Frequency domain calculations of eccentric orbit EMRIs on Schwarzschild

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July 4, 2011

Motivation and Scope

- Calculate gravitational self-force for EMRIs

  SCO (mass = μ) orbiting MBH (M)
  BH perturbation theory, expansion in $\epsilon = \mu/M$
  Successive field and motion corrections

- Waveforms for use in LISA detection and parameter estimation

  Cumulative phase $\Phi = \kappa_1 \frac{1}{\epsilon} + \kappa_2 \epsilon^0 + \kappa_3 \epsilon + \cdots$,
  $\Phi \sim 10^6$, $\delta\Phi \lesssim 0.1$ $\rightarrow$ need for second order

  But transient resonances $\rightarrow \epsilon^{-1/2}$ terms for generic orbits on Kerr
  Flanagan and Hinderer (2010), arXiv:1009.4923v2 [gr-qc]

- Present scope: Eccentric orbits on Schwarzschild

  Accurate 1st-order RWZ perturbations (done)
  Metric in RW gauge (done)
  Hopper and Evans (2010), PRD 82:084010; gr-qc:1006.4907

  Gauge generators Ξ_μ and metric in Lorenz gauge (in progress)
  1st-order self-force correction with high accuracy (coming)

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Accuracy Requirements

Example: Assume $\epsilon = \mu/M = 10^{-6}, \quad \Delta\Phi/\Phi \simeq 10^{-7}$

(metric errors)	$\mathcal{O}(1)$		$\mathcal{O}(10^{-6})$		$\mathcal{O}(10^{-12})$	
	$\vert$		$\vert$		$\vert$	
Metric :	$g_{\mu\nu}$	$+$	${}_1p_{\mu\nu}$	$+$	${}_2p_{\mu\nu}$	$+ \cdots$
			$\vert$		$\vert$	
Self − force :			${}_1f_{\mu}$	$+$	${}_2f_{\mu}$	$+ \cdots$
			$\vert$		$\vert$	
(self − force errors)			$\mathcal{O}(1)$		$\mathcal{O}(10^{-6})$	$\mathcal{O}(10^{-12})$

Analytic and Numerical Approach

High-level summary

(1) Use RWZ formalism; find $\Psi^{\rm odd}_{\ell m} = \Psi^{\rm CPM}_{\ell m}(t,r), \quad \Psi^{\rm even}_{\ell m} = \Psi^{\rm ZM}_{\ell m}(t,r)$

(2) Obtain metric in RW gauge

(3) Obtain gauge generators to go from RW to Lorenz gauge

(4) Obtain metric in Lorenz gauge

(5) (Next) use to obtain the self-force

...but how can we do with requisite accuracy

Analytic and Numerical Approach

- Frequency domain (FD) solution for master functions (Step 1)

  Standard Green function method for $R_{\ell mn}(r)$
  Find normalized homogeneous solutions $R^{\pm}_{\ell mn}$ (normalized)

- Time domain (TD) solution for master functions and metric (Step 2)

  Sum $R^{\pm}_{\ell mn}$, return to TD with $\Psi^{\pm}_{\ell m}(t,r)$
  Extend homogeneous solutions (EHS) to $r = r_p(t)$ to find $\Psi_{\ell m}(t,r)$
  Avoid Gibbs phenomenon
  Barack, Ori, Sago (2008), PRD 78:084021
  Reconstruct TD metric $p^{\rm RW}_{\mu\nu}$ in RW gauge

- FD solution for gauge generator amplitudes $\xi^{\ell mn}_{(i)}$ (Step 3)

  Solve separated equations
  Sago, Nakano, and Sasaki (2003), PRD 67:104017

- TD solution for Ξ_μ and metric $p^{\rm L}_{\mu\nu}$ in Lorenz gauge (Step 4)

  Using EHS and (new) EPS methods

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Master Functions and Regge-Wheeler Metric

- Schwarzschild geometry:

$$ds^2 = -f\ dt^2 + \frac{1}{f}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad f \equiv 1 - \frac{2M}{r}.$$

- $\Psi_{\ell m}^{\text{even}}$, $\Psi_{\ell m}^{\text{odd}}$ functions of $p_{\mu\nu}^{\ell m}$ and derivatives in RW gauge
- Both satisfy wave equations

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_\ell(r)\right]\Psi_{\ell m}(t,r) = S_{\ell m}(t,r)$$

- Singular sources

$$S_{\ell m}(t,r) = \tilde{G}_{\ell m}(t)\ \delta[r - r_p(t)] + \tilde{F}_{\ell m}(t)\ \delta'[r - r_p(t)]$$

- $p_{\mu\nu}^{\ell m}$ recovered from $\Psi_{\ell m}^{\text{even/odd}}$, their derivatives, and delta sources

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Master Functions

- $\Psi_{\ell m}^{\text{even/odd}}$ are linear combinations of $h_{\mu\nu}^{\ell m}$ and its derivatives in Regge-Wheeler gauge

- Even-parity master function is the Zerilli-Moncrief function

$$\Psi_{\text{even}}(t,r) \equiv \frac{2r}{\ell(\ell+1)}\left[K + \frac{1}{\Lambda}\left(f^2 h_{rr} - rf\partial_r K\right)\right],$$

- Odd-parity master function is the Cunningham-Price-Moncrief function

$$\Psi_{\text{odd}}(t,r) \equiv \frac{r}{\lambda}\left[\partial_r h_t - \partial_t h_r - \frac{2}{r}h_t\right].$$

Frequency Domain Formalism

Fourier coefficients for field and source in eccentric orbit:

$$R_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} dt\ \Psi_{\ell m}(t,r)e^{i\omega_{mn}t}$$

$$Z_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} dt\ S_{\ell m}(t,r)e^{i\omega_{mn}t}.$$

An ODE for each ℓ, m, n:

$$\frac{d^2 R_{\ell mn}(r)}{dr_*^2} + \Big[\omega_{mn}^2 - V_\ell(r)\Big]R_{\ell mn}(r) = Z_{\ell mn}(r), \quad \omega_{mn} \equiv m\Omega_\varphi + n\Omega_r.$$

Ω_r : Radial libration frequency, Ω_φ : Average angular rate of advance

$$\Psi_{\ell m}(t,r) = \sum_{-\infty}^{+\infty} R_{\ell mn}(r)\, e^{-i\omega mnt}, \quad S_{\ell m}(t,r) = \sum_{-\infty}^{+\infty} Z_{\ell mn}(r)\, e^{-i\omega mnt}$$

