Self-force calculations with the m-mode scheme

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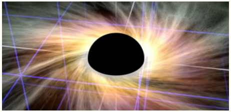

Talk overview

1. Basics of m-mode regularization
   - Motivation
   - Puncture scheme: local to global
   - m-modes & 2+1D evolution & worldtube
   - Mode sum convergence
2. Scalar-field SF on Kerr spacetime
   - Equatorial circular orbits
   - Finite difference scheme
   - Results
3. Gravitational SF in Lorenz gauge
   - Formalism + first (partial) implementation on Schw.
   - Problems: $m = 0$ and $m = 1$ modes
   - Gauge modes and conservation laws
4. Prospects

Motivation: Why calculate SF on Kerr?

(and why use the m-mode scheme?)

1. to compare key gauge-invariants with PN and NR:
   - ISCO shift
   - periastron advance
2. to 'solve' the two-body problem in GR:
   - Calibration for EOB theory
   - SF + symmetric mass-ratio $\Rightarrow$ IMRI models?
3. to understand orbital resonances!
4. for self-consistent evolutions (+ to check osculating approx.)
5. to model EMRIs for LISA.

The m-mode scheme

- Schw. $\Rightarrow$ l-mode regularization $\Rightarrow$ easy!
  - decompose $\bar{h}_{ab}$ in tensor spherical harmonics $Y_{ab}^{lm(i)}$
  - use Lorenz gauge $\nabla^a \bar{h}_{ab} = 0$ with gauge constraint damping
  - solve 1+1D or ODEs in freq. domain
- Kerr $\Rightarrow$ hard choices ...
  - spin-weighted spheroidal harmonics (Teukolsky) ...
    Radiation gauge $\rightarrow$ Lorenz gauge?
    $l = 0, 1$ modes?
  - tensor spheroidal harmonics ... [don't exist?]
  - Full 3+1D approach ... expensive!
  - m-mode + 2+1D evolution ... practical compromise.

$$\Phi_{\mathcal{R}} = \sum_{m=-\infty}^{\infty} \Phi_{\mathcal{R}}^m e^{im\varphi}, \quad F_r^m = q\partial_r \Phi_{\mathcal{R}}^m, \quad F_r = \sum_{m=-\infty}^{\infty} F_r^m$$

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Puncture scheme

[Barack/Golbourn/Sago … Detweiler/Vega … Wardell]

- Local approximation Φ_P for Detweiler-Whiting S field Φ_S
- Covariant expansion → power series approximation in
  - coordinate difference $\delta x^a = x^a - \bar{x}^a$, where
  - field point x, worldline point $\bar{x}$
- Classification: nth order expansion iff

$$\Phi_P^{[n]} - \Phi_S \sim \mathcal{O}(|\delta x| \delta x^{n-2})$$

- 2nd, 3rd, & 4th order expansions now available.
- From local expansion $\Phi_P^{[n]}$ to global puncture field $\Phi_{\mathcal{P}}^{[n]}$:
  - Let $\bar{x}$ become a function of x
  - e.g. set same BL time coordinate, $\bar{t} = t$
  - Periodic continuation:
    e.g. $\delta\varphi^2 \to 2(1 - \cos\delta\varphi) = \delta\varphi^2 + \mathcal{O}(\varphi^2)$

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Residual field + modal decomposition

- Introduce residual field: $\Phi_{\mathcal{R}}^{[n]} = \Phi - \Phi_{\mathcal{P}}^{[n]}$
- Residual field obeys wave equation,

$$\Box \Phi_{\mathcal{R}} = S_{\text{eff}}$$

with effective source $S_{\text{eff}} = \int_\gamma \delta(x - \bar{x}(\tau)) d\tau - \Box \Phi_{\mathcal{R}}^{n}$.

- Regularity: $S_{\text{eff}} \sim \mathcal{O}\left(|\delta x| \delta x^{n-4}\right)$
- Decomposition in m modes:

$$\Phi_{\mathcal{R}} = \sum_{m=-\infty}^{\infty} \Phi_{\mathcal{R}}^{m} e^{im\varphi}, \quad \{\Phi_{\mathcal{P}}^{m}, S_{\text{eff}}^{m}\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\Phi_{\mathcal{P}}, S_{\text{eff}}\} \, e^{-im\varphi} d\varphi$$

- 2+1D wave equations:

$$\Box^{m} \Phi_{\mathcal{R}}^{m} = S_{\text{eff}}^{m}$$

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Mode sums and convergence

- Field is real $\Rightarrow \Phi^{-m} = \Phi^{m*}$
- SF from mode sums, e.g.

$$F_r = q \sum_{m=0}^{\infty} \partial_r \tilde{\Phi}^m_{\mathcal{R}}$$

where

$$\tilde{\Phi}^m_{\mathcal{R}} = \begin{cases} \Phi^m_{\mathcal{R}}, & m = 0 \\ 2\,\mathrm{Re}\left(\Phi^m_{\mathcal{R}}\, e^{im\bar{\varphi}(t)}\right), & m \neq 0 \end{cases}$$

- Power law convergence $F^m_r \sim m^{-\zeta}$ in large-m regime
- Convergence rate ζ depends on order n of puncture.
- $\zeta = n$ for n even, and $\zeta = n - 1$ for n odd.

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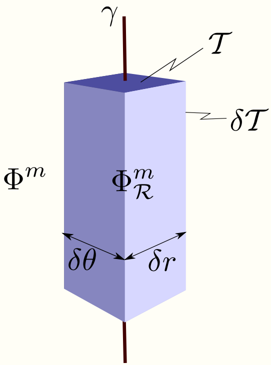
Puncture order and m-mode convergence

For circular orbits, F_r is conservative and F_φ is dissipative.

punc. order	$\Phi_{\mathcal{R}}$	C	$S_{\rm eff}$	Φ_R^m	F_r^m	F_φ^m
1	$\delta x/\lvert\delta x\rvert$	C^{-1}	$1/\delta x^2$	m^{-2}	—	—
2	$\lvert\delta x\rvert$	C^0	$1/\lvert\delta x\rvert$	m^{-2}	m^{-2}	$e^{-\lambda m}$
3	$\lvert\delta x\rvert\,\delta x$	C^1	$\delta x/\lvert\delta x\rvert$	m^{-4}	m^{-2}	$e^{-\lambda m}$
4	$\lvert\delta x\rvert\,\delta x^2$	C^2	$\lvert\delta x\rvert$	m^{-4}	m^{-4}	$e^{-\lambda m}$

World-tube construction



- Worldtube $\mathcal{T}$ of fixed dimensions δr, $\delta\theta$
- Outside: $\Box_m \Phi^m = 0$
- Inside: $\Box_m \Phi^m_{\mathcal{R}} = S^m_{\text{eff}}$
- Across boundary $\delta\mathcal{T}$: $\Phi^m_{\mathcal{R}} = \Phi^m - \Phi^m_{\mathcal{P}}$

Scalar-field SF for circular orbits on Kerr

- The first results for circular orbits on Kerr [freq.-domain] were presented by N. Warburton at Capra 12.
- We presented preliminary m-mode time-domain results at Capra 13. Papers: arXiv:1010.5255, arXiv:1107.0012.
- Parameters a and r_0.
- Boyer-Lindquist coords, and $d\varphi = d\phi + a/\Delta dr$
- Kerr wave equation ($\Psi = \Phi^m/r$, $\Delta = r^2 - 2Mr + a^2$):

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{(r^2+a^2)^2}{\Sigma^2}\frac{\partial^2\Psi}{\partial r_*^2} + \frac{4iamMr}{\Sigma^2}\frac{\partial\Psi}{\partial t} + \left[\frac{2a^2\Delta}{r\Sigma^2} - \frac{2iam(r^2+a^2)}{\Sigma^2}\right]\frac{\partial\Psi}{\partial r_*}$$
$$-\frac{\Delta}{\Sigma^2}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta} - r\left(\frac{\Delta}{r^2}\right)' - \frac{2iam}{r}\right]\Psi = 0$$

