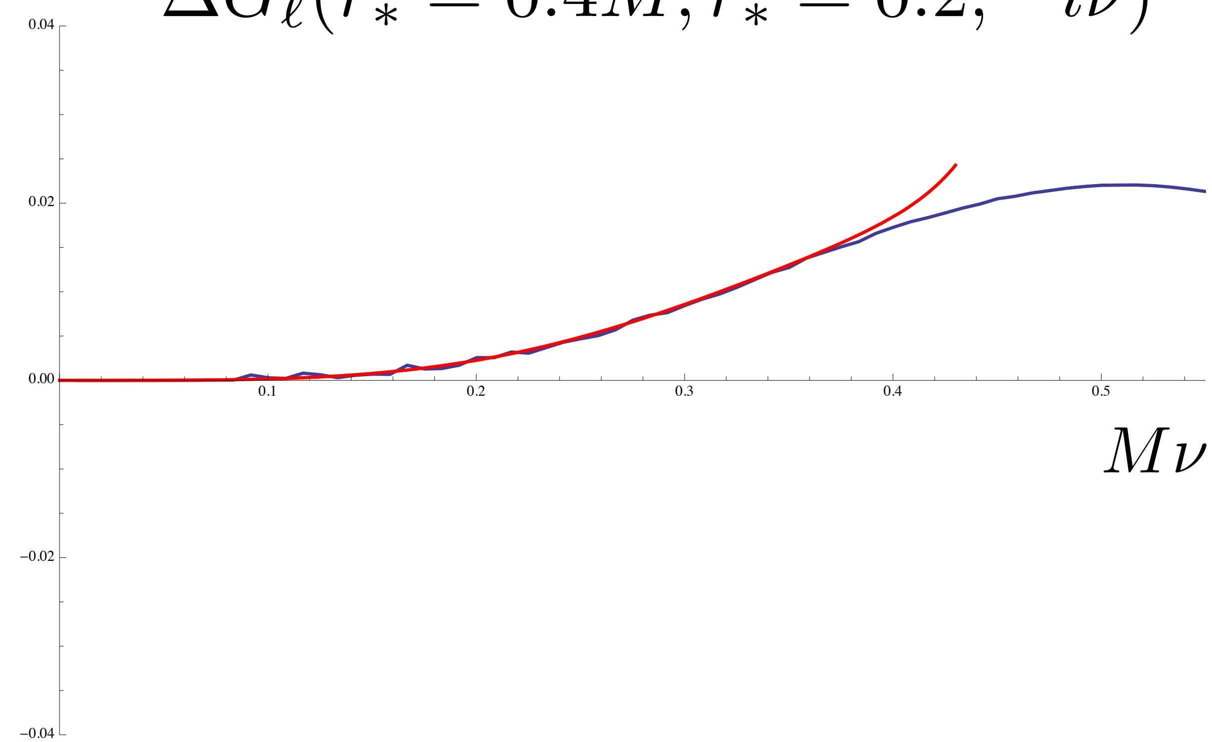
$q(\nu)$



$\mathrm{Re}(W), \mathrm{Im}(W), |W|$

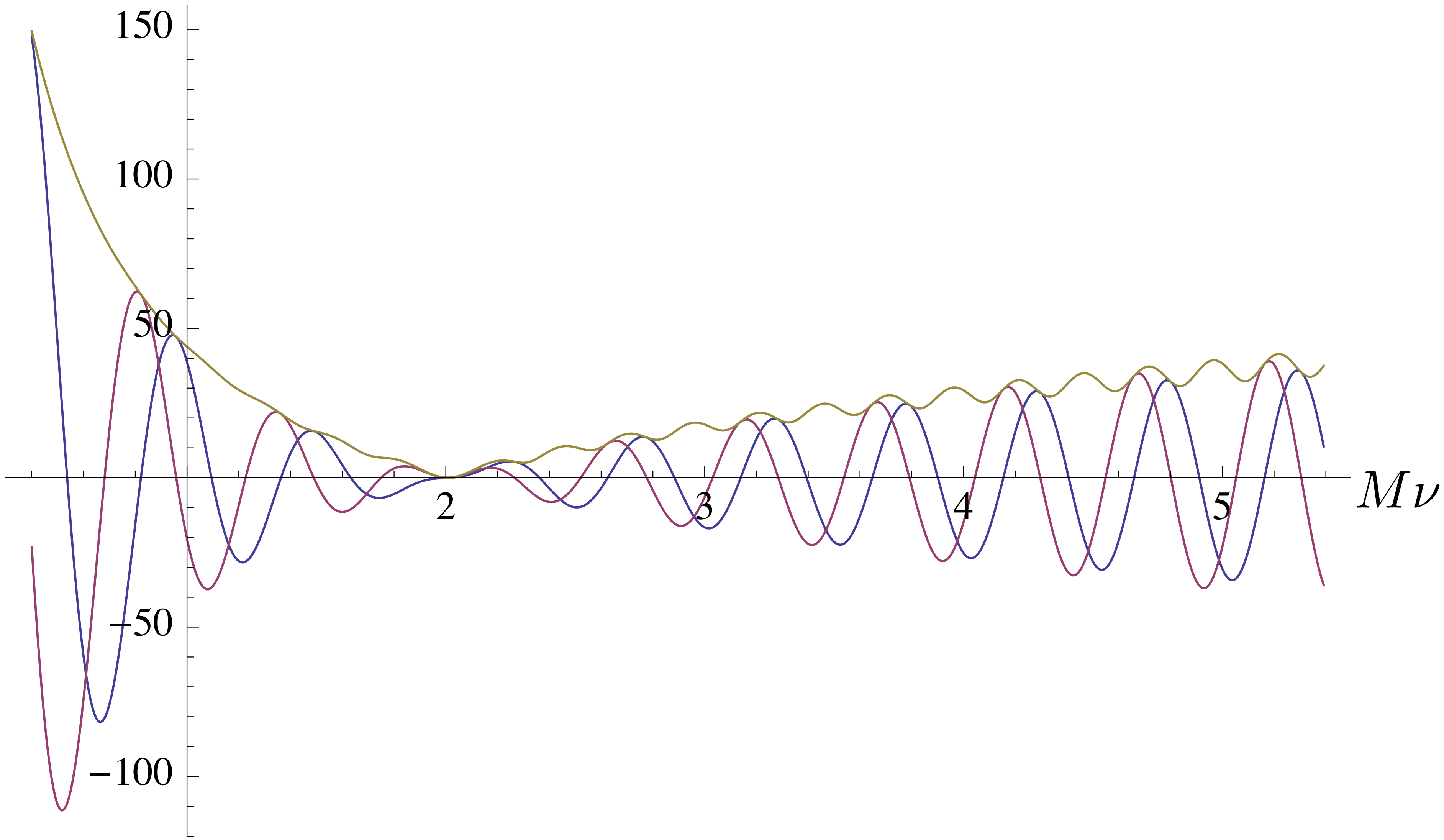


$\Delta G_\ell(r_* = 0.4M, r_* = 0.2; -i\nu)$