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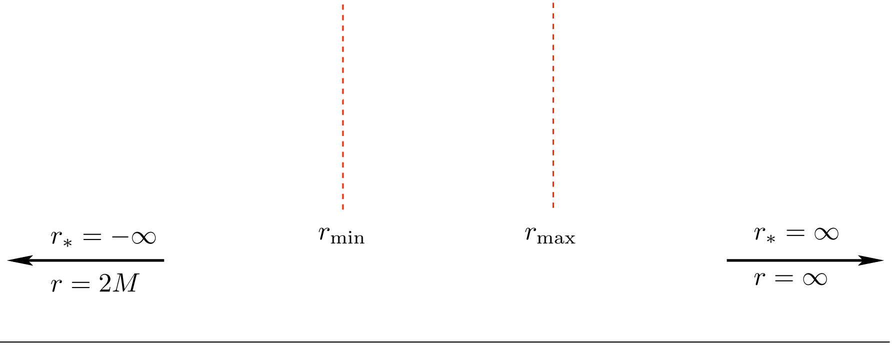
Standard Frequency Domain Method



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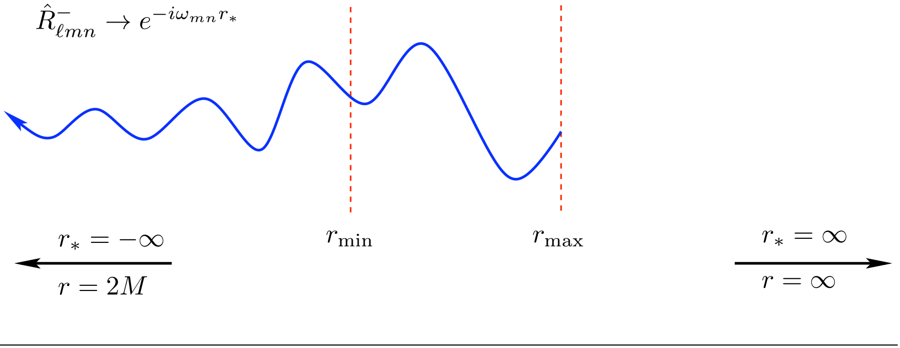

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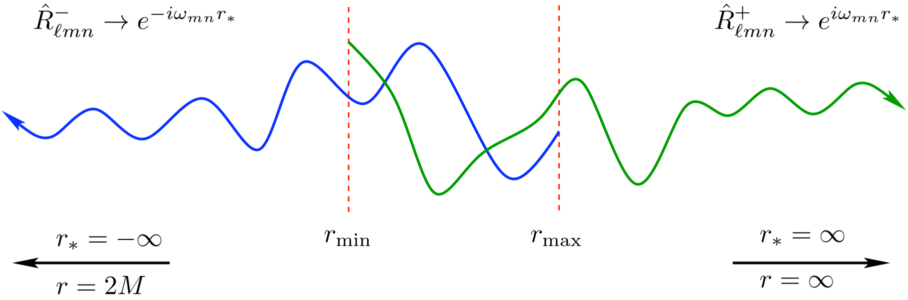

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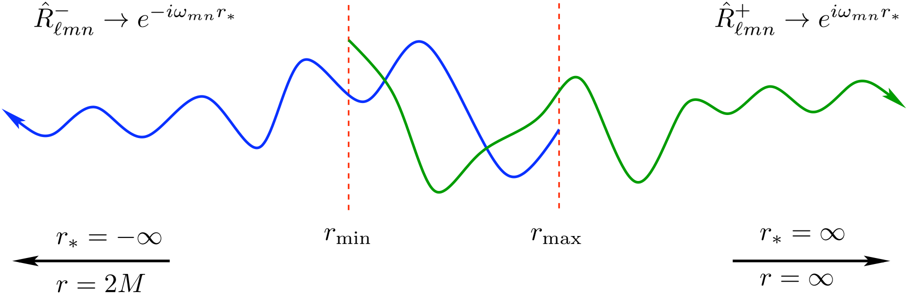

$$R^{\rm std}_{\ell mn}(r) = \hat{R}^+_{\ell mn}(r) \int_{r_{\min}}^{r} \frac{\hat{R}^-_{\ell mn}(r') Z_{\ell mn}(r')}{W_{\ell mn}} dr' + \hat{R}^-_{\ell mn}(r) \int_{r}^{r_{\max}} \frac{\hat{R}^+_{\ell mn}(r') Z_{\ell mn}(r')}{W_{\ell mn}} dr'$$

Standard Frequency Domain Method

- Normalized homogeneous solutions outside $r_{\min}$ and $r_{\max}$:

$$R^{\pm}_{\ell mn} \equiv C^{\pm}_{\ell mn} \hat{R}^{\pm}_{\ell mn}$$

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allow computation of emitted radiation

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- **Gibbs phenomenon appears in TD.**

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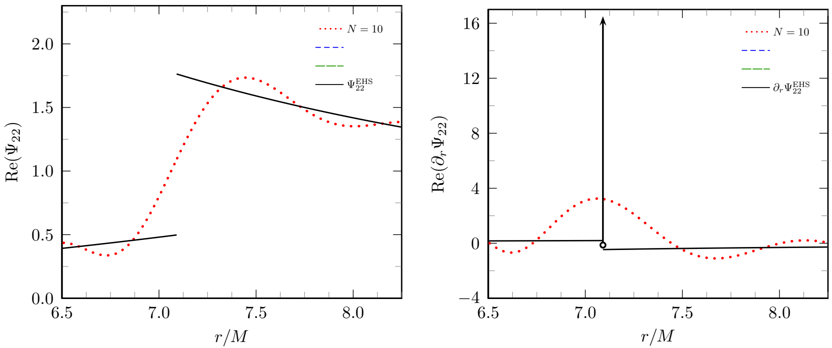
Standard Frequency Domain Method

$\Psi_{22}(t_0, r)$ $\partial_r \Psi_{22}(t_0, r)$



$p = 7.50478$

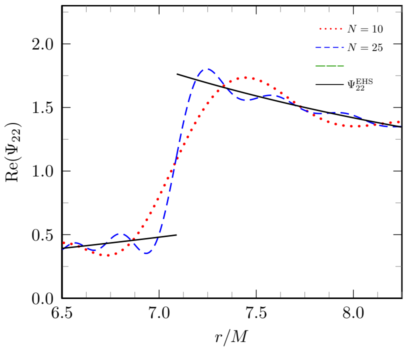
$e = 0.188917$

$t_0 = 51.78M$

$$\Psi_{\ell m}(t_0, r) = \sum_{n=-10}^{10} R^{\text{std}}_{\ell mn}(r) e^{-i\omega_{mn} t_0}$$

Standard Frequency Domain Method

$\Psi_{22}(t_0, r)$



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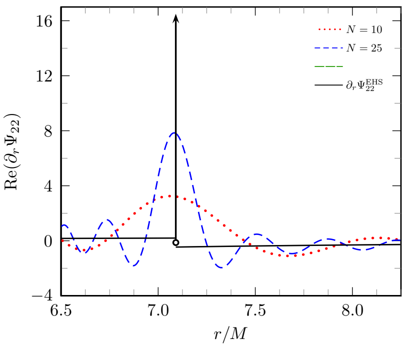

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$$\Psi_{\ell m}(t_0, r) = \sum_{n=-25}^{25} R^{\rm std}_{\ell mn}(r) e^{-i\omega_{mn} t_0}$$