- where

$$\Sigma^2 \equiv (r^2+a^2)^2 - a^2\Delta\sin^2\theta$$

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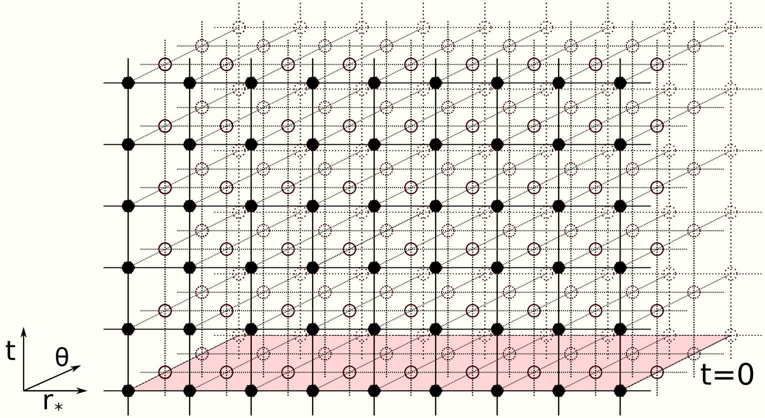
Finite difference method

2+1D grid



Finite difference method

- I used the method of lines with first-order variables

$$\Psi^m, \qquad \Pi^m \equiv \partial_t \Psi^m$$

- Grid spacings $\triangle t$, $\triangle r_*$, $\triangle\theta$.
- Second-order-accurate finite differences on spatial slices, e.g.

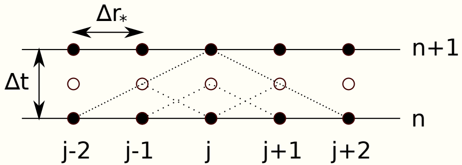
$$\frac{\partial^2\Psi}{\partial\theta^2} \approx \frac{\Psi(\theta+\triangle\theta) - 2\Psi(\theta) + \Psi(\theta-\triangle\theta)}{\triangle\theta^2} + \mathcal{O}(\triangle\theta^2)$$

- Why not fourth-order? Laziness, because S_{eff} is only C^0 on worldline
- Fourth-order Runge-Kutta time-step – why? stable.

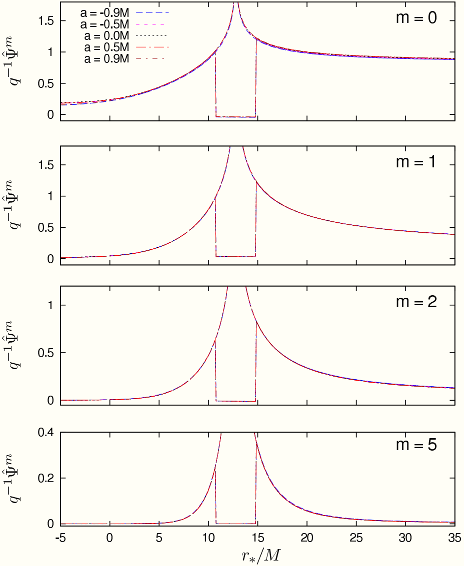
Finite difference method

2+1D grid

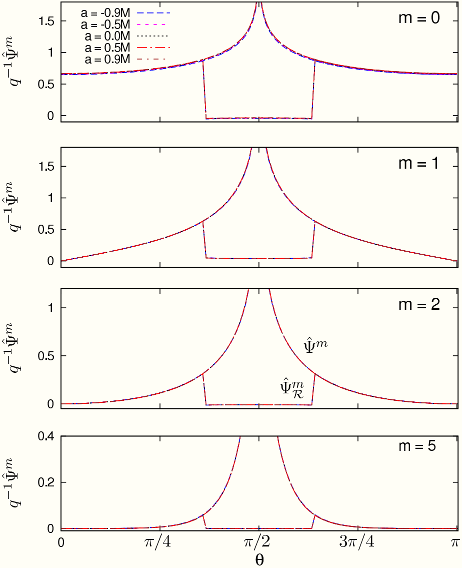
- Method of lines w. second-order finite differences on spatial slices
- 4th-order Runge-Kutta time steps
- Boundary conditions at poles: $\Psi^{m=0} = 0$, $\partial_\theta \Psi^{m\neq 0} = 0$.
- No spatial boundary conditions!
- Simple but runtime scales badly: $\sim t^2_{\max}/(\triangle t \triangle r_* \triangle \theta)$



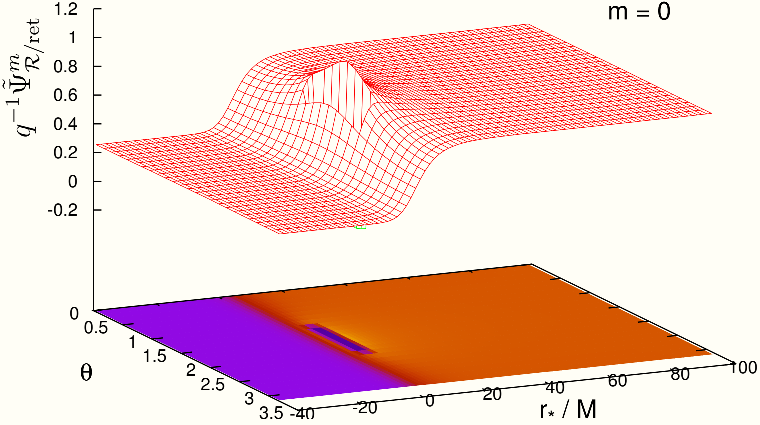
Spatial profile of modes: r_*



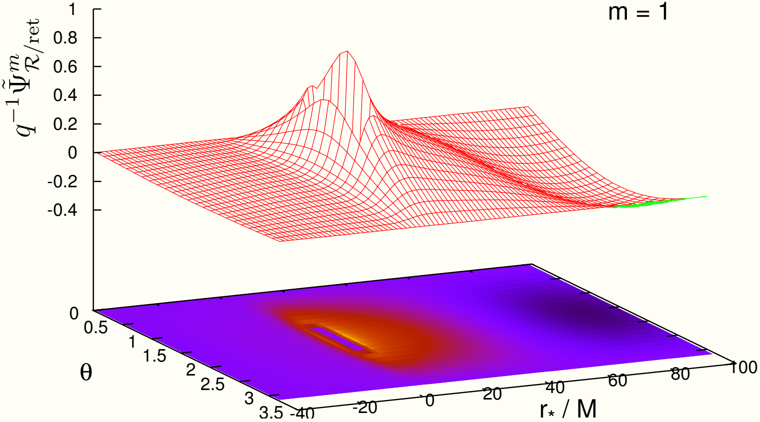
Spatial profile of modes: θ



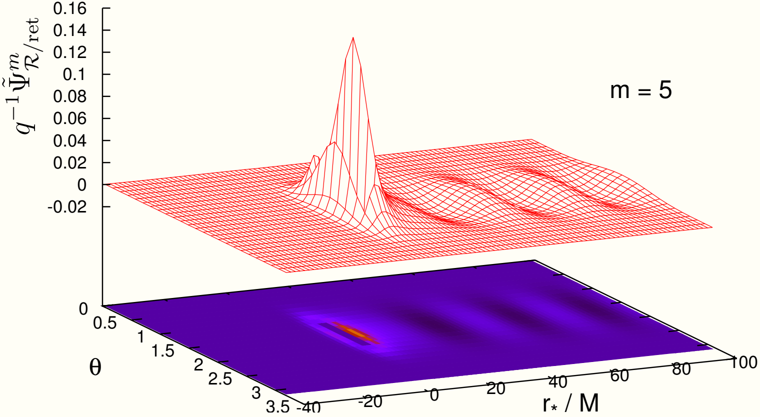
Spatial profiles: r_* and θ ($m = 0$)



