EOB and GSF

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Analytical Relativity of Relativistic Binary Systems

Aims:

- Analytical description of binary systems: binary black holes (BBH), binary neutron stars (BNS), black-hole-neutron-star (BHNS), ...

- Describe both comparable-mass or extreme-mass-ratio systems

- Describe both the *dynamics and the gravitational radiation (and the effect of gravitational radiation on the dynamics)*

- Currently focussed on *quasi-circular motion*, but this is not a limitation of principle

- Motivations: LIGO/VIRGO/... as well as LISA

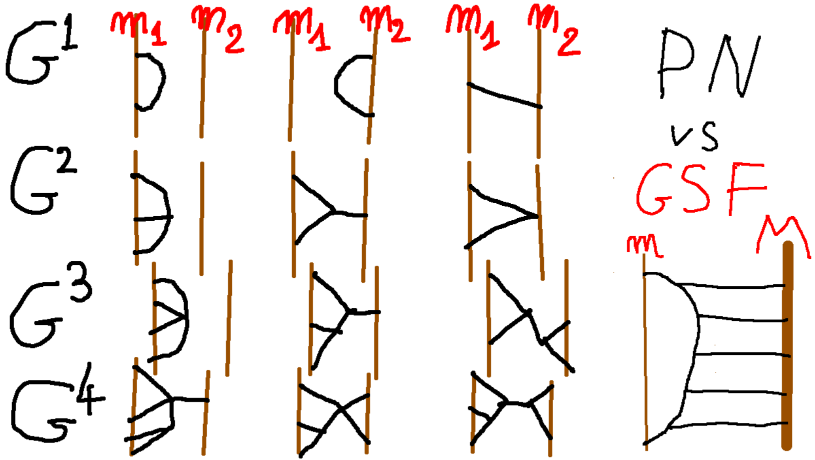
The Problem of Motion in General Relativity: Basic Strategies and Technical Tools

- Post-Minkowskian approach:
  $g_{\mu\nu}(x) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + \cdots + G^n h^{(n)}_{\mu\nu}(x) + \cdots$
- Post-Newtonian approach: $\partial_0 \ll \partial_i$, $GM/rc^2 \sim v^2/c^2 \ll 1$
- Multi-chart approach: body-charts, near-zone-chart, wave-zone-chart
- Matching of Asymptotic Expansions: body zone/near zone; and near-zone/wave zone
- Skeletonized description of strongly self-gravitating bodies
- Arnowitt-Deser-Misner (ADM) Hamiltonian approach
- Multipolar Post-Minkowskian approach in exterior zone (Thorne 80, Blanchet-Damour 86)

- Effective (skeletonized) action for BHs up to 5PN (Damour 82):
  $S = \int d^D x \sqrt{g} R(g)/16\pi G - \sum_A m_A \sqrt{-g_{\mu\nu}(y_A) dy_A^\mu dy_A^\nu}$
- Dimensional continuation (and regularization): $D \equiv d + 1 = 4 + \epsilon$
- Effective action for extended bodies (D+Esposito-Farese 98, Goldberger-Rothstein 06, D+Nagar 09)
  $S = \int d^D x \sqrt{g} R(g)/16\pi G - \sum_A m_A \sqrt{-g_{\mu\nu}(y_A) dy_A^\mu dy_A^\nu} +$
  $\sum_A \frac{1}{4} \mu_A \int ds_A [u_A^\mu u_A^\nu R_{\mu\alpha\nu\beta}(y_A)]^2 + \cdots$
- x-space integrals done using Riesz' formula ($\int d^d x r_1^a r_2^b = C(a,b,d) r_{12}^{a+b+d}$) and its generalizations (e.g. $\int d^d x \, (r_1^{2-d} r_2^{2-d} r_3^{2-d})$), together with local expansions (in dimension d near each "point mass" $x = y_A$).
- Multipole expansion of exterior metric, taking into account hereditary effects (tails, ...)
- Relativistic generalization of the "quadrupole formula" (D+Iyer 91, Blanchet 95, Blanchet+D+Esposito-Farese+Iyer 05):
  $I_L \sim \int d^d x \, [x^L(\tau^{00} + \tau^{ii}) + \cdots]$

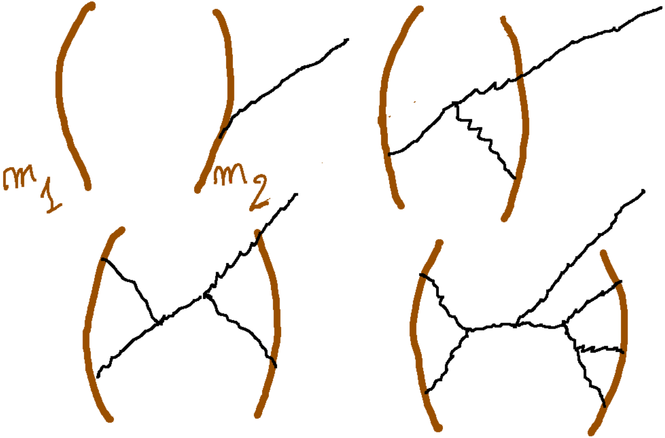
Structure of 3PN dynamics (EOM or H)

(Jaranowski-Schäfer 98, Blanchet-Faye 01,
Damour-Jaranowski-Schäfer 01, Itoh-Futamase 03,
Blanchet-Damour-Esposito-Farese 04, Foffa-Sturani 11)



Structure of 3PN radiation

(Blanchet-Iyer-Joguet 02, BDEFI 04, BF-Iyer-Sinha 08)



2-BODY TAYLOR-EXPANDED 3PN HAMILTONIAN

$$H_{\rm N}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \frac{\mathbf{p}_a^2}{2\,m_a} - \frac{1}{2}\sum_a \sum_{b \neq a} \frac{G\,m_a\,m_b}{r_{ab}}.$$

Newtonian Hamiltonian

$$H_{1\rm PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[-12\,\frac{\mathbf{p}_1^2}{m_1^2} + 14\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right] + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \longleftrightarrow 2),$$

