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Green Functions Beyond Caustics in Plane Wave Spacetimes

Theodore Drivas [1] Abraham Harte[2]

[1]Department of Physics, University of Chicago, Chicago, Illinois, USA

[2]Max Planck Institut für Gravitationsphysik, Albert Einstein Institut, Potsdam, Germany

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Goal:

Improve the general understanding of the Green's function associated with wave equations related to curved spacetimes.

We will study the singular structure of the Green's function *globally* in plane wave spacetimes.

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What is the difficulty?

Locally it is known that the Green's function is characterized by the Hadamard form: singular portion that lies on the light cone and a smooth (tail) portion that lies inside of it.

Globally the solution is more complicated. In particular, if null geodesics cross, the above characterization is no longer valid.

Previous work

Alternating Singular Structure

Within a normal neighborhood, the retarded/advanced Green's functions take on the "Hadamard form". This prescribes a direct (singular) portion on the light cone and a (smooth) tail term inside.

At a conjugate point, or a focal point of a 1 parameter family of null curves, the nature of that singularity changes.

Recently, the singular part of the Green's function has been observed to have a repeating 2-fold or 4-fold structure in some particular examples. (S. Dolan & A. Ottewill 2011, M. Casals et al 2009, B. Wardell PhD thesis, etc.)

Why is this difficult to do?

1. The Hadamard representation of the Green's function is strictly valid inside the normal neighborhood of the source. If not, its arguments may be multivalued and ill defined.
2. In general, it is not possible to calculate the necessary bitensors appearing in the representation (Synge's world function, van Vleck) explicitly.
3. It is unclear why on some spacetimes, the Green's function has the 2-fold and some the 4-fold structure.

Why work in Plane Wave Spacetimes?

1. Any two non-conjugate points are connected either by no geodesics or a single unique geodesic. This allows one to extend the definition of Synge's world function in a natural way.
2. The scalar wave equation admit solutions that satisfy Huygen's principle. No tails in the Hadamard form.
3. Penrose limits may provide means for understanding wave propagation through caustics on general spacetime.

Penrose limits

- (Essentially) Zoom in on a small neighborhood about a null geodesic in any spacetime. There is a sense in which this neighborhood has the geometry of a plane wave spacetime.
- In principle, it may be possible to take any spacetime, find its corresponding plane wave limit, understand wave propagation on this plane wave spacetime and use this prescription to pass through caustics in the general spacetime.

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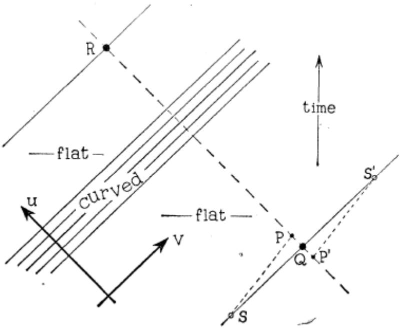
General Plane Waves

The Metric

A general plane wave metric can be written as

$$\mathrm{d}s^2 = -2\mathrm{d}u\mathrm{d}v + H_{ij}(u)x^i x^j \mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2, \qquad (1)$$

where $H_{ij}(u)$ is an arbitrary symmetric 2×2 matrix.



Geodesic Structure

So long as we are not on a constant u hyper-surface, the u-component of the geodesic depends linearly on the affine parameter s,

$$z^u(s) = s. \tag{2}$$

The spatial components of the geodesics satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} z_i(s) = H_{ij}(s) z^j(s) \tag{3}$$

The v component of any geodesic is

$$z^v(s) = a + bs + \frac{1}{2} z^i(s) \dot{z}_i(s), \tag{4}$$

where a and b are arbitrary constants. z^v is constructed from the spacial components of the geodesic, z^i.