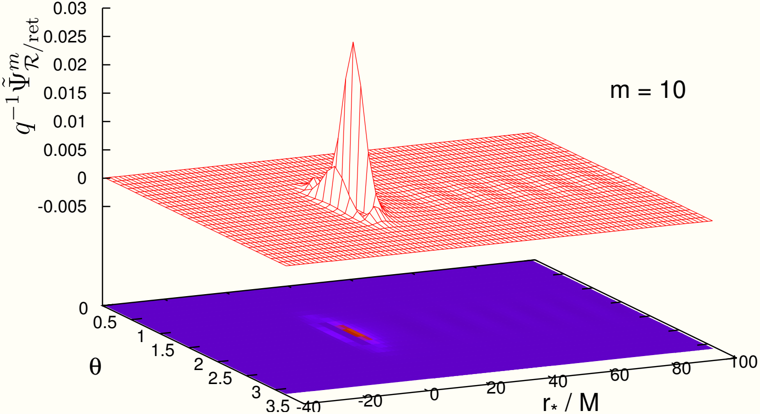
Spatial profiles: r_* and θ $(m = 1)$



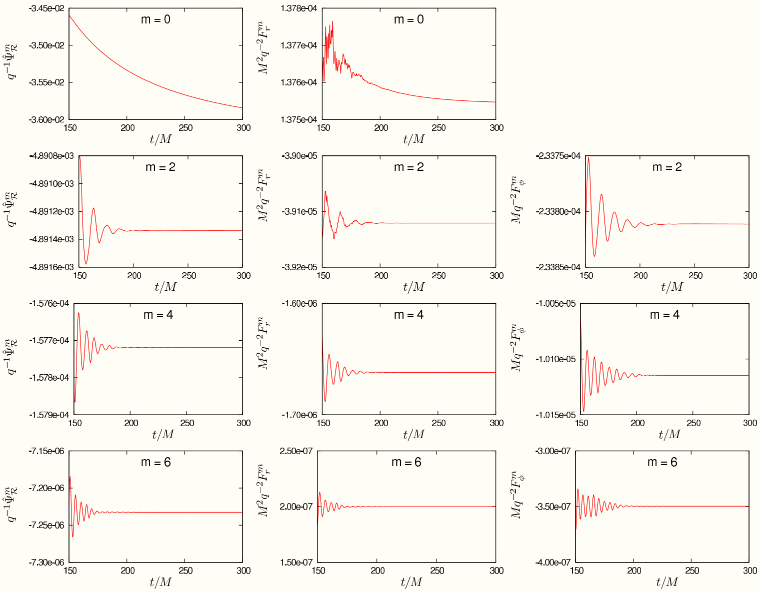
Spatial profiles: r_* and θ ($m = 5$)



Spatial profiles: r_* and θ ($m = 10$)

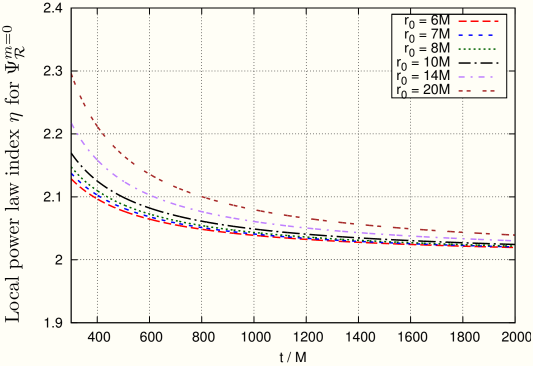


Time evolution of m-modes on worldline



Low-m modes and power law relaxation

- Low-m modes take longest to relax
- Fit power-law decay model
- e.g. for $m = 0$, $\tilde{\Phi}^m_{\mathcal{R}}(t) = \tilde{\Phi}^m_{\mathcal{R}}(\infty) + c_2 t^{-\eta} + \ldots$



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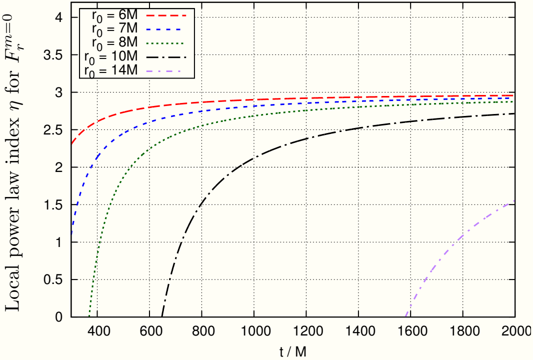


Richardson extrapolation (I)

Extrapolation to infinite resolution

- Results depends on grid resolution x, e.g. :

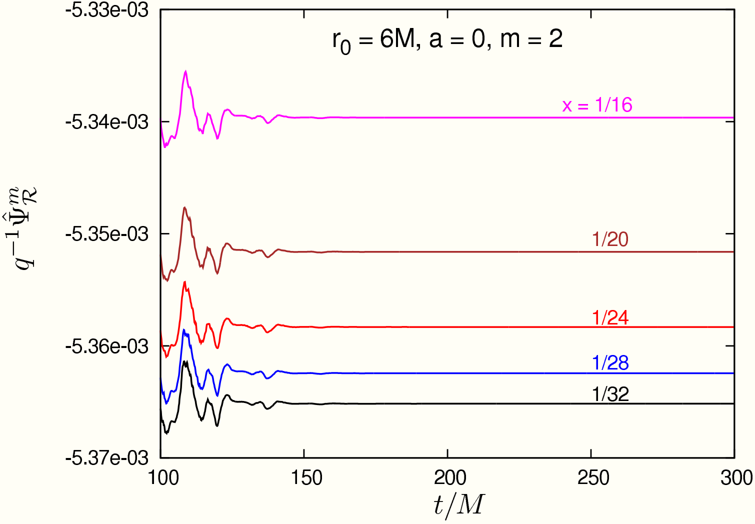
$$\triangle t = xM, \quad \triangle r_* = xM, \quad \triangle\theta = \pi x/6$$

- Second-order-accurate FD method $\Rightarrow$ error $\mathcal{O}(x^2)$

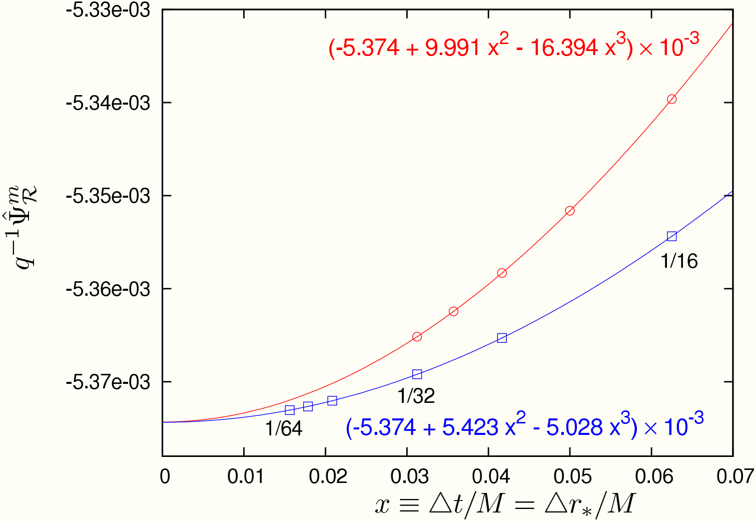
$$\Psi^m(x) = \Psi^m(x=0) + c_2 x^2 + c_3 x^3 + \ldots$$

- Fit results of runs at various resolutions to this model, and extrapolate to $x = 0$

Richardson extrapolation (II)

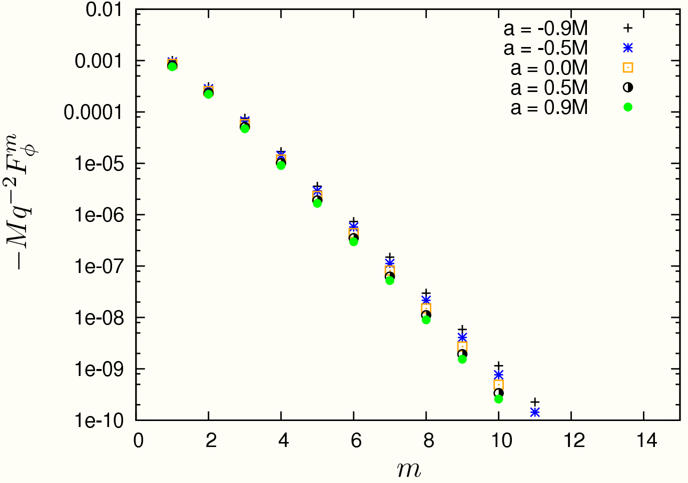


Richardson extrapolation (III)