1PN

$$\begin{aligned}
H_{2\rm PN}(\mathbf{x}_a, \mathbf{p}_a) = {} & \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[5\,\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.\\
& \left. -6\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]\\
& +\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2) + 6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\
& -\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2} + (1 \longleftrightarrow 2).
\end{aligned}$$

2PN

$$\begin{aligned}
H_{3\rm PN}^{\rm reg}(\mathbf{x}_a, \mathbf{p}_a) = {} & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left[-14\,\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\,\frac{\big((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\,\mathbf{p}_1^2\,\mathbf{p}_2^2\big)\mathbf{p}_1^2}{m_1^4m_2^2} + \frac{\big(\mathbf{p}_1^2\,\mathbf{p}_2^2 - 2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\right.\\
& -10\,\frac{\big(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\big)\mathbf{p}_1^2}{m_1^4m_2^2} + 24\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} + 2\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
& +\frac{\big(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2 - 10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} + 6\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\\
& \left.+15\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} - 18\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right]\\
& +\frac{G^2m_1m_2}{r_{12}^2}\left[\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2}\right.\\
& +\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} - \frac{1}{8}m_1\frac{\big(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
& -\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\
& \left.-\frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right] + \frac{G^3m_1m_2}{r_{12}^3}\left[-\frac{1}{48}\left(466\,m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1m_2 + 150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
& +\frac{1}{16}\left(77(m_1^2+m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(61\,m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
& \left.+\frac{1}{16}\left(21(m_1^2+m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\
& +\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right] + (1 \longleftrightarrow 2). \qquad (12)
\end{aligned}$$

3PN

2-body Taylor-expanded waveform

TAYLOR-EXPANDED 3PN WAVEFORM

Blanchet, Iyer&Joguet, 02; Blanchet, Damour, Iyer&Esposito-Farese, 04; Kidder07; Blanchet et al.,08

$$\begin{aligned}
h^{22} = -8\sqrt{\frac{\pi}{5}}\frac{G\nu m}{c^2 R}e^{-2i\phi}x\Big\{&1 - x\left(\frac{107}{42} - \frac{55}{42}\nu\right) + x^{3/2}\left[2\pi + 6i\ln\left(\frac{x}{x_0}\right)\right] - x^2\left(\frac{2173}{1512} + \frac{1069}{216}\nu - \frac{2047}{1512}\nu^2\right)\\
&- x^{5/2}\left[\left(\frac{107}{21} - \frac{34}{21}\nu\right)\pi + 24i\nu + \left(\frac{107i}{7} - \frac{34i}{7}\nu\right)\ln\left(\frac{x}{x_0}\right)\right]\\
&+ x^3\Big[\frac{27\,027\,409}{646\,800} - \frac{856}{105}\gamma_E + \frac{2}{3}\pi^2 - \frac{1712}{105}\ln 2 - \frac{428}{105}\ln x\\
&- 18\left[\ln\left(\frac{x}{x_0}\right)\right]^2 - \left(\frac{278\,185}{33\,264} - \frac{41}{96}\pi^2\right)\nu - \frac{20\,261}{2772}\nu^2 + \frac{114\,635}{99\,792}\nu^3 + \frac{428i}{105}\pi + 12i\pi\ln\left(\frac{x}{x_0}\right)\Big] + O(\epsilon^{7/2})\Big\}.
\end{aligned}$$

$$x = (M\Omega)^{2/3} \sim v^2/c^2$$

$$M = m_1 + m_2$$

$$\nu = \frac{m_1 m_2}{M^2}$$

PN-Expanded equations of motion

$a = 1, 2; \; i = 1, 2, 3$

$$\frac{d^2 \, z_a^i}{dt^2} = A_a^i(z_b, v_b) = A_a^{i\,\mathrm{cons}} + A_a^{iRR}$$

$$A^{\mathrm{cons}} = A_0 + c^{-2} \, A_2 + c^{-4} \, A_4 + c^{-6} \, A_6$$

$$A^{RR} = c^{-5} \, A_5 + c^{-7} \, A_7$$

Need to use balance equations to improve A^{RR} to higher fractional PN accuracy: fractional 3.5 PN

$$\Rightarrow \quad A^{RR} = c^{-5}[1 + c^{-2} + c^{-3} + c^{-4} + c^{-5} + c^{-6} + c^{-7}] = c^{-5} + \ldots + c^{-12}$$

Effective-one-body (EOB) approach to the general relativistic two-body problem

(Buonanno+D 99, 00, DJS00, D01, D+Nagar 07, D+Iyer+Nagar 08)

key ideas:

(1) Replace two-body dynamics (m_1, m_2) by dynamics of a particle $(\mu \equiv m_1\, m_2/(m_1 + m_2))$ in an effective metric $g_{\mu\nu}^{\rm eff}(u)$, with

$$u \equiv GM/c^2 R\,, \quad M \equiv m_1 + m_2$$

(2) Systematically use RESUMMATION of PN expressions (both $g_{\mu\nu}^{\rm eff}$ and $\mathcal{F}_{RR}$) based on various physical requirements

(3) Require continuous deformation w.r.t. $\nu \equiv \mu/M \equiv m_1\, m_2/(m_1 + m_2)^2$ in the interval $0 \leq \nu \leq \frac{1}{4}$

Two-body/EOB "correspondence"

Mapping real 2-body (m_1, m_2) c.o.m dynamics $\rightarrow$ "effective" dynamics of one body, of mass $\mu \equiv m_1\, m_2 / M$ following a (Finsler-generalized) geodesic in the "external" (spherically symmetric) metric

$$g^{\text{ext}}_{\mu\nu}\, dx^\mu\, dx^\nu = -A(R)\, c^2\, dT^2 + B(R)\, dR^2 + R^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

with

$$\begin{aligned} A(R) &= 1 + a_1 \frac{GM}{c^2 R} + a_2 \left(\frac{GM}{c^2 R}\right)^2 + a_3 \left(\frac{GM}{c^2 R}\right)^3 + \cdots ; \\ B(R) &= 1 + b_1 \frac{GM}{c^2 R} + b_2 \left(\frac{GM}{c^2 R}\right)^2 + \cdots \end{aligned}$$