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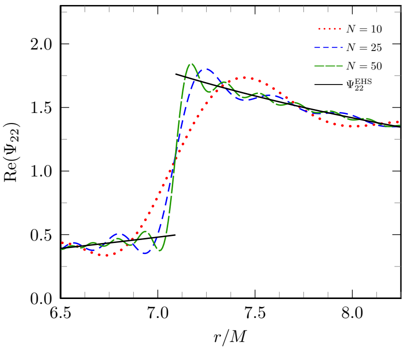
$\Psi_{22}(t_0, r)$



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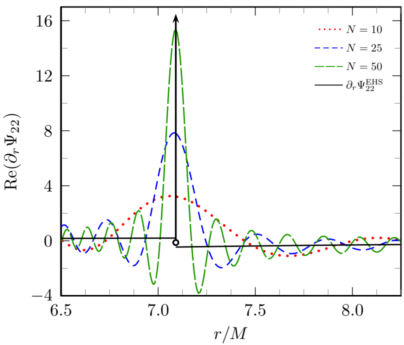

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$$\Psi_{\ell m}(t_0, r) = \sum_{n=-50}^{50} R^{\rm std}_{\ell mn}(r) e^{-i\omega_{mn} t_0}$$

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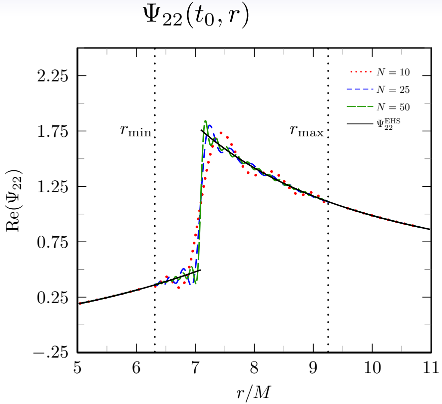

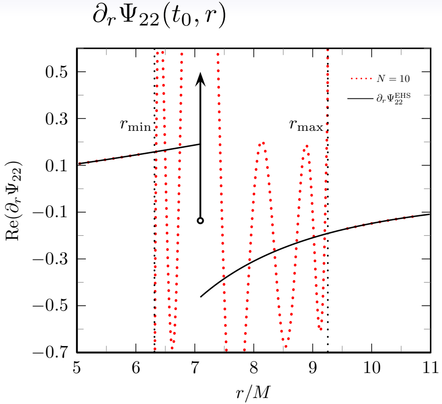

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Method of Extended Homogeneous Solutions

- Barack, Ori, and Sago (2008) demonstrated method for scalar charge on Schwarzschild and for monopole moment
- We applied EHS to "all" radiative gravitational modes (RWZ)
- Define *Time-Domain Extended Homogeneous Solutions:*

$$\Psi^{\pm}_{\ell m}(t,r) = \sum_n R^{\pm}_{\ell mn} e^{-i\omega_{mn} t} = \sum_n C^{\pm}_{\ell mn} \hat{R}^{\pm}_{\ell mn} e^{-i\omega_{mn} t}$$

- Valid in entire range: $2M < r < \infty$
- Argument: Solution to TD RWZ equations is given by

$$\Psi_{\ell m}(t,r) = \Psi^{+}_{\ell m}(t,r)\, \theta\left[r - r_p(t)\right] + \Psi^{-}_{\ell m}(t,r)\, \theta\left[r_p(t) - r\right]$$

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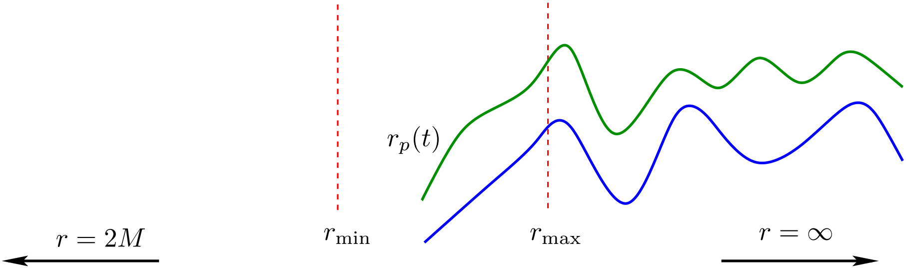
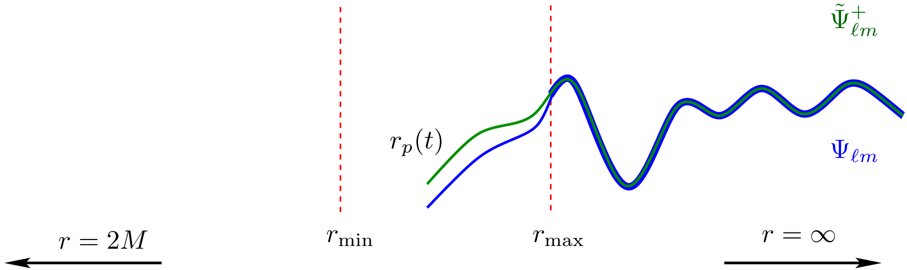


- $\Psi^{\rm std}_{\ell m}$ and $\Psi^{+}_{\ell m}$ can be found in TD from Fourier sums $(r > r_p(t))$
- They coincide throughout $r > r_{\rm max}(t)$ since there $R^{\rm std} = R^{+}$
- While $R^{\rm std} \neq R^{+}$ for $r_p(t) < r < r_{\rm max}$, $\Psi^{\rm std}$ and Ψ^{+} satisfy the same source-free wave equation
- Therefore, they must coincide for all of $r > r_p(t)$.
- The same is true for $\Psi_{\ell m}$ and $\tilde{\Psi}^{-}_{\ell m}$ throughout $r < r_p(t)$.

Method of Extended Homogeneous Solutions



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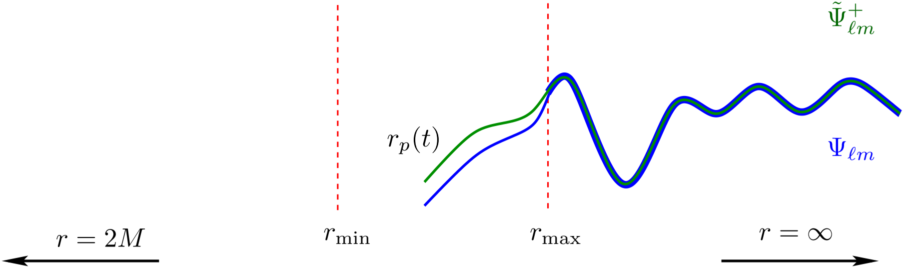
- While $R^{\rm std} \neq R^{+}$ for $r_p(t) < r < r_{\rm max}$, $\Psi^{\rm std}$ and Ψ^{+} satisfy the same source-free wave equation
- Therefore, they must coincide for all of $r > r_p(t)$.
- The same is true for $\Psi_{\ell m}$ and $\tilde{\Psi}^{-}_{\ell m}$ throughout $r < r_p(t)$.