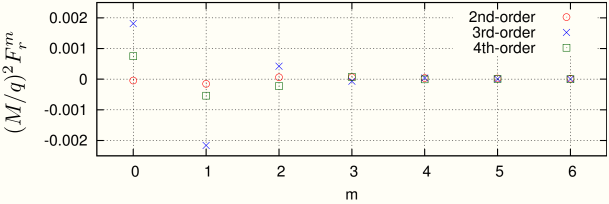
Modal convergence: F_ϕ^m

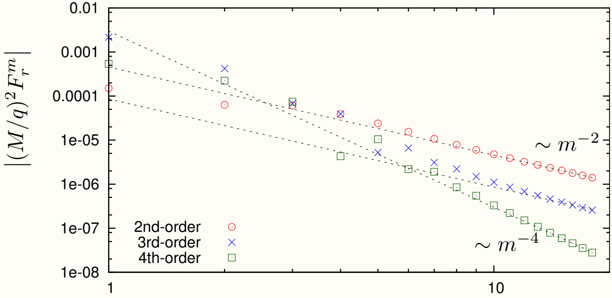
- Exponential convergence of dissipative component



Modal convergence: F_r^m

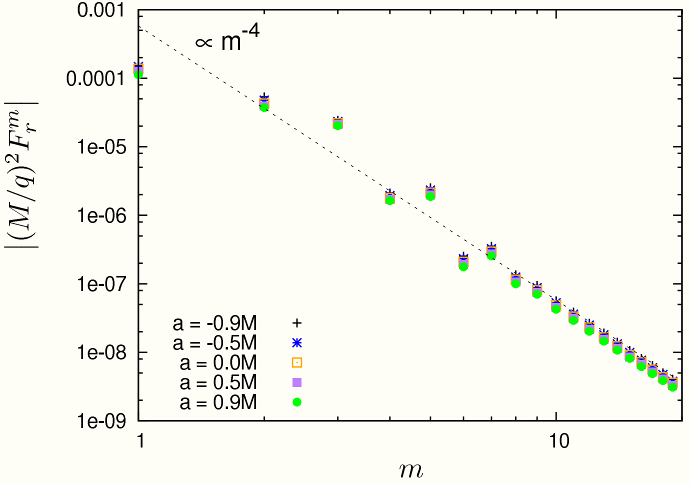
- Power-law convergence of conservative component
- Puncture orders $n = 2$, 3 and 4





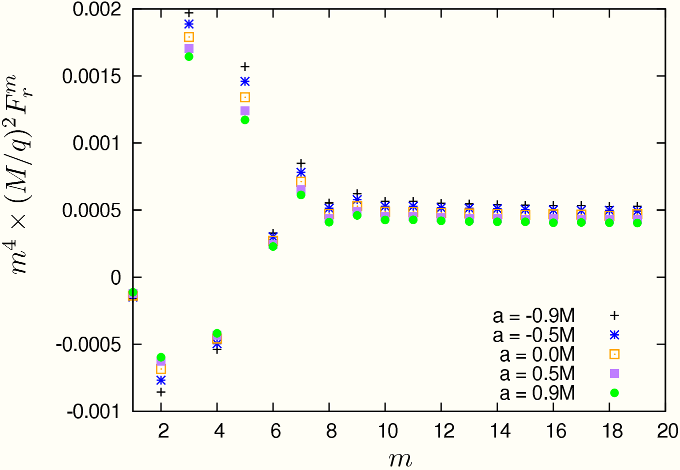
Modal convergence: F_r^m

4th-order puncture ... m^{-4} convergence



Modal convergence: F_r^m

rescaled variable $m^4 F_r^m$



Results: scalar-field SF on Kerr

F_r for circular orbits in equatorial plane

Radial component of SF, $(M^2/q^2)F_r^{\rm self}$			
	$r_0 = 6M$	$r_0 = 10M$	$r_0 = r_{\rm isco}$
$a = -0.9M$	– –	4.941(1) 4.39995 $\times 10^{-5}$	9.6074(7) 9.607001 $\times 10^{-5}$
$a = -0.7M$	– –	4.102(1) 4.100712 $\times 10^{-5}$	1.1077(2) 1.107625 $\times 10^{-4}$
$a = -0.5M$	– –	3.290(1) 3.28942 $\times 10^{-5}$	1.2751(2) 1.275170 $\times 10^{-4}$
$a = \quad 0M$	1.6771(2) 1.677283 $\times 10^{-4}$	1.379(1) 1.378448 $\times 10^{-5}$	1.6771(2) 1.677283 $\times 10^{-4}$
$a = +0.5M$	−2.423(4) −2.421685 $\times 10^{-5}$	−4.028(9) −4.03517 $\times 10^{-6}$	−6.925(5) −6.922147 $\times 10^{-5}$
$a = +0.7M$	−9.530(3) −9.528095 $\times 10^{-5}$	−1.0913(9) −1.091819 $\times 10^{-5}$	−1.0886(4) −1.088457 $\times 10^{-3}$
$a = +0.9M$	−1.6458(5) −1.645525 $\times 10^{-4}$	−1.767(1) −1.768232 $\times 10^{-5}$	−1.1344(9) −1.133673 $\times 10^{-2}$

Results: scalar-field SF on Kerr

F_ϕ for circular orbits in equatorial plane

Angular component of SF, $-(M/q^2)F_\phi^{\rm self}$			
	$r_0 = 6M$	$r_0 = 10M$	$r_0 = r_{\rm isco}$
$a = -0.9M$	– –	1.41470(1) 1.414708 $\times 10^{-3}$	2.18835(1) 2.188351 $\times 10^{-3}$
$a = -0.7M$	– –	1.35624(1) 1.356244 $\times 10^{-3}$	2.57803(1) 2.578045 $\times 10^{-3}$
$a = -0.5M$	– –	1.30226(1) 1.302267 $\times 10^{-3}$	3.08354(1) 3.083542 $\times 10^{-3}$
$a = \ \ 0M$	5.304230(3) 5.3042317 $\times 10^{-3}$	1.18592(1) 1.185926 $\times 10^{-3}$	5.30423(1) 5.304232 $\times 10^{-3}$
$a = +0.5M$	4.230745(3) 4.230749 $\times 10^{-3}$	1.09349(1) 1.093493 $\times 10^{-3}$	1.18357(4) 1.183567 $\times 10^{-2}$
$a = +0.7M$	3.928695(3) 3.928698 $\times 10^{-3}$	1.06216(1) 1.062163 $\times 10^{-3}$	1.94873(1) 1.948731 $\times 10^{-2}$
$a = +0.9M$	3.676723(8) 3.676726 $\times 10^{-3}$	1.03344(1) 1.0334444 $\times 10^{-3}$	4.5079(2) 4.508170 $\times 10^{-2}$

Perturbation theory (I)

- Einstein equations :

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$

- Vacuum background + stress-energy $T_{ab} \propto$ 'small' parameter $\mu = m/M$
- Metric split : background + perturbation :

$$g_{ab} = \hat{g}_{ab} + \mu h_{ab}$$

- Trace-reversed perturbation $\bar{h}_{ab}$:

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} h$$

Perturbation theory (II)

- 4 Lorenz gauge conditions :

$$\mathcal{Z}^b \equiv \nabla_a \bar{h}^{ab} = 0$$

- 10 Linearized Einstein equations :

$$\mathcal{W}_{ab} \equiv \nabla^c \nabla_c \bar{h}_{ab} + 2\, R^c{}_a{}^d{}_b\, \bar{h}_{cd} = -16\pi T_{ab}$$

- m-mode decomposition (Schw. $f = 1 - 2M/r$):

$$\bar{h}_{ab} = \alpha_{ab}(r,\theta) u_{ab}(r,\theta,t) e^{im\phi}, \qquad \text{(no sum)}$$