"Dictionary" between $H_{2\text{-body}}(q, p)$ and EOB Hamiltonian-Jacobi:

$$0 = \mu^2 + g^{\mu\nu}_{\text{eff}}(x)\, p_\mu\, p_\nu + Q(p)$$

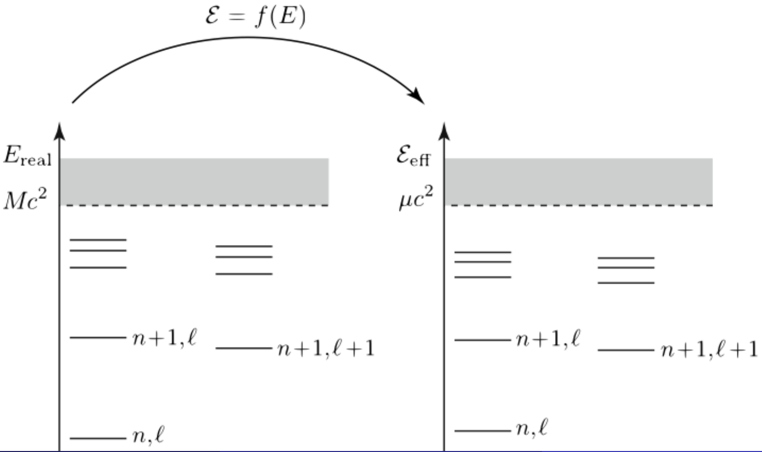
where $p_\mu = \partial S(x) / \partial x^\mu$ and

$$Q(p) = A^{\mu\nu\rho\sigma}(x)\, p_\mu\, p_\nu\, p_\rho\, p_\sigma + \ldots$$

Dictionary: Think quantum-mechanically (J.A. Wheeler)

Sommerfeld's Old Quantum Mechanics:

$$I_\varphi = \ell\, h = \oint p_\varphi \, d\varphi \,, \quad N = nh = I_\varphi + \oint p_r \, dr$$



The EOB energy map $(c = 1)$

$$\mathcal{E}_{\rm eff} = \frac{E_{\rm real}^2 - m_1^2 - m_2^2}{2(m_1 + m_2)} = \frac{s - m_1^2 - m_2^2}{2M}$$

EOB Hamiltonian

$$H_{\rm EOB} = M \sqrt{1 + 2\nu(\widehat{H}_{\rm eff} - 1)}$$

where

$$M \equiv m_1 + m_2\,, \quad \mu \equiv \frac{m_1\, m_2}{M}\,, \quad \nu \equiv \frac{\mu}{M}\,, \quad \widehat{H}_{\rm eff} = \frac{H_{\rm eff}}{\mu}$$

Explicit form of the EOB effective Hamiltonian

$$ds^2_{\text{eff}} = g^{\text{eff}}_{\mu\nu}(x)\,dx^\mu dx^\nu = -A(r;\nu)\,dt^2 + B(r,\nu)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

in terms of $P_{R_*} = \left(\frac{A}{B}\right)^{1/2} P_R$ and rescaled variables $r \equiv R/GM$, $p_{r_*} = P_{R_*}/\mu$, $p_\varphi = P_\varphi/\mu$

$$\widehat{H}_{\text{eff}} = \sqrt{p^2_{r_*} + A(r;\nu)\left(1 + \frac{p^2_\varphi}{r^2} + Q(r, p_{r^*})\right)}$$

where, with $u \equiv GM/c^2 R \equiv 1/r$, $\widehat{Q}(r, p_{r_*}) = 2(4-3\nu)\,\nu\, u^2\, p^4_{r_*} + \dots$

$$A(u;\nu) = 1 - 2u + 2\nu\, u^3 + \nu\, a_4\, u^4 + \mathcal{O}(u^5)$$

$$(A(r)\,B(r))^{-1} \equiv \overline{D}(u;\nu) = \overline{D}(u;\nu) = 1 + 6\nu\, u^2 + 2(26-3\nu)\,\nu\, u^3 + \mathcal{O}(u^4)$$

where

$$a_4 = \frac{94}{3} - \frac{41\,\pi^2}{32} \simeq 18.6879027 \text{ crucial 3PN information}$$

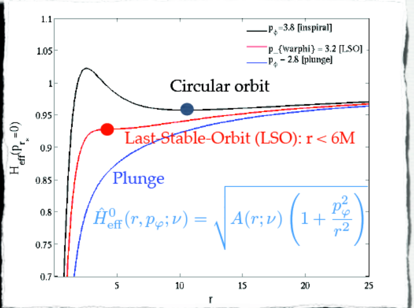
HAMILTON'S EQUATIONS & RADIATION REACTION

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi},$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi .$$



- The system must radiate angular momentum
- How?Use PN-based (Taylor-expanded) radiation reaction force (ang-mom flux)
- Need flux resummation

$$\hat{\mathcal{F}}_\varphi^{\rm Taylor} = -\frac{32}{5}\nu\Omega^5 r_\Omega^4 \hat{F}^{\rm Taylor}(v_\varphi)$$

NEW: resummation multipole by multipole (parameter free)

OLD: Padé resummation (parameter dependent)

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WAVEFORM RESUMMATION PROCEDURE

- Resummation of the waveform multipole by multipole
- Factorized (multipolar) waveform at highest available PN order (from Blanchet et al.)