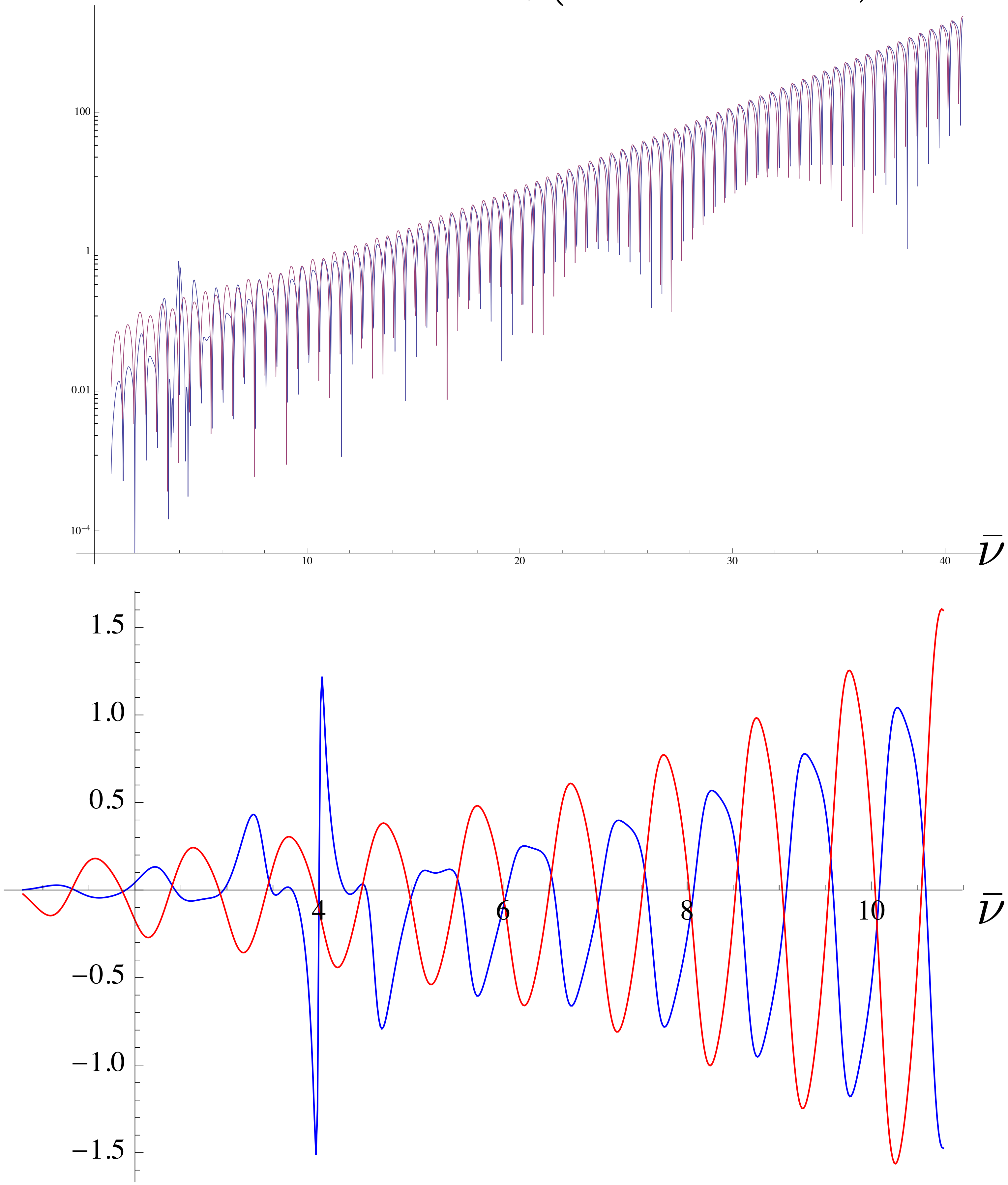


Mid-ν regime

$\mathrm{Re}(W), \mathrm{Im}(W), |W|$



$$\Delta G_\ell(r_* = 0.4M, r_* = 0.2; -i\nu)$$