- 10 wave equations (in vacuum)

$$f \Box_{sc} u_{ab} + \mathcal{M}_{ab}(u_{cd,t}, u_{cd,r_*}, u_{cd,\theta}, u_{cd}) = 0$$

2+1D Wave Equations

$$f\Box_{sc}u_{ab}+\mathcal{M}_{ab}(\dot{u}_{cd,t},u_{cd,r*},u_{cd,\theta},u_{cd})=0$$

$$\begin{aligned}
\mathcal{M}_{00} &= \frac{2\left(2r^2(\dot{u}_{01}-u'_{00})+u_{00}-u_{11}\right)}{r^4}+\frac{4f\left(u_{00}-u_{11}\right)}{r^3}+\frac{2f^2\left(u_{22}+u_{33}\right)}{r^3}\\
\mathcal{M}_{01} &= -\frac{2f^2\left(\cos\theta u_{02}+imu_{03}\right)}{r^2\sin\theta}+\frac{2(\dot{u}_{00}+\dot{u}_{11}-2u'_{01})}{r^2}-\frac{2f^2(u_{01}+\partial_\theta u_{02})}{r^2}\\
\mathcal{M}_{02} &= -\frac{f(u_{02}+2im\cos\theta u_{03})}{r^2\sin^2\theta}+\frac{2(\dot{u}_{12}-u'_{02})}{r^2}+\frac{f[(4+r)u_{02}+2r\partial_\theta u_{01}]}{r^3}-\frac{f^2u_{02}}{r^2}\\
\mathcal{M}_{03} &= -\frac{f(u_{03}-2im\cos\theta u_{02})}{r^2\sin^2\theta}+\frac{2fimu_{01}}{r^2\sin\theta}+\frac{2(\dot{u}_{13}-u'_{03})}{r^2}+\frac{f(4+r)u_{03}}{r^3}-\frac{f^2u_{03}}{r^2}\\
\mathcal{M}_{11} &= -\frac{4f^2(\cos\theta u_{12}+imu_{13})}{r^2\sin\theta}+\frac{2[2r^2(\dot{u}_{01}-u'_{11})+u_{11}-u_{00}]}{r^4}-\frac{4f(u_{00}-u_{11})}{r^3}\\
&\quad-\frac{2f^2(2ru_{11}+u_{22}+u_{33}+2r\partial_\theta u_{12})}{r^3}+\frac{2f^3(u_{22}+u_{33})}{r^2}\\
\mathcal{M}_{12} &= -\frac{f(u_{12}+2im\cos\theta u_{13})}{r^2\sin^2\theta}-\frac{2f^2[\cos\theta(u_{22}-u_{33})+imu_{23}]}{r^2\sin\theta}+\frac{2(\dot{u}_{02}-u'_{12})}{r^2}\\
&\quad+\frac{f[(4+r)u_{12}+2r\partial_\theta u_{11}]}{r^3}-\frac{f^2(5u_{12}+2\partial_\theta u_{22})}{r^2}
\end{aligned}$$

2+1D Wave Equations

$$f\Box_{sc}u_{ab} + \mathcal{M}_{ab}(\dot{u}_{cd,t}, u_{cd,r*}, u_{cd,\theta}, u_{cd}) = 0$$

$$\begin{aligned}
\mathcal{M}_{13} &= -\frac{f(u_{13} - 2im\cos\theta u_{12})}{r^2\sin^2\theta} - \frac{2f[2f\cos\theta u_{23} + im(fu_{33} - u_{11})]}{r^2\sin\theta} + \frac{2(\dot{u}_{03} - u'_{13})}{r^2} \\
&\quad + \frac{f(4+r)u_{13}}{r^3} - \frac{f^2(5u_{13} + 2\partial_\theta u_{23})}{r^2} \\
\mathcal{M}_{22} &= -\frac{2f[u_{22} - u_{33} + 2im\cos\theta u_{23}]}{r^2\sin^2\theta} + \frac{2(u_{00} - u_{11})}{r^3} + \frac{2f(u_{11} + u_{22} + 2\partial_\theta u_{12})}{r^2} \\
&\quad - \frac{2f^2(u_{22} + u_{33})}{r^2} \\
\mathcal{M}_{23} &= -\frac{2f[2u_{23} - im\cos\theta(u_{22} - u_{33})]}{r^2\sin^2\theta} - \frac{2f(\cos\theta u_{13} - imu_{12})}{r^2\sin\theta} + \frac{2f(u_{23} + \partial_\theta u_{13})}{r^2} \\
\mathcal{M}_{33} &= \frac{2f(u_{22} - u_{33} + 2im\cos\theta u_{23})}{r^2\sin^2\theta} + \frac{4f(\cos\theta u_{12} + imu_{13})}{r^2\sin\theta} + \frac{2(u_{00} - u_{11})}{r^3} \\
&\quad + \frac{2f(u_{11} + u_{33})}{r^2} - \frac{2f^2(u_{22} + u_{33})}{r^2}.
\end{aligned}$$

Gauge constraint damping

- Imperfect, gauge-violating initial data

$$\Rightarrow \mathcal{Z}^a \equiv \nabla_b \bar{h}^{ab} \neq 0.$$

- Gauge-violation itself obeys a wave equation:

$$\Box \mathcal{Z}^a = 0.$$

- How to drive system towards Lorenz gauge solution $\mathcal{Z}^a = 0$?
- Gauge Constraint Damping: add extra term to wave equations featuring gauge violation vector $\mathcal{Z}_a$, i.e.

$$\Box \bar{h}_{ab} + 2R^c{}_a{}^d{}_b \bar{h}_{cd} + f' \left(t_a \mathcal{Z}_b + t_b \mathcal{Z}_a \right) = 0.$$

so that $\mathcal{Z}_a$ obeys a damped wave equation

2nd-order puncture scheme

- Barack, Golbourn & Sago (2007) give a 2nd-order puncture formulation:

$$\bar{h}^P_{ab}(x) = \frac{\mu}{\epsilon_P^{[2]}}\chi_{ab}, \qquad \chi_{ab} = \Big[u_a u_b + (\Gamma^c_{ad}u_b + \Gamma^c_{bd}u_a)u_c\delta x^d\Big]_{x=\bar{x}}$$

- For circular orbits in eq. plane, this reduces to

$$\begin{aligned}
\chi_{00} &= C_{00} + D_{00}\delta r \\
\chi_{01} &= D_{01}\sin\delta\phi \\
\chi_{03} &= C_{03} + D_{03}\delta r \\
\chi_{13} &= D_{13}\sin\delta\phi \\
\chi_{33} &= C_{33} + D_{33}\delta r
\end{aligned}$$

2nd-order puncture scheme

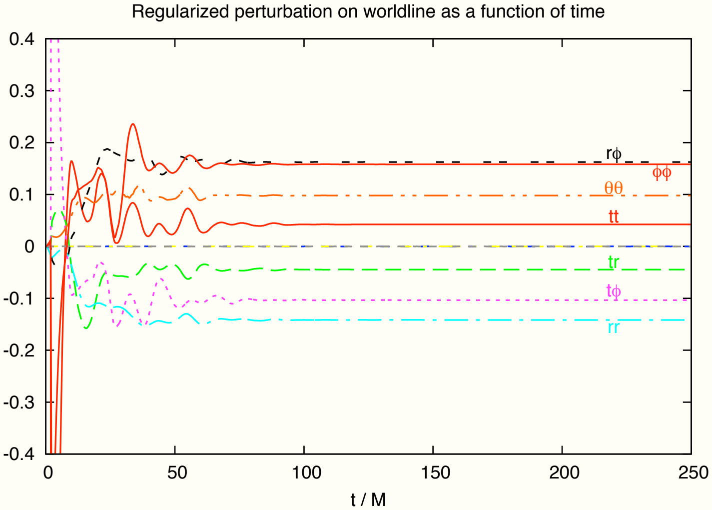
- Effective source: $S^{\rm eff}_{ab} = \left[\Box + 2R^{c}{}_{a}{}^{d}{}_{b}\right] \bar{h}^{P}_{cd}$
- m-mode decomposition: $\bar{h}^{P(m)}_{ab}$ and $S^{{\rm eff}(m)}_{ab}$
- Puncture and source found in terms of 'symmetric' elliptic integrals $I^m_1, \ldots, I^m_5$ …
- … and antisymmetric integrals $J^m_1, \ldots, J^m_5$ …