Next-to-quasi-circular correction

$$h_{\ell m} = \underbrace{h_{\ell m}^{(N,\epsilon)}}\,\underbrace{\hat{h}_{\ell m}^{(\epsilon)}}_{\text{PN-correction}}\,\underbrace{f_{\ell m}^{NQC}}$$

Newtonian x PN x NQC

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\text{eff}}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

Remnant phase and modulus corrections: "improved" PN series

The "Tail factor"

Effective source:
EOB (effective) energy (even-parity modes)
EOB angular momentum (odd-parity modes)

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell + 1)} e^{\pi \hat{k}} e^{2i\hat{\hat{k}} \ln(2kr_0)}$$

Resums an infinite number of leading logarithms in tail effects

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RESUMMING RADIATION REACTION

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\,\Omega}\sum_{\ell=2}^{\ell_{\max}}\sum_{m=1}^{\ell}(m\,\Omega)^2\,|R\,h_{\ell m}^{(\epsilon)}|^2$$

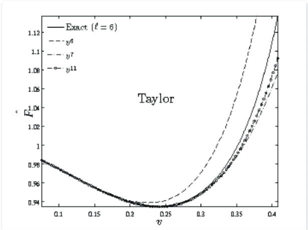
Residual amplitude correction:

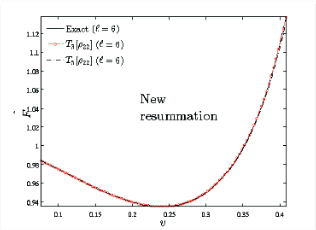
$$\begin{aligned}\rho_{22}(x;\nu) = 1 &+ \left(\frac{55\nu}{84}-\frac{43}{42}\right)x + \left(\frac{19583\nu^2}{42336}-\frac{33025\nu}{21168}-\frac{20555}{10584}\right)x^2\\ &+\left(\frac{10620745\nu^3}{39118464}-\frac{6292061\nu^2}{3259872}+\frac{41\pi^2\nu}{192}-\frac{48993925\nu}{9779616}-\frac{428}{105}\mathrm{eulerlog}_2(x)+\frac{1556919113}{122245200}\right)x^3\\ &+\left(\frac{9202}{2205}\mathrm{eulerlog}_2(x)-\frac{387216563023}{160190110080}\right)x^4+\left(\frac{439877}{55566}\mathrm{eulerlog}_2(x)-\frac{16094530514677}{533967033600}\right)x^5+\mathcal{O}(x^6),\end{aligned}$$

EFFECTIVENESS OF FLUX RESUMMATION

Test-mass

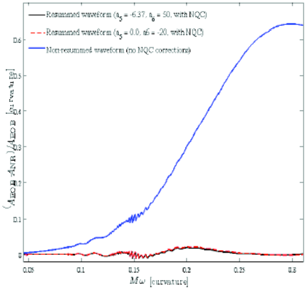
Comparing fluxes, circular orbits)



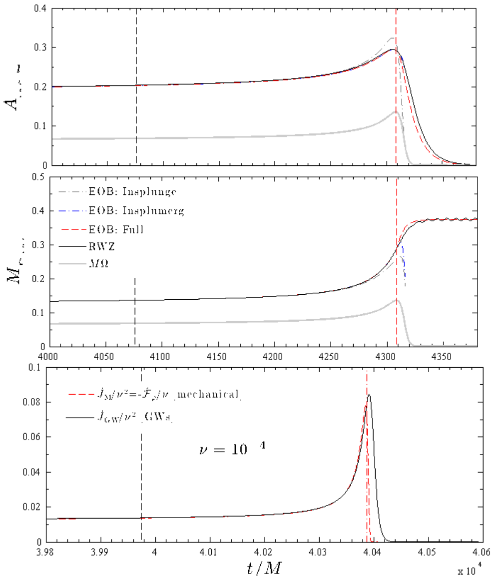


Equal-mass

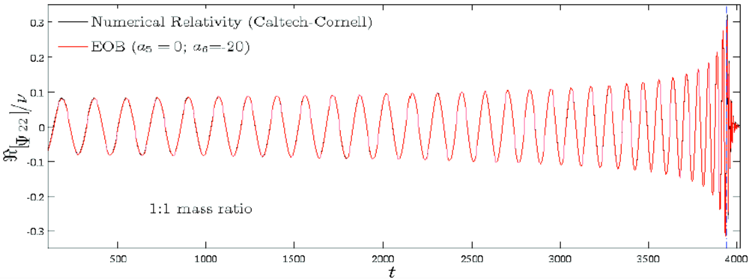
(Comparing non-resummed & EOB-resummed *amplitudes* to Caltech-Cornell BBH data)



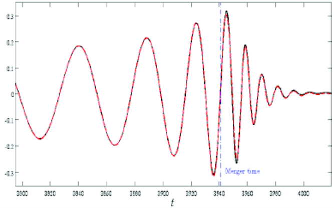
EMRI limit (Bernuzzi-Nagar-Zenginoglu)



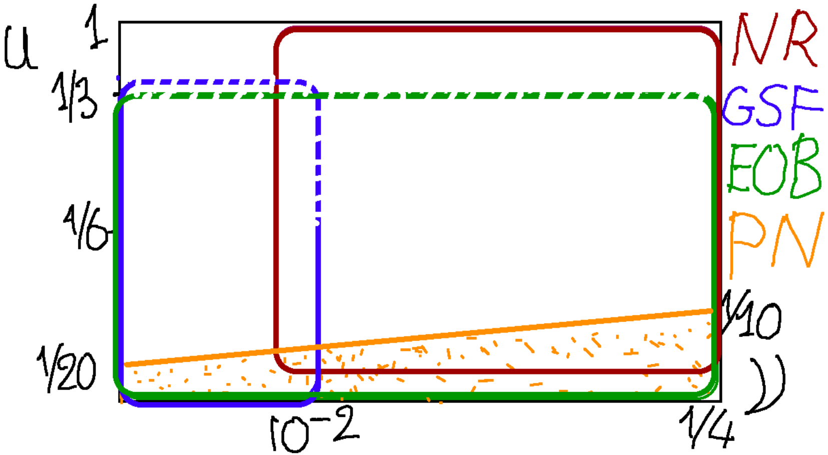
COMPARISON EOB VS NR WAVEFORMS



- Two unknown EOB parameters
- NR calibration of the maximum GW amplitude
- Need to tune only 1 parameter
- Banana-like “best-region” in the plane (a_5, a_6) extending from (0, -20) to (-36, 520) where EOB/NR phase difference < 0.02 rad
- Consistency with 2:1 mass ratio case (not shown)