Method of Extended Homogeneous Solutions



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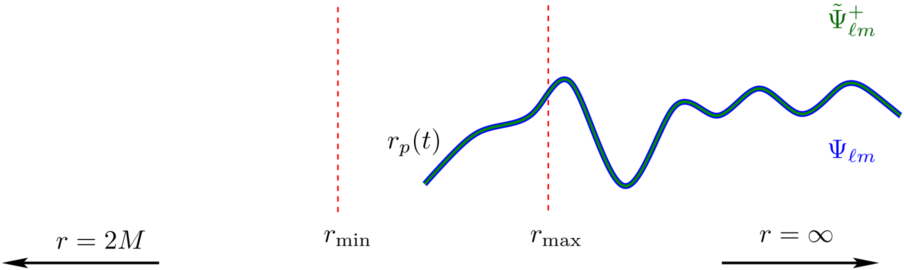

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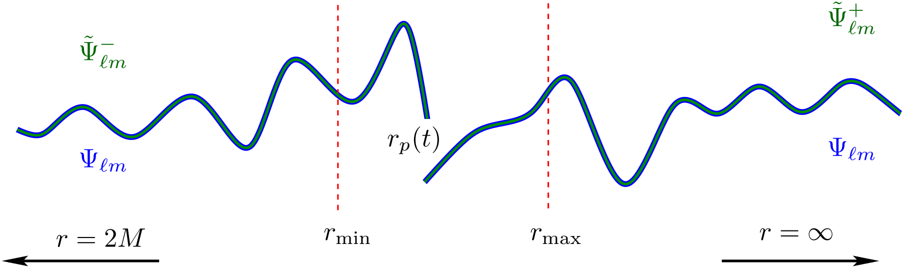

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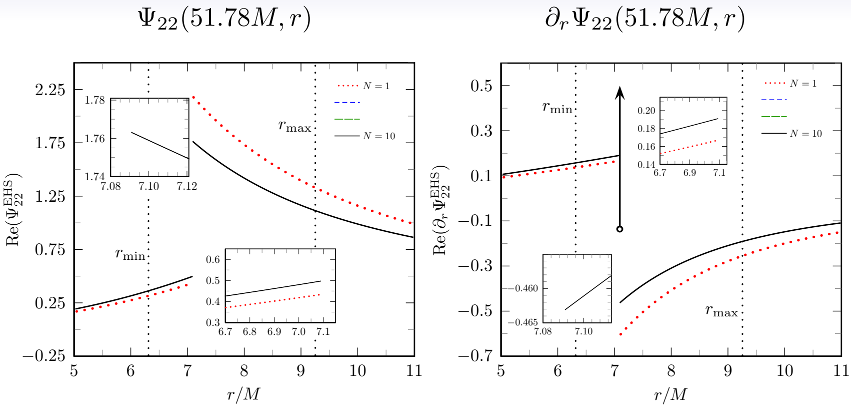
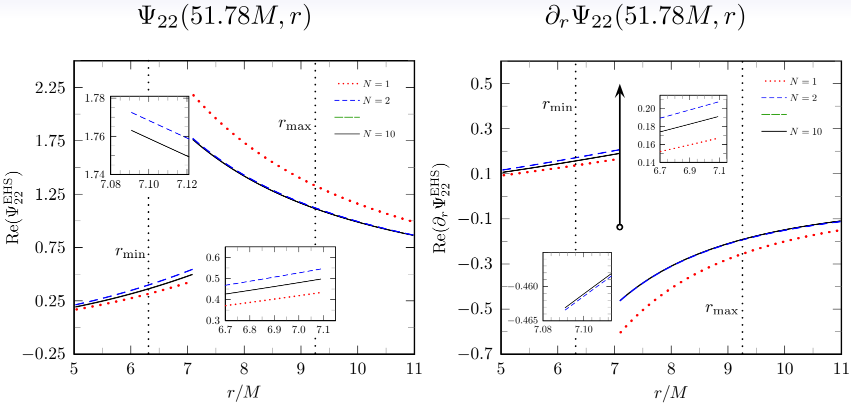
Master Functions via Extended Homogeneous Solns



$$p = 7.50478$$
$$e = 0.188917$$

$$\Psi^{\pm}_{\ell m}(t_p, r) = \sum_{n=-1}^{1} R^{\pm}_{\ell mn} e^{-i\omega_{mn} t}$$

Master Functions via Extended Homogeneous Solns



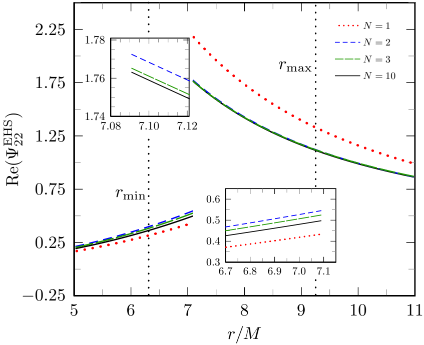
$p = 7.50478$

$e = 0.188917$

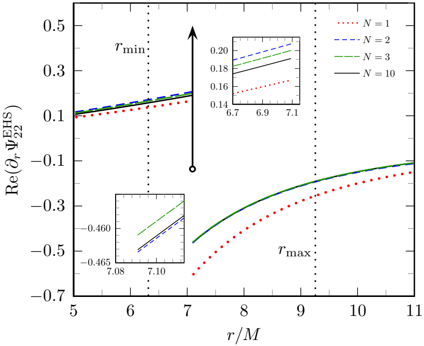
$$\Psi^{\pm}_{\ell m}(t_p, r) = \sum_{n=-2}^{2} R^{\pm}_{\ell mn} e^{-i\omega_{mn} t}$$

Master Functions via Extended Homogeneous Solns

$\Psi_{22}(51.78M, r)$



$\partial_r \Psi_{22}(51.78M, r)$

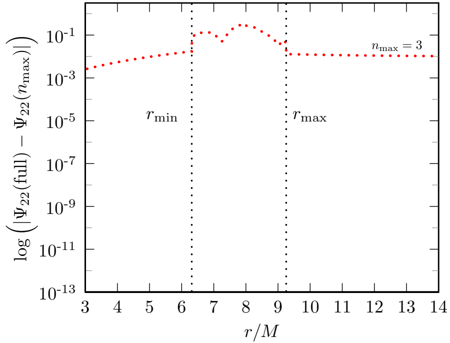


$p = 7.50478$
$e = 0.188917$

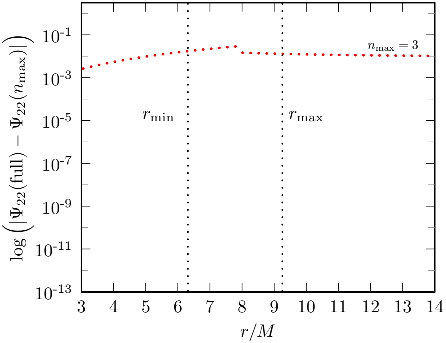
$$\Psi^{\pm}_{\ell m}(t_p, r) = \sum_{n=-3}^{3} R^{\pm}_{\ell m n} e^{-i\omega_{mn} t}$$