$$\begin{aligned}
\int_{-\pi}^{\pi} \epsilon_P^{-3} \sin\delta\phi\, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i}{B^{3/2}\rho}\left[q^m_{1K} K(i/\rho) + \rho^2 q^m_{1E} E(i/\rho)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-3} \sin\delta\phi\, \cos\delta\phi\, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i\gamma}{B^{3/2}}\left[q^m_{2K} K(\gamma) + q^m_{2E} E(\gamma)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-5} \sin\delta\phi\, \cos^2(\delta\phi/2)\, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i\gamma}{B^{5/2}}\left[q^m_{3K} K(\gamma) + \rho^{-2} q^m_{3E} E(\gamma)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-5} \sin\delta\phi\, \sin^2(\delta\phi)\, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i}{B^{5/2}\rho}\left[q^m_{4K} K(i/\rho) + \rho^2 q^m_{4E} E(i/\rho)\right] \\
\int_{-\pi}^{\pi} \epsilon_P^{-5} \sin\delta\phi\, \sin^2(\delta\phi/2)\, e^{-im\delta\phi} d(\delta\phi) &= \frac{-i\gamma^2}{B^{5/2}\rho}\left[q^m_{5K} K(i/\rho) + \rho^2 q^m_{5E} E(i/\rho)\right]
\end{aligned}$$

- Wardell and co. developing a 4th-order scheme

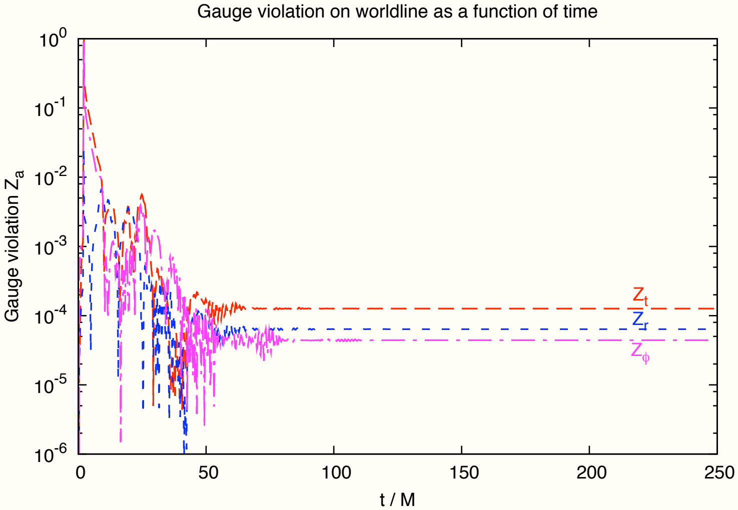
Metric Perturbations : Time evolution

$r_0 = 7M$, $m = 2$



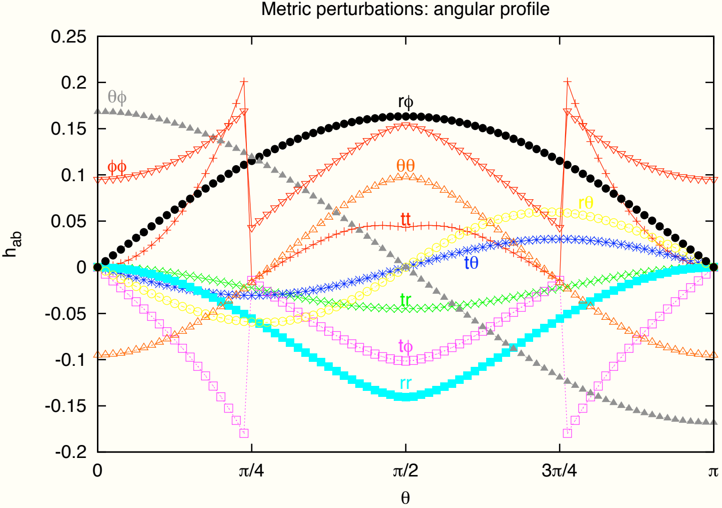
Gauge Violation : Time evolution

$r_0 = 7M$, $m = 2$



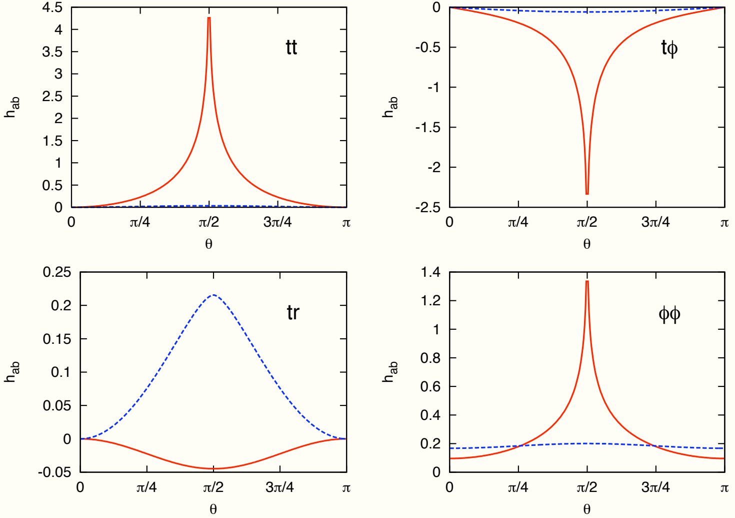
Metric Perturbations : Angular Profile

$r_0 = 7M$, $m = 2$



Metric Perturbations : Angular Profile

$r_0 = 7M$, $m = 2$



l-mode and m-modes

- Project out: m-modes $u^m_{ab}(t, r, \theta)$ onto lm modes $h^{(i)}_{lm}(t, r)$ of Barack/Lousto/Sago.
- Use tensor spherical harmonics $i = 1 \ldots 10$,

$$h^{(1)}_{lm}(r,t) = 2\pi \int_0^\pi \sin x \, (u_{00} + u_{11}) \, Y^*_{lm}(x) dx \quad (1)$$

$$h^{(2)}_{lm}(r,t) = 2\pi \int_0^\pi \sin x \, 2u_{01} Y^*_{lm}(x) dx \quad (2)$$

$$h^{(3)}_{lm}(r,t) = 2\pi \int_0^\pi \sin x \, (u_{00} - u_{11}) \, Y^*_{lm}(x) dx \quad (3)$$

$$h^{(4)}_{lm}(r,t) = 4\pi \int_0^\pi [\sin x \, u_{02} \, \partial_x - imu_{03}] \, Y^*_{lm} dx \quad (4)$$

$$h^{(5)}_{lm}(r,t) = 4\pi \int_0^\pi [\sin x \, u_{12} \, \partial_x - imu_{13}] \, Y^*_{lm} dx \quad (5)$$

$$h^{(6)}_{lm}(r,t) = 2\pi \int_0^\pi \sin x \, (u_{22} + u_{33}) \, Y^*_{lm} \quad (6)$$

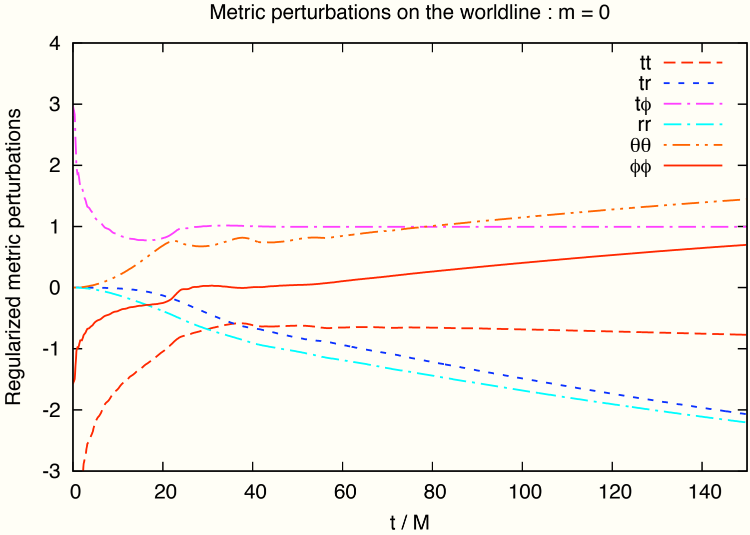
$$h^{(7)}_{lm}(r,t) = 2\pi \int_0^\pi [\sin x (u_{22} - u_{33}) D_2 + 2u_{23} D_1] \, Y^*_{lm} dx \quad (7)$$