On the EOB/GSF synergy



Uncertainties in the knowledge of the A function

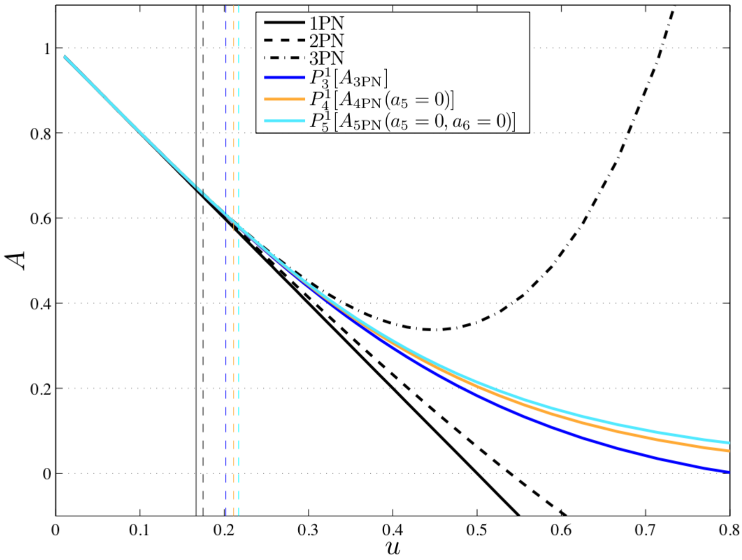
In the absence of information, use Padé resummation to go beyond PN knowledge

$$\begin{aligned} A_3^1(u,\nu) \equiv P_3^1[A_{\rm 3PN}] &= P_3^1[1 - 2u + 2\nu u^3 + a_4 \nu u^4] \\ &= \frac{1 + n_1(\nu) u}{1 + d_1(\nu) u + d_2(\nu) u^2 + d_3(\nu) u^3} \end{aligned}$$

Improved parametrization: $4\mathrm{PN}(a_5) + 5\mathrm{PN}(a_6)$, tuned to comparable-mass NR data

$$A_5^1(u,\nu) \equiv P_5^1[A_{\rm 3PN} + a_5\, \nu\, u^5 + a_6\, \nu\, u^6]$$

Uncertainties in the knowledge of the A function



EOB information contained in $\delta^{\mathrm{GSF}}\,\Omega_{\mathrm{LSO}}$ (Barack-Sago 09)

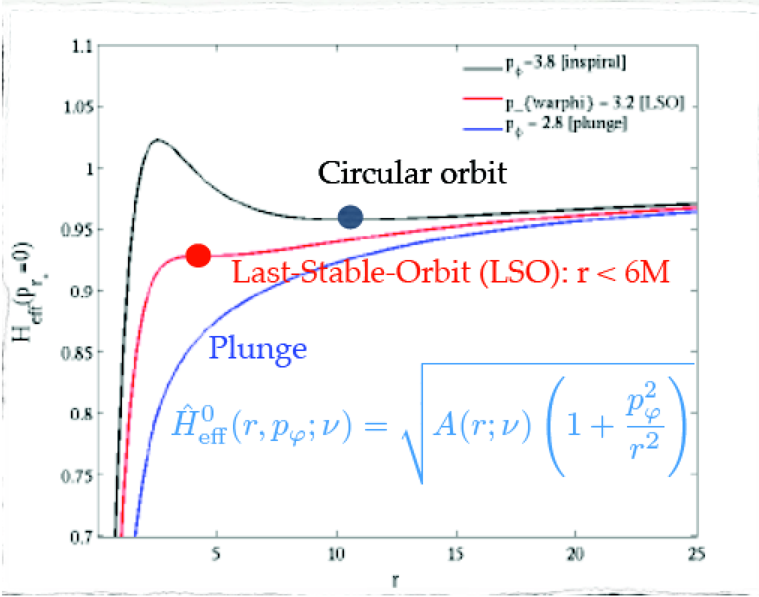
Most useful:

$$
\begin{aligned}
G(m_1+m_2)\,\Omega^{\infty}_{\mathrm{LSO}} &= 6^{-3/2}\left[1+\nu\left(c^{\mathrm{BS}}_{\Omega}+1-\frac{1}{\sqrt{18}}\right)+\mathcal{O}(\nu^2)\right] \\
&= 6^{-3/2}[1+1.25120\,\nu+\mathcal{O}(\nu^2)]
\end{aligned}
$$

or

$$
x_{\mathrm{LSO}} \equiv (\mathrm{GM}\,\Omega^{\infty}_{\mathrm{LSO}})^{2/3} = \frac{1}{6}\,[1+0.83413\,\nu+\mathcal{O}(\nu^2)]
$$

EOB theory of LSO



EOB theory of LSO

Dependence on $\delta_{\mathrm{GSF}}\, x_{\mathrm{LSO}}$ on EOB A potential.
Consider the 1GSF expansion of the function $A(u,\nu)$:

$$A(u,\nu) = 1 - 2u + \nu\, a(u) + \mathcal{O}(\nu^2)$$

$$x_{\mathrm{LSO}} = \frac{1}{6}\left[1 + c_x^{\mathrm{EOB}}\,\nu + \mathcal{O}(\nu^2)\right]$$

with (Damour 09)

$$c_x^{\mathrm{EOB}} = \frac{2}{3}\left(1 - \sqrt{\frac{8}{9}}\right) + a\left(\frac{1}{6}\right) + \frac{1}{6}\, a'\left(\frac{1}{6}\right) + \frac{1}{18}\, a''\left(\frac{1}{6}\right)$$

PN expansion (here without logs)

$$a(u) = \sum_{n\geq 3} a_n\, u^n = a_3\, u^3 + a_4\, u^4 + a_5\, u^5 + a_6\, u^6 + a_7\, u^7 + \dots$$