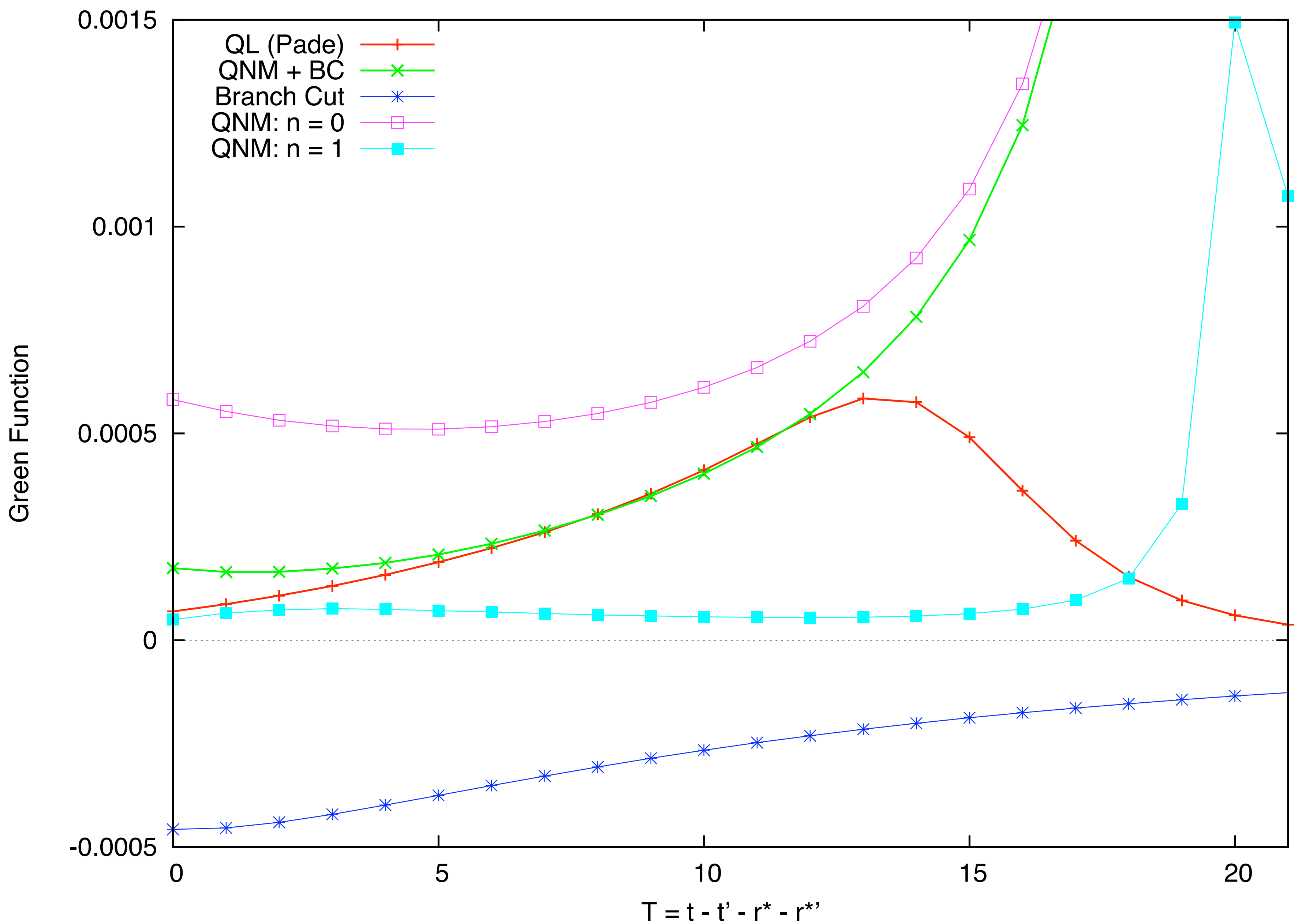
$s = 0, \ell = 1$

$s = 2, \ell = 2$

Matching QNM+BC to QL?

Static scalar charge

$$r = r' = 10M$$



Conclusions

- First method for the BC Green function modes on the NIA

- We can now calculate the full BC Green function exactly

- Next: try to match QNM+BC (‘distant past’) to QuasiLocal part