Fourier Covergence of the Master Functions

Standard Method



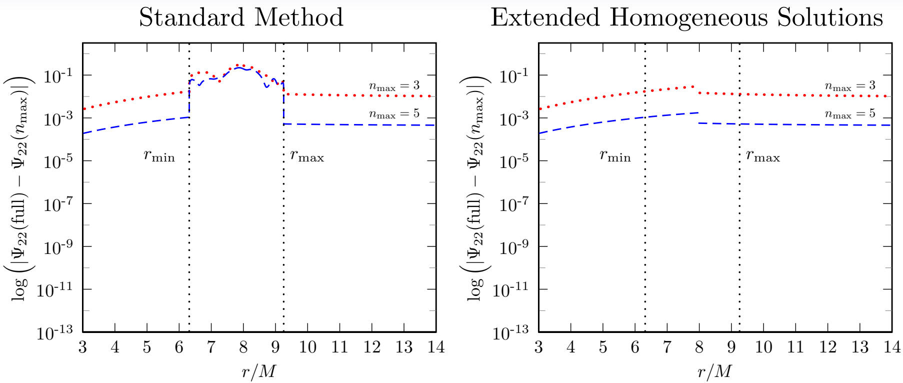
Extended Homogeneous Solutions



$$p = 7.50478$$
$$e = 0.188917$$
$$t = 80.62M$$

$$\Psi_{\ell m}(t_p, r) = \sum_{n=-3}^{3} R_{\ell m n}(r) e^{-i\omega_{mn} t}$$

Fourier Covergence of the Master Functions



$$p = 7.50478$$
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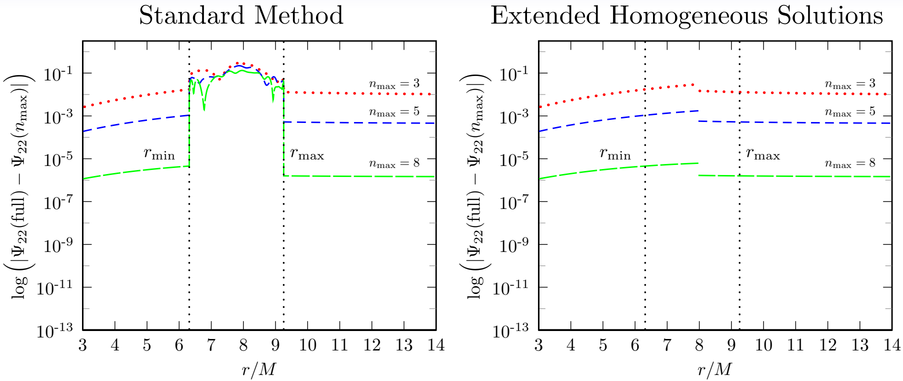
$$t = 80.62M$$

$$\Psi_{\ell m}(t_p, r) = \sum_{n=-5}^{5} R_{\ell mn}(r) e^{-i\omega_{mn} t}$$

Fourier Covergence of the Master Functions



$$p = 7.50478$$
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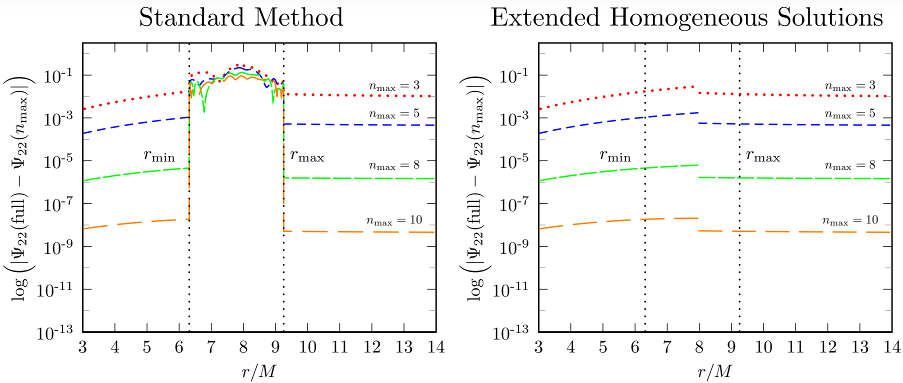
$$\Psi_{\ell m}(t_p, r) = \sum_{n=-8}^{8} R_{\ell mn}(r) e^{-i\omega_{mn} t}$$

Fourier Covergence of the Master Functions



$$p = 7.50478$$
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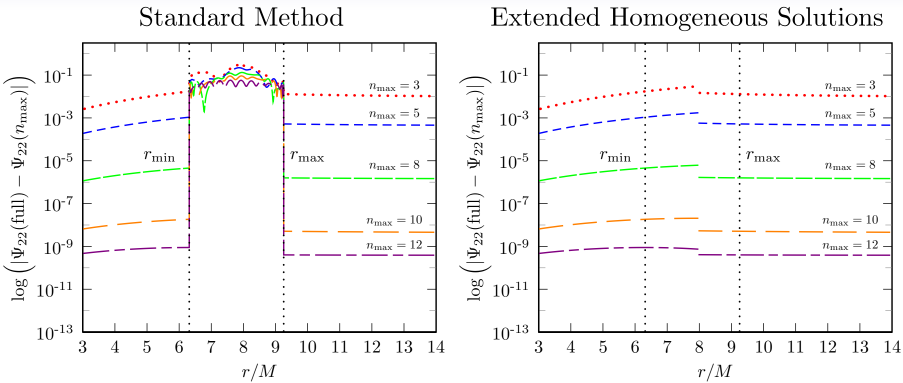
$$t = 80.62M$$

$$\Psi_{\ell m}(t_p, r) = \sum_{n=-10}^{10} R_{\ell mn}(r) e^{-i\omega_{mn} t}$$

Fourier Covergence of the Master Functions



$$p = 7.50478$$
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$$\Psi_{\ell m}(t_p, r) = \sum_{n=-12}^{12} R_{\ell mn}(r) e^{-i\omega_{mn} t}$$

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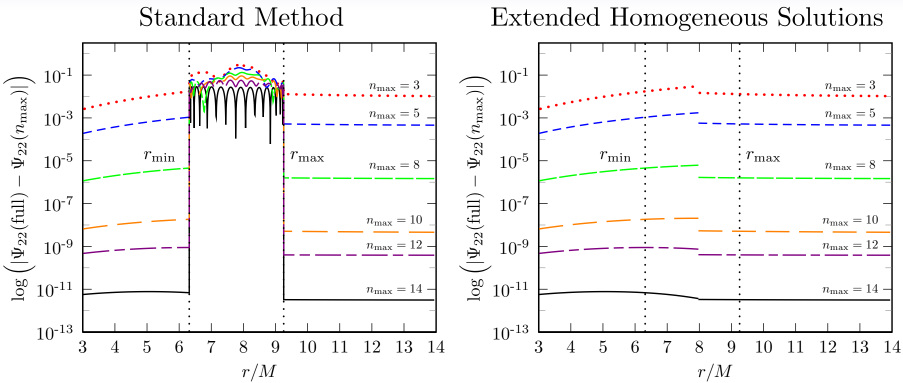

$$p = 7.50478$$
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$$\Psi^{\pm}_{\ell m}(t_p, r) = \sum_{n=-14}^{14} R^{\pm}_{\ell mn}(r) e^{-i\omega_{mn}t}$$