Comparison with l-modes

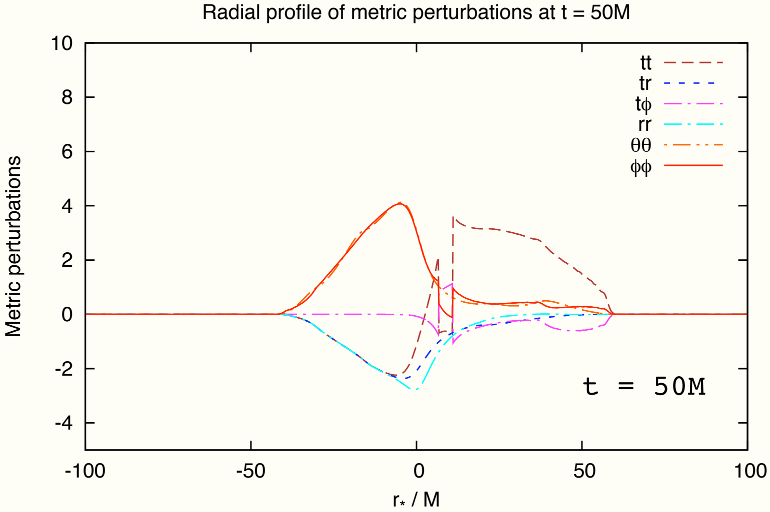
Projection from m modes onto lm modes of Barack/Lousto/Sago

	$l = 2$, $m = 2$	
$i = 1$	3.1246	$-0.2630i$
	3.1246	$-0.2632i$
$i = 2$	-0.2316	$0.9755i$
	-0.2312	$0.9758i$
$i = 3$	5.3159	$0.6164i$
	5.3162	$0.6162i$
$i = 4$	-0.9269	$9.4275i$
	-0.9249	$9.4292i$
$i = 5$	-2.3297	$-2.5279i$
	-2.3310	$-2.5279i$
$i = 6$	1.5471	$0.6009i$
	1.5468	$0.6006i$
$i = 7$	-5.3326	$-5.2205i$
	-5.3319	$-5.2190i$

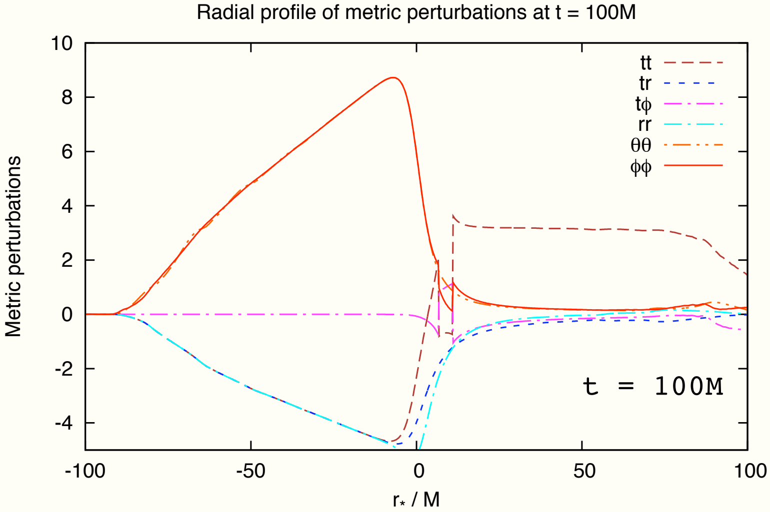
Time Evolution : $m = 0$ mode



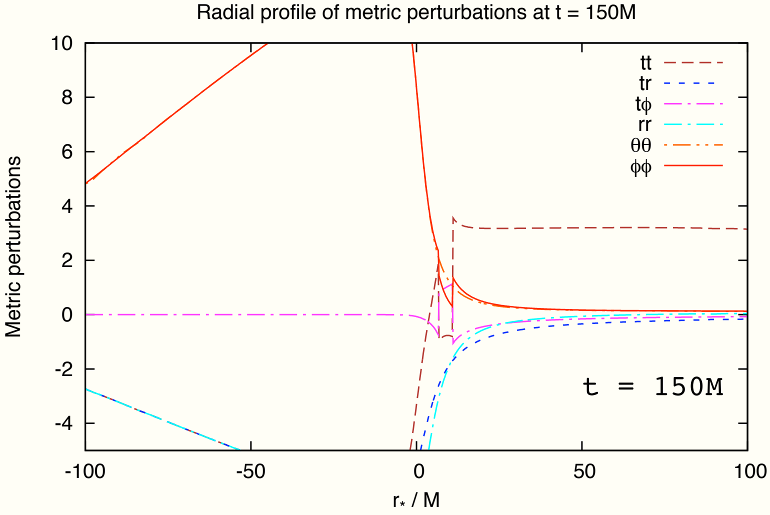
Radial Profile : $m = 0$ mode



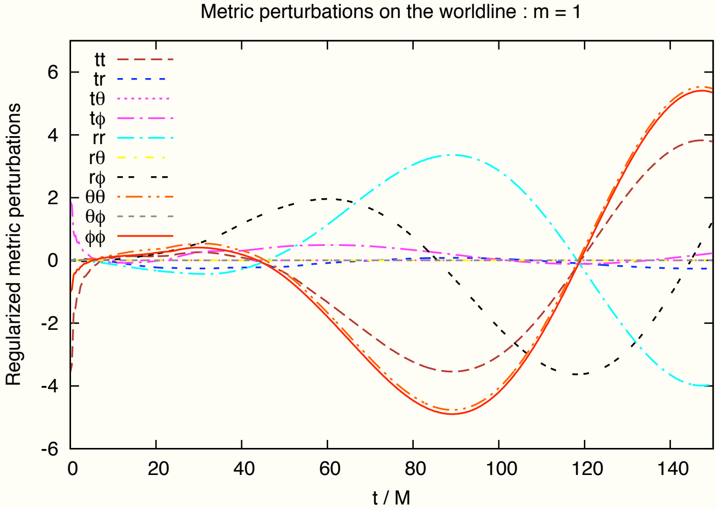
Radial Profile : $m = 0$ mode



Radial Profile : $m = 0$ mode



Time Evolution : $m = 1$ mode



The non-radiative multipoles problem

- The growing solutions arise even in vacuum.
- The growing solutions are (locally) Lorenz-gauge
- The growing solutions violate physical boundary conditions at horizon
- Even though the modes grow, the trace $h = -\bar{h}_a^a$ does not. (Recall $\Box h = 0$ in vacuum).
- The problem is entirely in $l = m = 0$ and $l = m = 1$ modes in Schw.
- Q. Why has no-one evolved Schw. $l = 0$ and $l = 1$ modes in time-domain?
- A. Negative potentials ($r < 3M$), unstable evolutions.