Known 3PN terms:

$$c_x^{\rm 3PN} = 0.038127 + 0.148148 + 0.418171 = 0.604446$$

which represents 72.5% of the GSF result $c_x^{\rm BS} = 0.83413$

$\Rightarrow$ information about $4{\rm PN} + 5{\rm PN} + \dots$ terms in $a(u)$:

$$a_5 + 0.242754\, a_6 \simeq 38.84(7)$$

can formally complement the knowledge acquired from EOB/NR comparison

$\Rightarrow$

$$a_5^{\cap} \simeq -22.3\,, \qquad a_6^{\cap} \simeq +252$$

EOB/GSF synergy from small-eccentricity orbits

EOB theory of periastron advance (Damour 09)

$$W(x) \equiv \left(\frac{\Omega_r}{\Omega_\varphi}\right)^2 = 1 - 6x + \nu\,\rho(x) + \mathcal{O}(\nu^2)$$

$$\rho(x) \equiv \rho_E(x) + \rho_a(x) + \rho_d(x)$$

$$\rho_E(x) = 4x\left(1 - \frac{1-2x}{\sqrt{1-3x}}\right)$$

$$\rho_a(x) = a(x) + x\,a'(x) + \frac{1}{2}\,x(1-2x)\,a''(x)$$

$$\rho_d(x) = (1-6x)\,\bar{d}(x)$$

where $\bar{d}(x)$ is the coefficient of the 1GSF expansion of the second EOB potential:

$$\overline{D}(u,\nu) = 1 + \nu\,\overline{d}(u) + \mathcal{O}(\nu^2)$$

3PN expansion of the function $\rho(x)$

$$\rho^{\mathrm{3PN}}(x) = \rho_2\, x^2 + \rho_3\, x^3 \ ,$$

with

$$\begin{aligned} \rho_2 &= 14\,, \\ \rho_3 &= \frac{397}{2} - \frac{123}{16}\,\pi^2 = 122.627416 \end{aligned}$$

Numerically

$$c_x[\rho^{\mathrm{3PN}}] = \frac{\rho_2}{6^2} + \frac{\rho_3}{6^3} = 0.956608$$

now larger than c_x^{BS} by 14.67%

Example of unreliability of using (non resummed) PN expansions for estimating physical quantities in the strong-field regime (here the LSO)

Precession effect of GSF and the EOB formalism

$$W(x) \equiv \left(\frac{\Omega_r}{\Omega_\varphi}\right)^2 = 1 - 6x + \nu\,\rho(x) + \mathcal{O}(\nu^2)$$

GSF result for $\rho(x)$

$$\rho(x) = f_{r0}(x)\tilde{F}^r_{\rm circ} + f_{r1}(x)\tilde{F}^r_1 + f_{\varphi 1}(x)\tilde{F}_{\varphi 1} + f_{(\alpha)}(x)$$

with $\tilde{F}^r_{\rm circ} \equiv \nu^{-2} F^r_{\rm circ}$, etc.

$$\begin{aligned} F^r &= F^r_{\rm circ} + e F^r_1 \cos\omega_r\tau, \\ F_t &= e\omega_r F_{t1} \sin\omega_r\tau, \\ F_\varphi &= e\omega_r F_{\varphi 1} \sin\omega_r\tau \end{aligned}$$

and $f_{r0}(x) = -\frac{2(1-3x)(1-x)}{x^2(1-2x)}$, etc.

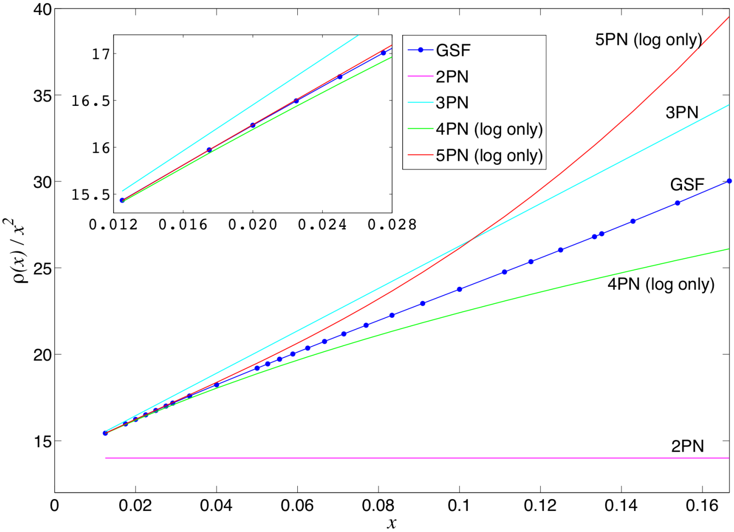
Numerical GSF data for $\rho(x)/x^2$ compared to various EOB/PN approximations

PN expansion of $\rho(x)$ [with logarithms (Damour; Blanchet-Detweiler-LeTiec-Whiting; Barack-Damour-Sago)]

$$\rho^{\mathrm{PN}}(x) = \rho_2 x^2 + \rho_3 x^3 + (\rho_4^{\mathrm{c}} + \rho_4^{\mathrm{log}} \ln x) x^4 + (\rho_5^{\mathrm{c}} + \rho_5^{\mathrm{log}} \ln x) x^5 + O(x^{6+0})$$

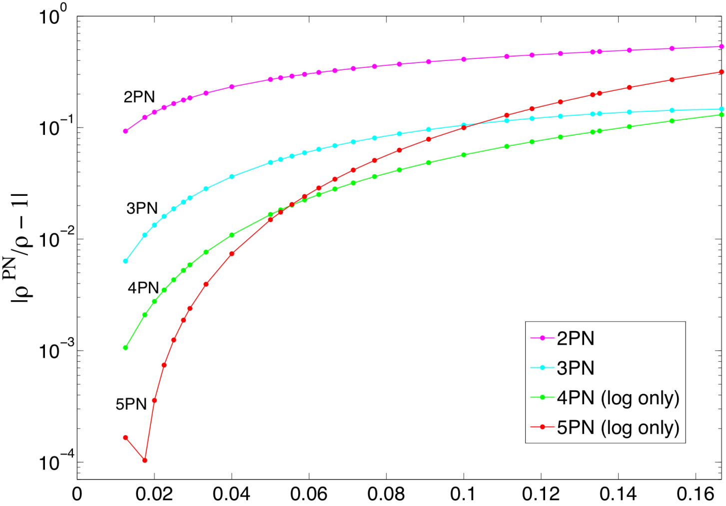
where $\rho_4^{\mathrm{log}} \simeq 167.466\ldots$ and $\rho_5^{\mathrm{log}} \simeq -1619.428\ldots$ are known, but ρ_4^c, ρ_5^c are not

Numerical GSF data for $\rho(x)/x^2$ compared to various EOB/PN approximations



How to make the best use of GSF data

Try to extract unknown higher PN coefficients from GSF data:
? strength of successive PN signals in $\rho(x)$