Metric Perturbation Reconstruction

- Even-parity sector:

$$K(t,r) = f\partial_r\Psi_{\rm even} + A\,\Psi_{\rm even} - \frac{r^2 f^2}{(\lambda+1)\Lambda}Q^{tt},$$

$$h_{rr}(t,r) = \frac{\Lambda}{f^2}\left[\frac{\lambda+1}{r}\Psi_{\rm even} - K\right] + \frac{r}{f}\partial_r K,$$

$$h_{tr}(t,r) = r\partial_t\partial_r\Psi_{\rm even} + rB\,\partial_t\Psi_{\rm even} - \frac{r^2}{\lambda+1}\left[Q^{tr} + \frac{rf}{\Lambda}\partial_t Q^{tt}\right],$$

$$h_{tt}(t,r) = f^2 h_{rr} + fQ^{\sharp},$$

$$\Lambda(r) \equiv \lambda + \frac{3M}{r}, \qquad \lambda \equiv \frac{(\ell+2)\,(\ell-1)}{2}.$$

$$A \equiv \frac{1}{r\Lambda}\left[\lambda(\lambda+1) + \frac{3M}{r}\left(\lambda + \frac{2M}{r}\right)\right],\ B \equiv \frac{1}{rf\Lambda}\left[\lambda\left(1 - \frac{3M}{r}\right) - \frac{3M^2}{r^2}\right]$$

- Q's are the even-parity source terms $\propto \delta\left[r - r_p(t)\right]$

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Metric Perturbation Reconstruction

- Odd-parity sector:

$$h_t(t,r) = \frac{f}{2}\partial_r\left(r\Psi_{\rm odd}\right) - \frac{r^2 f}{2\lambda}P^t,$$
$$h_r(t,r) = \frac{r}{2f}\partial_t\Psi_{\rm odd} + \frac{r^2}{2\lambda f}P^r,$$

- P's are odd-parity source terms $\propto \delta\left[r - r_p(t)\right]$
- Again, assume $\Psi = \Psi^+\theta(z) + \Psi^-\theta(-z), \qquad z \equiv r - r_p(t)$
- Both of the odd-parity amplitudes are C^{-1}, with no singular terms:

$$h_t^\pm(t,r) = \frac{f}{2}\partial_r\left(r\Psi^\pm\right), \qquad h_t^{\rm S}(t) = 0,$$
$$h_r^\pm(t,r) = \frac{r}{2f}\partial_t\Psi^\pm, \qquad h_r^{\rm S}(t) = 0.$$

Metric Perturbation Reconstruction

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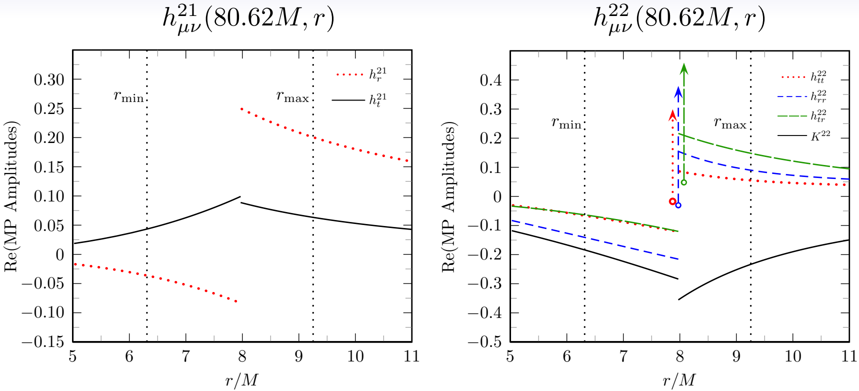
Metric Perturbation Reconstruction



$h^{\ell m}_{\mu\nu}$ recovered from: $\Psi_{\ell m}$, $\partial_\mu \Psi_{\ell m}$ and $T_{\mu\nu}$

- Even parity metric amplitudes h_{tt}, h_{tr}, and h_{rr} are point singular
  Time dependent singular amplitudes are computable
- Metric amplitudes K (even) and h_r (odd) and h_t (odd) are C^{-1}

Transforming to Lorenz gauge

- Gauge transformation from Regge-Wheeler (RW) to Lorenz (L)

$$x^\mu_{\rm RW} \to x^\mu_{\rm L} = x^\mu_{\rm RW} + \Xi^\mu, \qquad |\Xi^\mu| \sim |p_{\mu\nu}| \ll 1$$

- Metric perturbation transforms as

$$p^{\rm RW}_{\mu\nu} \to p^{\rm L}_{\mu\nu} = p^{\rm RW}_{\mu\nu} - \Xi_{\mu|\nu} - \Xi_{\nu|\mu}$$

Stroke $|$ is covariant differentiation with respect to $g_{\mu\nu}$

- Demand $p^{\rm L}_{\mu\nu}$ satisfy the Lorenz gauge condition, $\bar{p}^{\rm L}_{\mu\nu}{}^{|\nu} = 0$
- Therefore

$$\Xi_{\mu|\nu}{}^{\nu} = \bar{p}^{{\rm RW}|\nu}_{\mu\nu} = p^{{\rm RW}|\nu}_{\mu\nu} - \frac{1}{2} g^{\alpha\beta} p^{\rm RW}_{\alpha\beta|\mu}$$

Equations for the Gauge Generator Amplitudes

- Define spherical harmonic amplitudes

$$\Xi_t = \xi_t(t,r)Y^{\ell m}$$
$$\Xi_r = \xi_r(t,r)Y^{\ell m}$$
$$\Xi_A = \xi_{(e)}(t,r)Y_A^{\ell m} + \xi_{(o)}(t,r)X_A^{\ell m}$$

- One, separate odd-parity equation

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - f(r)\frac{\ell(\ell+1)}{r^2}\right]\xi_{(o)}(t,r) = 2f\Psi_{RW} + fp(t)\delta[r - r_p(t)]$$

- Three, coupled even-parity equations

$$\Box\xi_t + M_t(\xi_t, \xi_r) = F_t(\Psi_{ZM}) + \text{singular term}$$
$$\Box\xi_r + M_r(\xi_r, \xi_t, \xi_{(e)}) = F_r(\Psi_{ZM}) + \text{singular term}$$
$$\Box\xi_{(e)} + M_{(e)}(\xi_{(e)}, \xi_r) = F_{(e)}(\Psi_{ZM}) + \text{singular term}$$