Lorenz-Gauge Monopole Modes

- Pure-gauge modes generated by gauge vectors X_a

$$h_{ab} = X_{(a;b)} \quad \Rightarrow \quad \bar{h}_{ab} = X_{(a;b)} - \frac{1}{2} g_{ab} X^{c}_{;c}$$

- Lorenz-gauge $\bar{h}^{;b}_{ab} = 0 \Rightarrow X_{a;b}{}^{b} = 0$
- Two scalar monopole gauge modes $X_a = \Phi_{;a} \Rightarrow (\Box\Phi)_{;a} = 0$ $\Rightarrow \Box\Phi = \{0, \text{const.}\}$
- Trace : $h = X^{a}_{;a} = \Box\Phi = \{0, \text{const}\}$ $\Rightarrow$ Trace-free, static scalar gauge mode $\Phi_0 = \frac{1}{2}\ln f$
- Pseudo-static mode $\Phi = t \times \Phi_0 = \frac{t}{2}\ln f$, $\Box\Phi = 0$,

$$u_{00}, u_{11}, u_{22} \propto t, \qquad u_{01} \neq 0.$$

Pseudo-static modes

- Pseudo-static (i.e. linearly-growing) locally Lorenz-gauge modes in monopole
- How do they arise in time domain?
- Low multipole equations hide non-radiative quasi-local quantities, $\partial_t Q = 0$
- i.e. Conservation laws due to symmetries of Ricci-flat background
- After reducing degrees of freedom, find wave equation with negative potential.
- Growing solutions violate physical boundary conditions at horizon.

Conservation Law (I)

- Symmetries: Background spacetime has Killing vectors ξ_a :

$$\nabla_a \xi_b + \nabla_b \xi_a = 0$$

- Stress-energy is conserved, $\nabla_a T^{ab} = 0$, so we can construct a conserved vector:

$$j^a \equiv T^{ab}\xi_b \quad \Rightarrow \quad \nabla_a \, j^a = 0.$$

- The vector $j_a = (-16\pi)^{-1}\mathcal{W}_{ab}\xi^b$ can be written

$$j^a = \nabla_b F^{ab}, \qquad \text{where} \quad F_{ab} = -F_{ba}$$

- i.e. the divergence of an antisymmetric tensor F^{ab} where

$$\boxed{(-16\pi)F_{ab} = \bar{h}_{ac;b}\xi^c - \bar{h}_{bc;a}\xi^c - \bar{h}_{ac}\xi^c{}_{;b} + \bar{h}_{bc}\xi^c{}_{;a}}$$

- Apply Stokes' theorem $\Rightarrow$ Conserved integrals on two-surfaces

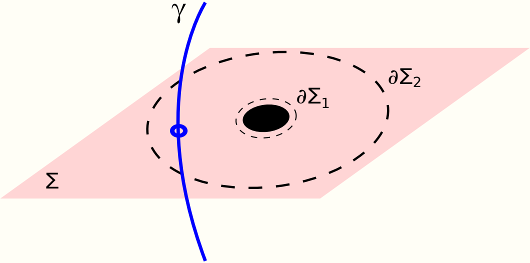
Conservation Law (II)

- Gauss's theorem:

$$\int_{\Sigma_1} j^a d\Sigma_a = \int_{\Sigma_2} j^a d\Sigma_a$$

- Stokes' theorem ($j^a = F^{ab}{}_{;b}$):

$$\int_{\Sigma} F^{ab}{}_{;b} d\Sigma_a = \frac{1}{2}\int_{\partial\Sigma} F^{ab} dS_{ab}$$



Conservation Law (III)

- Integrate on constant-t hypersurfaces, on concentric spheres:
- $\xi_a^{(t)} \Rightarrow$ Energy $\mathcal{E}$, $\quad \xi_a^{(\phi)} \Rightarrow$ Ang. Mom. $\mathcal{L}_z$
- Energy is in monopole

$$4\pi \left[r^2 F_{01}^{(t)} \right]_{r_1}^{r_2} = \begin{cases} \mathcal{E} \equiv -u_t, & r_1 < r_0 < r_2, \\ 0, & \text{otherwise.} \end{cases}$$

- Locally conserved quantity in monopole ($l = m = 0$) equations:

$$r^2 \left(\bar{h}_{tt,r} - \bar{h}_{tr,t} \right) - 2f^{-1}\bar{h}_{tt} + 2f\bar{h}_{rr} = \begin{cases} -4\mathcal{E}, & r > r_0, \\ 0, & r < r_0. \end{cases}$$

Monopole equations

- Monopole has four equations $(u_{00}, u_{01}, u_{11}, u_{22} = u_{33})$ + two gauge constraints.
- Trace equation evolve stably
- Use conserved quantity $C = \begin{cases} -4\mathcal{E} & r < r_0 \\ 0 & r > r_0 \end{cases}$
- Hierarchical system of equations for $\{H, X, Y\}$

$$\begin{aligned} D^2 H &= 0 \\ D^2 X &= \frac{2f}{r^4} H - \frac{3fC}{r^3} \\ \left[D^2 - \frac{2f}{r^2} \left(1 - \frac{4M}{r} \right) \right] Y &= -\frac{4f}{r^2} H + \frac{2f}{r} C \end{aligned}$$

where $D^2 = -\partial_t^2 + \partial_{r^*}^2 - 2fM/r^3$

- $H = r\bar{h}_a^a$, $X = (2rf)^{-1} [u_{11} - (2r-3)u_{00}]$, $Y = rf^{-1}(u_{00} - u_{11})$.

Monopole equations

- H and X equations evolve stably. Y equation does not.
- Y equation resembles a Regge-Wheeler equation

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-V_{12}(r)\right]Y=\dots$$

where

$$V_{ls}(r)=f\left(\frac{l(l+1)}{r^2}-\frac{2M(1-s^2)}{r^3}\right)$$

i.e. here $l=1,\ s=2$.

- Potential turns negative within $r<3M \Rightarrow$ growing modes.
- In principle, Y can be recovered from H, X by integrating conservation law on spatial slices:

$$\frac{\partial}{\partial r^*}(rY)=r[C-2X-fH].$$

Metric components at the horizon

- Physical boundary conditions: consider ingoing AEF coordinates $v = t + r_*$, $R = r$
- $h_{vv} \sim h_{vR} \sim h_{RR} \sim h_{\theta\theta} \sim \mathcal{O}(1)$, i.e.

$$\begin{aligned} h_{vv} &= h_{tt} \\ h_{vR} &= h_{tr} - f^{-1} h_{tt} \\ h_{RR} &= h_{rr} - 2f^{-1} h_{tr} + f^{-2} h_{tt}. \end{aligned}$$

- Evolution scheme uses variables $u_{00} = r\bar{h}_{tt}$, $u_{01} = rf\bar{h}_{tr}$ and $u_{11} = rf^2\bar{h}_{rr}$, and allows them to be $\mathcal{O}(1)$ at horizon.
- Trace $-hr = f^{-1}(-u_{00} + u_{11}) + 2u_{22} \Rightarrow \boxed{u_{00} - u_{11} \sim f}$
- $h_{RR} \sim 1 \Rightarrow \boxed{u_{00} + u_{11} - 2u_{01} \sim f^2}$.
- Growing modes violate these conditions $\Rightarrow$ non-physical

Challenges for time-domain Lorenz gauge formulation

Open questions:

- How do we evolve $l = 0$, $l = 1$ modes in time domain in 1+1D?
- e.g. how do we eliminate trace-free, massless, locally-Lorenz gauge modes? (in monopole and dipole)
- How do we enforce the physical boundary condition at the horizon?

Ideas . . . :

- Change of metric basis? $\{\bar{h}_{tt}, \bar{h}_{tr}, \bar{h}_{rr}\} \to \{\bar{h}_{vv}, \bar{h}_{vR}, \bar{h}_{RR}\}$?
- Horizon-penetrating coordinates? Hyperboloidal slicing?
- Restricted set of variables, with reconstruction of metric by integrating conservation first-order ODEs?
- Additional constraint damping?

Conclusions + Prospects

State of play:

- First implementation of m-mode scheme in time domain in Kerr [arXiv:1010.5255, arXiv:1107.0012].
- m-mode agrees with l-mode sum reg. to high accuracy.
- 4th-order puncture schemes $\Rightarrow$ rapidly-convergent mode sums m^{-4}
- GSF on Schw. – agreement in $m \geq 2$ modes.

To be continued:

- Extensions to eccentric orbits, AMR, 4th-order accurate.
- Problem of unstable low multipoles remains to be solved.
- Towards practical scheme for time-domain GSF on Kerr.