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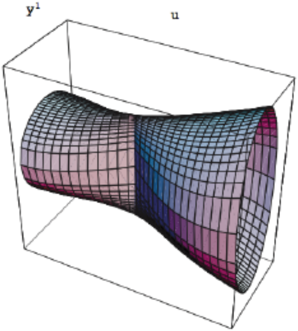
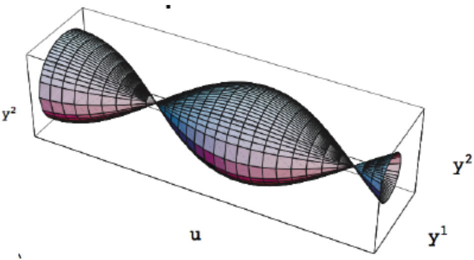
Example Plane Waves





Geodesic Deviation

The equation

Jacobi fields $\xi^a(s)$ on a geodesic $z(s)$ satisfy

$$\frac{\mathrm{D}^2}{\mathrm{d}s^2}\xi^a - R_{bcd}{}^{a}\xi^b\dot{z}^c\dot{z}^d = 0. \tag{5}$$

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Jacobi Fields

The general solution to the Jacobi field equation has the form

$$\xi^a(s) = A^a{}_{b'}(s,s')\xi^{b'}(s') + B^a{}_{b'}(s,s')\frac{\mathrm{D}}{\mathrm{d}s}\xi^{b'}(s') \tag{6}$$

for some "Jacobi Propagators" $A^a{}_{b'}(s,s')$ and $B^a{}_{b'}(s,s')$.

Conjugate Points

Definition

Two points $z(\tau)$ and $z(s')$ lying on a particular geodesic are said to be *conjugate* if there exists a nontrivial Jacobi field on that geodesic which vanishes at both $z(\tau)$ and $z(s')$.

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In plane wave spacetimes

It may be demonstrated that distinct points $z(\tau)$ and $z(s')$ in plane wave spacetimes are conjugate if and only if

$$\det \mathbf{B}(\tau, s') = 0 \iff \mathbf{B}^{-1}(\tau, s') \text{ diverges.} \tag{7}$$

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Conjugate Points cont.

Types of Conjugate Points

All geodesics (of any type) emanating from a particular point p' with spatial coordinates $\mathbf{x}'$ will be focused to

$$\mathbf{x} = \mathbf{A}(\tau, u')\mathbf{x}' \tag{8}$$

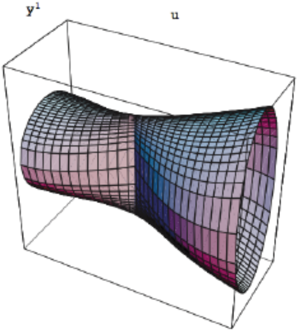
as they pass through any *degenerate* conjugate point at $u = \tau$.

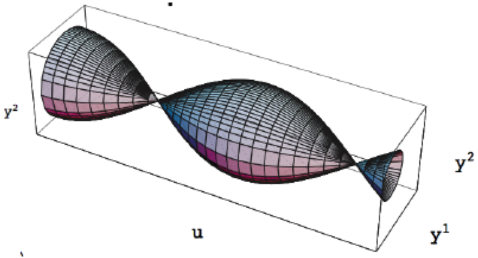
The spatial components of all geodesics starting at $\mathbf{x}'$ lie on the line

$$\mathbf{x} = \mathbf{A}(\tau, u')\mathbf{x}' + \lambda \mathbf{u} \tag{9}$$

as they pass through any *simple* conjugate point at $u = \tau$.

Example Plane Waves





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Conjugate Point Structure

- Fixing one point p', we say that there exist “conjugate null hyperplanes” which divide the spacetime into a collection of disjoint geodesically convex domains (effective normal neighborhoods).
- If a point p lies on one of these hyperplanes, it is either conjugate to p' or they are not connected by any geodesics.
- If a point p is not on one of these hypersurfaces, it is connected to p' by *exactly one geodesic*.

Bitensors

Synge's World function

Synge's world function measures half the squared geodesic distance between two points and it is typically restricted to the normal neighborhood of one of its arguments. We extend the world function globally:

$$\tilde{\sigma}(s,s') = \frac{1}{2}(s'-s)\int_s^{s'} g_{ab}(z(t))\dot{z}^a(t)\dot{z}^b(t)\mathrm{d}t. \tag{10}$$

$\tilde{\sigma}(s,s')$ reduces to the ordinary world function if the arguments are connected by a *unique* geodesic, which is a feature almost everywhere in plane wave spacetimes.