Alternative: Sago, Nakano, and Sasaki

- Formalism devised by Sago, Nakano, and Sasaki (2002)
- The odd-parity part same as before
- The even-parity splits further into scalar part and divergence-free vector part

$$\Xi^{\mu}_{\rm even} = {\Xi_{(s)}}^{|\mu} + \Xi^{\mu}_{(v)}$$

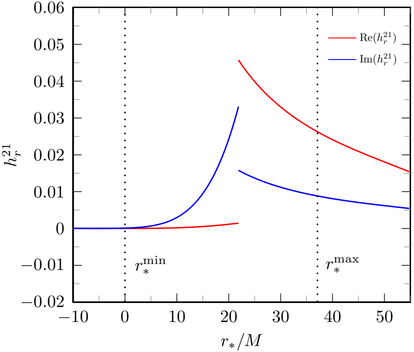
- $r\Xi_{(s)}$ satisfies 4th order wave equation
- Antisymmetric gradient of $\Xi_{(v)}$ projects to two scalars ϕ_0 and ϕ_2

  ϕ_0 and ϕ_2 enter into $s = -1$ and $s = +1$ Teukolsky equations
- A last equation sourced by ϕ_0 and ϕ_2 determines ξ_t, ξ_r, and $\xi_{(e)}$

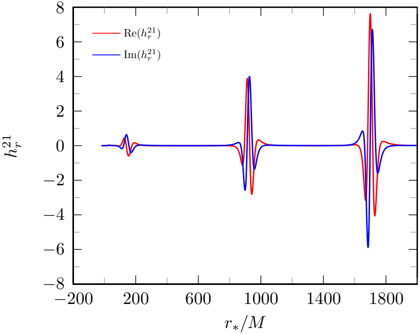
- Direct even-parity gauge solution: 3 coupled 2nd order equations
- SNS even-parity gauge solution: 1 separate 4th order, 3 separate 2nd order equations

Sneak Peek: $h_r^{\ell m}$ in Regge-Wheeler gauge

$h_r^{21}(t_\circ, r_*)$ locally



$h_r^{21}(t_\circ, r_*)$ asymptotically



$$p = 8.75455$$

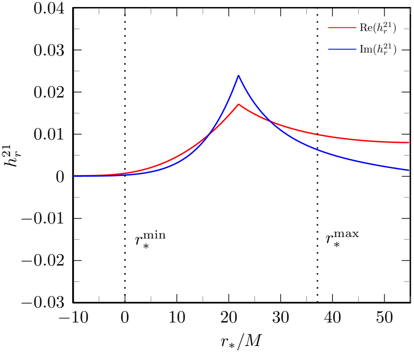
$$e = 0.764124$$

$$t_\circ = 143.45M$$

$$-50 \leq n \leq 50$$

Sneak Peek $h_r^{\ell m}$ in Lorenz gauge

$h_r^{21}(t_\circ, r_*)$ locally



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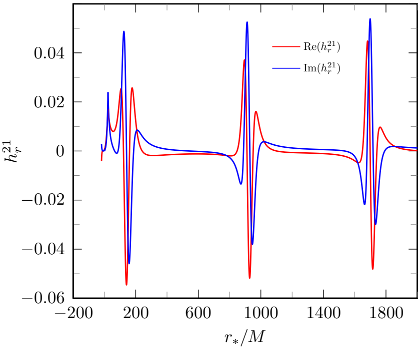

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$$-50 \leq n \leq 50$$

- Now C^0 at the particle
- Asymptotically $\sim$ wave

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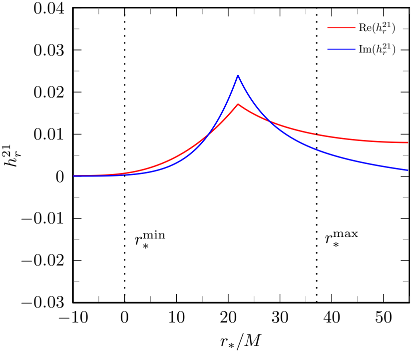

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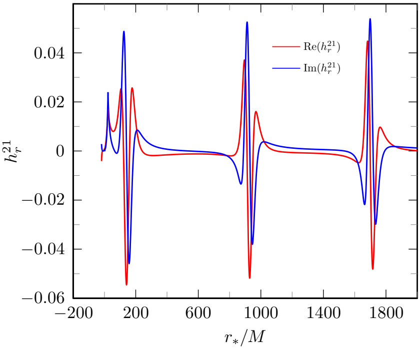

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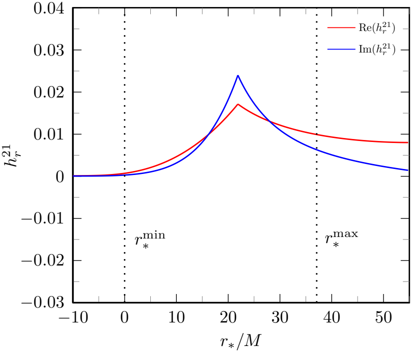

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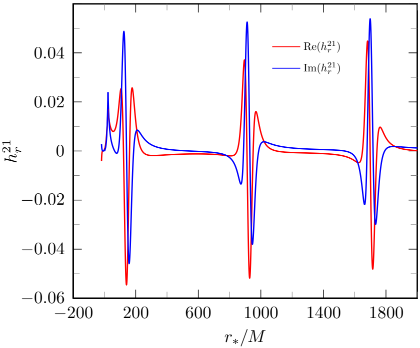

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Conclusions

- Highly accurate (10^{-12}) $\Psi_{\ell m}^{\mathrm{even/odd}}$ for eccentric orbits

  Use FD engine
- EHS method obtains master functions and metric (RW) accurate at $r = r_p(t)$
- TD jump conditions provide checks and convergence criteria
- Reasonable runtimes $\lesssim 6$ hours for $0 \le e \lesssim 0.8$
- Frequency domain fully competitive with time domain for most e.
- Avoids transients
- Gauge transformation to Lorenz in progress (Seth Hopper)
- Using Sago, Nakano, and Sasaki instead of direct, coupled 1+3 gauge equations
- Some new solution techniques coming out of latter effort

Jump Conditions in the Time Domain

- Now we equate the respective δ and δ' coefficients from the two sides, leaving

$$[\![\Psi]\!]_p(t) = \frac{\mathcal{E}^2}{f_p^2 U_p^2} \tilde{F}(t)$$

$$[\![\partial_r \Psi]\!]_p(t) = \frac{\mathcal{E}^2}{f_p^2 U_p^2} \left[\tilde{G}(t) + \frac{1}{U_p^2 r_p^2} \left(3M - \frac{\mathcal{L}^2}{r_p} + \frac{5M\mathcal{L}^2}{r_p^2} \right) \tilde{F}(t) - 2\dot{r}_p \frac{d}{dt} \left([\![\Psi]\!]_p \right) \right].$$

- $U^2(r, \mathcal{L}^2) \equiv f \left(1 + \mathcal{L}^2 / r^2 \right)$
- $\mathcal{E}$ and $\mathcal{L}$ are the specific energy and angular momentum
- These expressions provide a powerful check on the validity of our method.
- They also provide useful stopping conditions, telling us when we've computed enough frequency modes.