Quantitative test of GSF data against known EOB/PN terms 2PN coefficient $\rho_2 = 14$

fit model	fixed params.	ρ_2 (best fit)	χ^2/DoF	L^∞-norm
$\rho^{2\mathrm{PN}}$	none	21.5941	6.8×10^7	2.3×10^{-1}
$\rho^{3\mathrm{PN}}$	none	14.5748	5810	3.6×10^{-3}
$\rho^{4\mathrm{PN}}$	none	14.5135	5264	4.8×10^{-3}
$\rho^{4\mathrm{PN}+}$	none	13.9665	29.4	6.0×10^{-4}
$\rho^{5\mathrm{PN}}$	none	14.0544	4.08	2.0×10^{-4}
$\rho^{5\mathrm{PN}+}$	none	13.9721	0.74	4.5×10^{-5}
$\rho^{6\mathrm{PN}}$	none	14.0106	0.59	1.6×10^{-5}
$\rho^{6\mathrm{PN}+}$	none	13.9619	0.58	1.7×10^{-5}
$\rho^{7\mathrm{PN}}$	none	13.9527	0.61	1.7×10^{-5}
$\rho^{7\mathrm{PN}}$	ρ_3	13.9946	0.58	1.7×10^{-5}
$\rho^{7\mathrm{PN}}$	$\rho_3,\ \rho_4^{\log}$	14.0015	0.56	1.6×10^{-5}
$\rho^{7\mathrm{PN}}$	$\rho_3,\ \rho_4^{\log},\ \rho_5^{\log}$	14.00002	0.55	1.6×10^{-5}

Quantitative test of GSF data against the known EOB 3PN coefficient $\rho_3 = 122.6274$

fit model	fixed params.	ρ_3 (best fit)	χ^2/DoF	L^∞-norm
ρ^{3PN}	ρ_2	97.953	3.7×10^5	8.2×10^{-3}
ρ^{4PN}	ρ_2	106.936	4.9×10^4	1.2×10^{-2}
ρ^{4PN+}	ρ_2	122.458	20.5	4.4×10^{-4}
ρ^{5PN}	ρ_2	120.962	12.0	3.6×10^{-4}
ρ^{5PN+}	ρ_2	124.365	1.04	7.8×10^{-5}
ρ^{6PN}	ρ_2	122.256	0.57	1.6×10^{-5}
ρ^{6PN+}	ρ_2	123.758	0.57	1.6×10^{-5}
ρ^{7PN}	ρ_2	120.914	0.58	1.7×10^{-5}
ρ^{7PN}	$\rho_2,\ \rho_4^{\log}$	122.929	0.56	1.7×10^{-5}
ρ^{7PN}	$\rho_2,\ \rho_4^{\log},\ \rho_5^{\log}$	122.623	0.55	1.6×10^{-5}

$+$ Similar tests of the analytically determined $\rho_4^{\log}$ and $\rho_5^{\log}$

Determination of unknown EOB/PN parameters

Strategy: Fixing all known parameters $(\rho_2, \rho_3, \rho_4^{\log}, \rho_5^{\log})$ at their analytical values, one fits GSF data to some PN models which include a variety of the a priori most significant unknown higher-order parameters

Result:

$$\rho_4^c = 69^{+7}_{-4}, \qquad \rho_5^c = -4800^{+400}_{-1200}, \quad \rho_6^{\log} < 0$$

Implications for EOB theory

$$\begin{aligned} 10 a_5^c + \bar{d}_4^c + \frac{9}{2} a_5^{\log} &\simeq 518.6^{+7}_{-4}, \\ 14 a_5^c + 6 \bar{d}_4^c - 15 a_6^c - \bar{d}_5^c + 8 a_5^{\log} - \frac{11}{2} a_6^{\log} &\simeq 4779^{-400}_{+1200} \end{aligned}$$

First constraints on higher PN parameters of direct physical significance

Determination of the global strong-field behaviour of an EOB function

PN theory gives information only about the $x \to 0$ behaviour of some $f(x)$; EOB theory aims at extending into the strong-field region $x = \mathcal{O}(1)$ the PN knowledge of some $f(x)$: $A(x;\nu), B(x;\nu), \ldots$ However, EOB does this by trying some resummation methods (e.g. Padé $[A^{\rm PN}]$), and imposing some general requirements, but needs the help of strong-field data to improve or calibrate its resummed functions. E.g. NR data for constraining $a_5(\nu)$, $a_6(\nu)$ in Padé $[A^{\rm 3PN} + a_5(\nu)\, u^5 + a_6(\nu)\, u^6]$ (D+Nagar, Buonanno, Pan et al., ...)

Determination of the global strong-field behaviour of an EOB function

GSF data on small-eccentricity precession → first-ever determination of a combination of (ν-derivative of) EOB functions in the strong-field regime

$$\rho(x) = \rho_E(x) + \rho_a(x) + \rho_d(x)$$

$$\begin{aligned}
\rho_E(x) &= 4x\left(1 - \frac{1-2x}{\sqrt{1-3x}}\right), \\
\rho_a(x) &= a(x) + xa'(x) + \frac{1}{2}x(1-2x)a''(x), \\
\rho_d(x) &= (1-6x)\bar{d}(x)
\end{aligned}$$

where $a(x)$ and $\bar{d}(x)$ enter the 1GSF expansions:
$A(x,\nu) = 1 - 2u + \nu a(x) + \mathcal{O}(\nu^2)$, $\bar{D}(x,\nu) = 1 + \nu\bar{d}(x) + \mathcal{O}(\nu^2)$.