Synge's World function

In plane wave spacetimes, we can explicitly construct the world function and bitensors constructed from it:

$$\sigma(p,p') = \frac{1}{2}(s'-s)\left(z_i\dot{z}^i - 2z^v\right)\Big|_s^{s'} \tag{11}$$

$$= \frac{1}{2}(u-u')\Big[-2(v-v') + \mathbf{x}^\intercal\partial_u\mathbf{B}\mathbf{B}^{-1}\mathbf{x} \tag{12}$$

$$+ (\mathbf{x}')^\intercal\mathbf{B}^{-1}\mathbf{A}\mathbf{x}' - 2(\mathbf{x}')^\intercal\mathbf{B}^{-1}\mathbf{x}\Big], \tag{13}$$

and the van Vleck determinant

$$\Delta(p,p') = -\frac{\det(-\nabla_\mu\nabla_{\mu'}\sigma)}{\sqrt{-g}\sqrt{-g'}} = \frac{(u-u')^2}{\det\mathbf{B}}. \tag{14}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices which are related to the Jacobi propagators.

Wave Propagation in a Bulk Spacetime

General Form for Green's function

Choose the ansatz

$$G(p, p') = \sqrt{|\Delta|} f_m(\sigma).$$

if p is not conjugate to p'. Substituting this into the homogeneous wave equation $\Box G = 0$, we find that the general distributional form for f_m is

$$f_m = \alpha_m \delta(\sigma) + \text{p.v.}\left(\frac{\beta_m}{\sigma}\right) + \gamma_m.$$

with $\alpha_m, \beta_m, \gamma_m \in \mathbb{R}$.

Continuing $G(p, p')$

Insuring G is a solution

We require that $G(p, p')$ is a weak solution of the wave equation. That is, we must have $\langle \Box G, \phi \rangle = 0$ for any test function ϕ. This means that

$$\langle G(p, p'), \Box \phi(p) \rangle = -4\pi \phi(p'). \tag{15}$$

Continuing $G(p, p')$

Determining the Condition

$$\langle G(p,p'), \Box\phi(p)\rangle = \sum_n \int_{\mathbb{R}^3} \mathrm{d}x\mathrm{d}y\mathrm{d}v \int_{M_n(u')} G(p,p')\Box\phi(p)\mathrm{d}u$$

Conducting an integration by parts in u, we find

$$= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^3} \mathrm{d}x\mathrm{d}y\mathrm{d}v \Bigg[-2G(p,p')\partial_v\phi(p)\Bigg|_{\partial M_n(u')} \\ - \cancelto{0}{\int_{M_n(u')} \Box G(p,p')}\phi(p)\mathrm{d}u \Bigg].$$

Continuing through degenerate conjugate

Forward in time, the Green's function goes from (factoring out a $\sqrt{|\Delta|}/2$)

$$\delta(\sigma) \to -\delta(\sigma) \to \delta(\sigma) \to \dots \tag{16}$$

and backwards in time,

$$\dots \leftarrow -\delta(\sigma) \leftarrow \delta(\sigma). \tag{17}$$

Continuing through simple conjugate

Forward in time, the Green's function goes from

$$\delta(\sigma) \to -\mathsf{pv}\frac{1}{\pi\sigma} \to -\delta(\sigma) \to \mathsf{pv}\frac{1}{\pi\sigma} \to \delta(\sigma) \to \dots \tag{18}$$

and backwards in time,

$$\dots \leftarrow \mathsf{pv}\frac{1}{\pi\sigma} \leftarrow -\delta(\sigma) \leftarrow \mathsf{pv}\frac{1}{\pi\sigma} \leftarrow \delta(\sigma). \tag{19}$$

Conclusion

In this project we have...

1. Studied the general geodesic & conjugate point structure of plane wave spacetimes.
2. Found the general form for the (Hadamard type) Green's function for the wave equation on these spacetimes.
3. Proved a cyclicly alternating behavior of the Green's function as it passes through conjugate points for all plane wave spacetimes.

Future work

1. Perhaps an understanding of this phenomenon in plane wave spacetimes will result in an understanding in general via the Penrose limit.