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Jump Conditions in the Time Domain

- Now we equate the respective δ and δ' coefficients from the two sides, leaving

$$[\![\Psi]\!]_p(t) = \frac{\mathcal{E}^2}{f_p^2 U_p^2} \tilde{F}(t)$$

$$[\![\partial_r \Psi]\!]_p(t) = \frac{\mathcal{E}^2}{f_p^2 U_p^2} \left[\tilde{G}(t) + \frac{1}{U_p^2 r_p^2} \left(3M - \frac{\mathcal{L}^2}{r_p} + \frac{5M\mathcal{L}^2}{r_p^2} \right) \tilde{F}(t) - 2\dot{r}_p \frac{d}{dt} \left([\![\Psi]\!]_p \right) \right].$$

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Master Functions

2 degrees of freedom corresponding to 2 wave polarizations

$\Psi_{\ell m}^{\rm even/odd}$ are linear combinations of $h_{\mu\nu}^{\ell m}$ and its derivatives in Regge-Wheeler gauge:

$$\Psi_{\rm even}(t,r) \equiv \frac{2r}{\ell(\ell+1)}\left[K + \frac{1}{\Lambda}\left(f^2 h_{rr} - rf\partial_r K\right)\right],$$

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Both satisfy wave equations

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_\ell(r)\right]\Psi_{\ell m}(t,r) = S_{\ell m}(t,r), \quad r_* : \text{tortoise coord}$$

Singular sources: $S_{\ell m}(t,r) = \tilde{G}_{\ell m}(t)\ \delta[r - r_p(t)] + \tilde{F}_{\ell m}(t)\ \delta'[r - r_p(t)]$

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Metric Perturbation Reconstruction

- Even-parity sector:

$$K(t,r) = f\partial_r \Psi_{\text{even}} + A\,\Psi_{\text{even}} - \frac{r^2 f^2}{(\lambda+1)\Lambda} Q^{tt},$$

$$h_{rr}(t,r) = \frac{\Lambda}{f^2}\left[\frac{\lambda+1}{r}\Psi_{\text{even}} - K\right] + \frac{r}{f}\partial_r K,$$

$$h_{tr}(t,r) = r\partial_t\partial_r\Psi_{\text{even}} + rB\,\partial_t\Psi_{\text{even}} - \frac{r^2}{\lambda+1}\left[Q^{tr} + \frac{rf}{\Lambda}\partial_t Q^{tt}\right],$$

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$$\Lambda(r) \equiv \lambda + \frac{3M}{r}, \qquad \lambda \equiv \frac{(\ell+2)(\ell-1)}{2}.$$

$$A \equiv \frac{1}{r\Lambda}\left[\lambda(\lambda+1) + \frac{3M}{r}\left(\lambda + \frac{2M}{r}\right)\right], \; B \equiv \frac{1}{rf\Lambda}\left[\lambda\left(1 - \frac{3M}{r}\right) - \frac{3M^2}{r^2}\right]$$

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Metric Perturbation Reconstruction

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$$h_t(t,r) = \frac{f}{2}\partial_r\left(r\Psi_{\text{odd}}\right) - \frac{r^2 f}{2\lambda}P^t,$$
$$h_r(t,r) = \frac{r}{2f}\partial_t\Psi_{\text{odd}} + \frac{r^2}{2\lambda f}P^r,$$

- P's are odd-parity source terms $\propto \delta\left[r - r_p(t)\right]$
- Again, assume $\Psi = \Psi^+\theta(z) + \Psi^-\theta(-z), \qquad z \equiv r - r_p(t)$
- Both of the odd-parity amplitudes are C^{-1}, with no singular terms:

$$h_t^{\pm}(t,r) = \frac{f}{2}\partial_r\left(r\Psi^{\pm}\right), \qquad h_t^{\text{S}}(t) = 0,$$
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Black Hole Perturbation Theory

	Field Equations	Equations of Motion
0	$G_{\mu\nu}(g_{\alpha\beta}) = 0$	$\dfrac{Du^\mu}{d\tau} = 0$
	$\Box\bar{h}_{\alpha\beta} + 2R^{\gamma}{}_{\alpha}{}^{\delta}{}_{\beta}\bar{h}_{\gamma\delta} = -16\pi T_{\alpha\beta}$ $\bar{h}^{\alpha\beta}{}_{\vert\beta} = 0$ $\mathrm{g}_{\mu\nu} = g_{\mu\nu} + {}_1h_{\mu\nu}\cdots$	$\dfrac{Du^\mu}{d\tau} = -\dfrac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right) \times \left(2h_{\nu\lambda;\rho} - h_{\lambda\rho;\nu}\right)u^\lambda u^\rho$ MiSaTaQuWa Equations
	$\Box_2\bar{h} = (1 + {}_1\bar{h})T[z+\delta z] + \left(\nabla_1\bar{h}\right)^2$ (Schematic)	

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	$\Box \bar{h}_{\alpha\beta} + 2R^{\gamma}{}_{\alpha}{}^{\delta}{}_{\beta}\bar{h}_{\gamma\delta} = -16\pi T_{\alpha\beta}$ $\bar{h}^{\alpha\beta}{}_{\vert\beta} = 0$ $\mathrm{g}_{\mu\nu} = g_{\mu\nu} + {}_1h_{\mu\nu} \cdots$	$\dfrac{Du^\mu}{d\tau} = -\dfrac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right) \times \left(2h_{\nu\lambda;\rho} - h_{\lambda\rho;\nu}\right) u^\lambda u^\rho$ MiSaTaQuWa Equations
	$\Box_2 \bar{h} = (1 + {}_1\bar{h})T[z + \delta z] + \left(\nabla_1 \bar{h}\right)^2$ (Schematic)	

Black Hole Perturbation Theory

	Field Equations	Equations of Motion
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(Schematic)

Black Hole Perturbation Theory

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Analytic and Numerical Method

SH & CRE (2010)	Barack and Sago (2010)
Frequency domain	Time domain
RWZ formalism	Lorenz gauge
Master functions: $\Psi^{\rm odd}_{\ell m}$, $\Psi^{\rm even}_{\ell m}$	Metric perturbations: $h^{\ell m}_{\mu\nu}$
ODEs but multiple harmonics	Coupled PDEs
Exact bndy conditions, periodic	IV problem, transients
Smeared source, libration region	Moving delta source
ODE accuracy, TD convergence	4th-order PDE scheme
Gibbs problem (use EHS)	Time dep. jump conditions
Reconstructed metric (RW gauge)	Metric in Lorenz gauge
Gauge transform and self-force	Self-force
Kerr/higher order?	Kerr/higher order?