An efficient strategy for GSF/EOB synergy

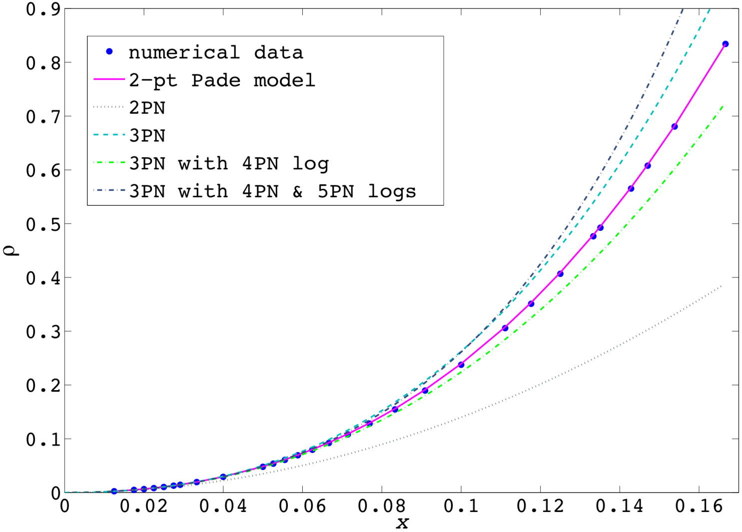
EOB resummation is using one-point Padé approximants of various fundamental functions.

Proposed new strategy (BDS10) for defining sufficiently accurate global representations of the strong-field behaviour of dynamically relevant functions based on combining a minimal amount of strong-field information with the available PN information: to use multiple-point Padé approximants constrained both by PN data at $x_1 = 0$ and GSF (or NR) data at some strong-field points, $x_2, x_3, \ldots$

Application to $\rho(x)$: combining PN knowledge of $\rho(x) = \rho_2\, x^2 + \rho_3\, x^3 + \mathcal{O}(x^4)$ with, e.g., GSF-computed values of $\rho(1/6)$ and $\rho'(1/6) \Rightarrow 4$ pieces of data allowing one to uniquely determine a 4-parameter Padé model:

$$\rho(x) = c_0 x^2 \left(\frac{1 + c_1 x}{1 + d_1 x + d_2 x^2} \right)$$

Two-point Padé model of $\rho(x)$ based on PN information at $x = 0$ and GSF data at $x = 1/6$



Possible future EOB/GSF synergies

- Study $\rho(x)$ for unstable circular orbits: $\frac{1}{6} < x < \frac{1}{3}$
- Unbound orbits: renormalize E_1^∞ and J_1^∞ into global $\mathcal{E}$ and $\mathcal{J}$ and compare the scattering angle $\theta(\mathcal{E}, \mathcal{J})$ to its EOB prediction
- Special zero-binding zoom-whirl motion $\Rightarrow a\left(\frac{1}{4}\right)$ and $a'\left(\frac{1}{4}\right)$ [then usable for the two-point Padé strategy]
- Compute conserved E_1, J_1 for bound orbits, extrapolate them to global $\mathcal{E}, \mathcal{J}$, and compare the *two* gauge-invariant functions of *two* gauge-invariant variables $\Omega_r = \Omega_r(\mathcal{E}, \mathcal{J})$, $\Omega_\varphi = \Omega_\varphi(\mathcal{E}, \mathcal{J})$ to EOB predictions
- In principle $\delta^{\rm GSF}$ of the BS singular curve in $(\Omega_r, \Omega_\varphi)$ plane can inform EOB

Possible future EOB/GSF synergies

- On the other hand, the gauge-invariant Detweiler redshift function $u^t(\Omega_\varphi^{\rm circ})$ (and its generalization $\langle u^t\rangle(\Omega_r, \Omega_\varphi)$ BS11) are not simply related to the dynamical EOB functions
- Go from 1GSF to 2GSF level $\Rightarrow$ compute $\mathcal{E} - M = e_1\,\nu + e_2\,\nu^2 + \mathcal{O}(\nu^3)$, $\mathcal{J} = j_1\,\nu + j_2\,\nu^2 + \mathcal{O}(\nu^3)$
- Spin-dependent effects: compute the strong-field behaviour of the $\mathcal{O}(\nu)$ terms in the two gyro-gravitomagnetic ratios $g_S^{\rm eff}$, $g_{S^*}^{\rm eff}$ (DJS08) and combine this information with the PN knowledge at NNLO (DJS08, Hartung-Steinhoff11, Nagar11)
- For non-conservative force: 2GSF $\Rightarrow$ $\mathcal{O}(\nu)$ fractional corrections to radiation reaction and waveform?

Conclusions

- There are many prospects for a fruitful synergy between EOB and GSF frameworks
- The first examples of this synergy have already shown how GSF can bring crucial strong-field data that can inform EOB:
- BS09 on $\delta^{\rm GSF}\Omega_{\rm LSO}$:

$\Rightarrow$ $a\left(\frac{1}{6}\right) + \frac{1}{6}\, a'\left(\frac{1}{6}\right) + \frac{1}{18}\, a''\left(\frac{1}{6}\right) \simeq 0.796$

$\Rightarrow$ help to complement NR data in determining $A^{\rm EOB}(u,\nu)$

- BDS10 on $\rho(x)$:

$\Rightarrow$ quantitative confirmation of PN terms (including 4PN and 5PN logs)

$\Rightarrow$ numerical determination of yet uncalculated PN terms: non-log terms at 4PN and 5PN

$\Rightarrow$ first determination of the (medium-)strong-field behaviour of a combination of the 1GSF coefficients in the ν expansion of EOB functions: $\rho(x) \sim \overline{d}(x)$ & $a(x)$ & $a'(x)$ & $a''(x)$

$\Rightarrow$ introduction of a new strategy for combining PN and a small sample of GSF data: multiple-point, GSF-informed Padé approximants

- LMBBPST11 recently considered an interesting multiple comparison: NR/PN/EOB/GSF. This comparison has confirmed two of the basic EOB tenets: (i) effectiveness of EOB resummation; and (ii) usefulness of taking $\nu = m_1 m_2/(m_1 + m_2)^2$ as continuous deformation parameter with $0 < \nu \leq \frac{1}{4}$
- The general PN-EOB fact that the terms of order $\mathcal{O}(\nu^2)$ (2GSF), or higher, in the basic EOB functions are subdominant makes 1GSF data useful for improving EOB. However, $\mathcal{O}(\nu^2)$ effects are known to be important in some strong-field quantities: e.g. $\widehat{\Omega}_{\rm LSO}(\nu) \simeq 6^{-3/2}[1 + 1.25\nu + 1.87\nu^2]$